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ON THE HEEGAARD GENUS OF A CLOSED, ORIENTABLE 3-MANIFOLD

By

R. CRAGGS

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We consider closed, compact, connected, orientable, pwl 3-manifolds and their subpolyhedra. By a Heegaard decomposition $(M; U, V)$ of a 3-manifold M, we understand a partition $M=U\cup V$ into a pair of cubes with handles U and V (regular neighborhoods of graphs) such that $U\cap V=BdU=BdV$. Notice that if we collapse U to a 1-dimensional graph G, we can reconstruct the decomposition, up to equivalence, by taking as U any regular neighborhood of G and as V the complement $M\backslash \operatorname{Int} U$. We will thus refer to a compact, connected graph G in a 3-manifold M as a *Heegaard graph* (see also [1]) if both a regular neighborhood $N(G)$ of G and the complement $M\backslash \operatorname{Int} N(G)$ are cubes with handles. Note that for a Heegaard graph G in M, if $\chi(G)=1-n$, then G defines a Heegaard decomposition of genus n.

We prove the following theorem:

Theorem. *Suppose that G is a Heegaard graph in the 3-manifold M with $\chi(G)=1-n$. Suppose further that G contains a connected subgraph G_0 with $\chi(G_0)=1-p$ such that G is contained in the interior of a 3-ball B in M.*

Then M has a Heegaard decomposition of genus at most $n-p$; that is, the Heegaard genus of M is at most $n-p$.

Proof. There is no loss in assuming that M is not a 3-sphere.

Step 1. We may assume that G and G_0 are cell complexes with one 0-cell and n and p 1-cells respectively:

Regard G as a simplicial complex. Locate a maximal tree T in G that intersects G_0 in a maximal tree T_0. Using the collapse $T\searrow T_0$ deform the 3-ball B so that it engulfs T and still contains G_0 in its interior. Now it is an easy matter to define a series of 2-dimensional deformations of G, all taking place inside B, so that all the 1-cells of $G\backslash T$ are pushed down and attached to a single vertex $t\in T$. Finally collapse T back to the vertex t. We are left with G and G_0 as the desired bouquets of loops. Since the deformations do not change the regular neighborhood type, it follows that the modified G is still a Heegaard graph. Note that $G_1=Cl(G_0\backslash G_0)$ is now connected.

Alternate Step 1. After the deformation of B mentioned in the last paragraph, shrink T to a point. With a suitable triangulation, M/T is pwl homeomorphic to M and B/T and G/T are polyhedra with G/T and $G_0/T=G_0/T_0$ bouquets of circles where $Cl(G/T\backslash G_0/T)$ is connected. Replace M, B, G, and G_0 by M/T, B/T, G/T, and G_0/T and proceed to Step 2.

Step 2. A split regular neighborhood for G:

Let D be a disk in $\operatorname{Int} B$ with $D\cap G=t\in\operatorname{Int} D$. Choose D so that locally it separates $G_0\backslash t$ from $G_1\backslash t$ in M. Choose a regular neighborhood U of G so that $U\cap D$ is a regular neighborhood of t in $\operatorname{Int} D$ with $U\searrow G\cup(U\cap D)\searrow G$ and so that the component of $U\backslash D$ containing $G_0\backslash t$ is contained in $\operatorname{Int} B$. Let U_0 and U_1 denote the closures of the components of $U\backslash D$ containing G_0 and G_1 respectively. Note that we have $U=U_0\cup U_1$ with $U_0\cap U_1$ a disk and that U_0 is contained in $\operatorname{Int} B$.

This next step is well known, but we include it for completeness.

Step 3. Some homology meridians on BdU_0:

We claim that there are p disjoint, homologously independent simple closed curves $J_1, \cdots, J_p$ on $(BdU_0)\backslash D$ that are homologous to zero in $B\backslash\operatorname{Int} U_0$. (Homology coefficients $\boldsymbol{Z}$ are assumed here.)

An application of the Mayer-Vietoris sequence produces an internal direct sum decomposition $H_1(BdU_0)=H\oplus K$ where H denotes the kernel of the (inclusion induced) homomorphism $H_1(BdU_0)\to H_1(B\backslash\operatorname{Int} U_0)$ and K denotes the kernel of the corresponding homomorphism $H_1(BdU_0)\to H_1(U_0)$. Now K is free of rank p; so H must also be free of rank p. Thus there is a basis $h_1, \cdots, h_p, k_1, \cdots, k_p$ for $H_1(BdU_0)$ so that $h_1, \cdots, h_p$ is a basis for H and $k_1, \cdots, k_p$ is a basis for K.

The intersection numbers $h_i\circ h_j$ and $k_i\circ k_j$ must be zero for all pairs i and j. If, say, $h_i\circ h_j$ were not zero, then we could push a cycle representing h_i slightly into $\operatorname{Int} U_0$ and there it would link a cycle representing h_j. But that would contradict the fact that a cycle representing h_j is homologously trivial in $B\backslash\operatorname{Int} U_0$. A similar argument applies to the numbers $k_i\circ k_j$.

Let h_j correspond to the jth column of the matrix $\begin{pmatrix} I_{p,p} \\ O_{p,p}\end{pmatrix}$ and k_j to the jth column of the matrix $\begin{pmatrix} O_{p,p} \\ I_{p,p}\end{pmatrix}$. Then a column vector $X=[x_1, \cdots, x_{2p}]^T$ corresponds to $\sum_{i=1}^{p} x_j h_j+x_{j+p}k_j$. With respect to this basis, intersection numbers are given by $X\circ Y=X^TRY$ where

$$R=\left(\begin{array}{c|c} 0 & -C^T \\ \hline C & 0 \end{array}\right)=\left(\begin{array}{c|c} 0 & (h_i\circ k_j) \\ \hline (k_i\circ h_j) & 0 \end{array}\right).$$

Now C is unimodular since the intersection pairing is a unimodular form. Let $k_1', \cdots, k_p'$ correspond to the successive columns of

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & C^{-1} \end{array}\right)\begin{pmatrix} 0 \\ I_{p,p} \end{pmatrix}.$$

With respect to the revised basis $h_1, \cdots, h_p, k'_1, \cdots, k'_p$, the intersection matrix is

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & C^{-1} \end{array}\right)\left(\begin{array}{c|c} 0 & -C^T \\ \hline C & 0 \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & (C^T)^{-1} \end{array}\right)=\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right).$$

Corresponding to the presentation

$$\pi_1(BdU_0)=\langle g_1, \cdots, g_{2p} | [g_1, g_{p+1}] \cdots [g_p, g_{2p}]\rangle$$

there are simple closed curves $S_1, \cdots, S_{2p}$ on $BdU_0 \backslash D$ representing the conjugacy classes of $g_1, \cdots, g_{2p}$ so that $S_1, \cdots, S_p$ are disjoint, $S_{p+1}, \cdots, S_{2p}$ are disjoint, and $S_i \cap S_{j+p}$ is either empty or a single point of transverse intersection accordingly as $i \neq j$ or $i=j$. Thus, when orientation matters are seen to, we may regard $g_1, \cdots, g_{2p}$ as a basis for $H_1(BdU_0)$ so that the intersection matrix with respect to this basis is $\left(\begin{array}{c|c} 0 & -I \\ \hline I & 0 \end{array}\right)$. The automorphism induced by $g_j \to h_j$, $g_{j+p} \to k'_j$, $j \leqq p$ preserves intersection numbers and so, by a theorem of Nielsen's, see [4, Ch. 3, Th. N13], the automorphism is induced by a homeomorphism h. We may assume that h is the identity on $D \cap BdU_0$.

The desired simple closed curves $J_1, \cdots, J_p$ may be taken to be $h(S_1), \cdots, h(S_p)$.

Step 4. Surgery on M:

Let W be a cube with p-handles and let $L_1, \cdots, L_p$ be a complete set of meridians on BdW. Let $f: BdW \to BdU_0$ be a homeomorphism chosen so that $f(L_i)=J_i$. Consider now the 3-manifold M' defined by

$$M'=W \underset{x=f(x)}{+} (U_1 \cup V)$$

where $+$ denotes disjoint union. Then M' results from a surgery that replaces U_0 by W.

The 3-ball B can be used to express M as the connected sum $M \# \Sigma$ where Σ denotes the 3-sphere and B is thought of as the part of Σ remaining after formation of connected sum. The surgery takes place inside B, so what we really have is $M'=M \# M_0$ where M_0 results from a corresponding surgery on Σ. A second application of the Mayer-Vietoris sequence reveals that $H_1(M_0)$ is free of rank p.

Step 5. Completion of the proof:

Given two Heegaard decompositions $(M_a; U_a, V_a)$ and $(M_b; U_b, V_b)$, the *sum* $(M_a; U_a, V_a) \# (M_b; U_b, V_b)$ is formed by locating 3-balls A and B in M_a and M_b so that $A \cap U_a$ and $B \cap U_b$ are disks properly embedded in A and B. Then

$M_a \backslash \mathrm{Int}\, A$ is attached to $M_b \backslash \mathrm{Int}\, B$ by means of a homeomorphism $f: BdA \to BdB$ that sends $(BdA) \cap U_a$ to $(BdB) \cap U_b$. This naturally forms, in addition to $M_a \# M_b$, two boundary sum $U_a \# U_b$ and $V_a \# V_b$, and the sum of decompositions is defined to be $(M_a \# M_b, U_a \# U_b, V_a \# V_b)$.

A theorem of Haken's [2, Sec. 7] together with the uniqueness of connected sum decompositions of 3-manifolds [3, 5] implies that any Heegaard decomposition of a non-trivial connected sum of 3-manifolds splits as a sum of decompositions of the component 3-manifolds. The application of Haken's theorem may require one to convert non-separating 2-spheres in a 3-manifold to separating 2-spheres, but this is easy to do.

We have a Heegaard decomposition $(M', W \underset{x=f(x)}{+} U_1, V)$ of M' of genus n. From the preceding remarks we see that this decomposition splits as a sum $(M; U_2, V_2) \# (M_0; U_3, V_3)$. The genus of $(M_0; U_3, V_3)$ is at least p since the first Betti number of M_0 is p. Thus the genus of $(M; U_2, V_2)$ is at most $n-p$ and so we may take $(M; U_2, V_2)$ to be the decomposition promised by our theorem.

Question: Is the Heegaard decomposition $(M; U_2, V_2)$ a summand of the original decomposition $(M; U, V)$? We conjecture that it is.

The following corollary to our theorem is almost immediate. We omit a proof:

Corollary. *Suppose that G is a Heegaard graph in the 3-manifold M with $\chi(G)=1-n$. Suppose further that there are disjoint 3-balls $B_0, \cdots, B_r$ in M and disjoint, connected subgraphs $G_0, G_1, \cdots, G_r$ of G such that $G_i \subseteq \mathrm{Int}\, B_i$ for each i. Then M has a Heegard decomposition of genus at most $n - \sum_{i=0}^{r} p_i$.*

References

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Department of Mathematics
University of Illinois
at Urbana-Champaign
1409 West Green Street
Urbana, Illinois 61801