YOKOHAMA MATHEMATICAL
JOURNAL VOL. 31, 1983

ON THE CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT RANDOM FIELDS

By

HIROSHI TAKAHATA

(Received October 12, 1982)

0. Introdution.

The central limit theorem (C. L. T.) for weakly dependent random fields has been studied by many authors (for example, [2], [7], [8], [9], [10] and [11]) and applied to Gibbs random fields.

The techniques of proof are divided into three types roughly. The first is calculation of semi-invariants ([2] and [8]), the second, Bernstein's blocking ([10] and [11]) and the third, the use of Chen's result [9].

In this paper we shall prove the C. L. T. for weakly dependent random fields by Stein's technique. In below we explain our method shortly. In [12] C. Stein characterizes the standard normal distribution $N(0, 1)$ as follows. A random variable W has the distribution $N(0, 1)$ iff for sufficiently many $f\in C_b^1$ ($\|f\|_\infty<\infty$, $\|f'\|_\infty<\infty$ and f' is continuous on R^1)

$$E\{f'(W)-Wf(W)\}=0. \tag{0.1}$$

From this fact it can be guessed that the distribution of a random variable W is close to $N(0, 1)$ if for each $f\in C_b^1$ $E\{f'(W)-Wf(W)\}$ is close to 0 in some sense. In fact, for each bounded continuous function h, define

$$f(x)=e^{x^2/2}\int_{-\infty}^{x}(h(t)-Nh)e^{-t^2/2}dt \tag{0.2}$$

where

$$Nh=(2\pi)^{-1/2}\int_{-\infty}^{\infty}h(x)e^{-x^2/2}dx,$$

then $f'(x)-xf(x)=h(x)-Nh$, hence we have

$$E\{f'(W)-Wf(W)\}=E\{h(W)\}-Nh. \tag{0.3}$$

This simple idea was, firstly, used in estimating of the error in the C. L. T. for weakly dependent random variables [12] by Stein himself (cf. [13]). And Erickson [6] and Chen ([4], [5] and etc.) developed the idea and obtained many good results which have been thought to be impossible to prove by "classical" blocking method. As to a detailed explanation of Stein's idea and related topics the readers should be referred to Chen [5].

The usefulness of Stein's method is not only in its simplicity but also in its generality. We shall consider a random field $X=\{X_a ; a\in Z^d\}$ (for the definition, see section 1) and the convergence for sums $S(V)=\sum_{a\in V} X_a$ $(V\subset Z^d)$. The proof by blocking needs some regularity of the shape of V, for example, cube. On the other hand the proof by calculating the semi-invariants needs rather strong conditions on the moments and the rate of the decay of mixing coefficients. In this paper we shall show that the C.L.T. for weakly dependent random fields can be proved under weaker conditions without assuming any regularity of the shape of V.

Our results are generalizations of results of Nahapetian [8] and Neaderhouser [11].

1. Preliminaries and results.

Let $X=\{X_a ; a\in Z^d\}$ $(d\geqq 1)$ be a family of random variables defined on a probability space $(\Omega, \mathcal{A}, P)$, indexed by elements in Z^d, with $E(X_a)=0$ for all $a\in Z^d$. We shall use the following notations and abbreviations in the sequel.

(N1) $D=Z^d$. $|a-b|=\max_{1\leqq i\leqq d}|a_i-b_i|$ for $a=(a_1, \cdots, a_d)$ and $b=(b_1, \cdots, b_d)$ in D.

(N2) $|V|=\#\{a ; a\in V\}$ the cardinality of $V\subset D$.

(N3) $\mathfrak{A}=\{V ; |V|<\infty, V\subset D\}$.

(N4) for $V_1, V_2\in\mathfrak{A}$, $d(V_1, V_2)=\min\{|a-b| ; a\in V_1, b\in V_2\}$.

(N5) for $V\in\mathfrak{A}$, $\mathfrak{F}(V)=\sigma\{X_a ; a\in V\}$: if $V=\phi$, $\mathfrak{F}(V)=\{\Omega, \phi\}$.

(N6) $S(V)=\sum_{a\in V} X_a$, $\sigma^2(V)=E(S(V)^2)$.

(N7) $\mathfrak{G}$ denotes the class of functions g with the following properties;

(1) for some $a\geqq 0$, $g(x)$ is defined on $(a, +\infty)$, and $g(x)\to+\infty$ as $x\to+\infty$.

(2) for any positive numbers a, b, c and d $(c<d)$

$$g(x)^c/g(x^a g(x)^{-b})^d\to 0 \quad \text{as} \quad x\to+\infty. \tag{1.1}$$

Remark. $\mathfrak{G}$ contains $\log x$, $\log\log x$ and etc.

Now we introduce two types of conditions on the weak dependence for X.

(M1). ϕ-mixing [10]: there exist a sequence $\{\phi(n)\}$ of nonnegative real numbers and a nondecreasing function M_1 defined on Z_+ such that $\phi(n)\downarrow 0$ as $n\to+\infty$ and that, for each $V_1, V_2\in\mathfrak{A}$

$$\sup_{A\in\mathfrak{F}(V_1)}|P(A|\mathfrak{F}(V_2))-P(A)|\leqq M_1(|V_1|)\phi(d(V_1, V_2)). \tag{1.2}$$

(M2). Strong mixing [11]: there exist a sequence $\{\alpha(n)\}$ of nonnegative real numbers and a function M_2 defined on Z_+^2, nondecreasing with respect to each variable, such that $\alpha(n)\downarrow 0$ as $n\uparrow +\infty$ and that, for each $V_1, V_2\in\mathfrak{A}$,

$$\sup_{A\in\mathcal{F}(V_1)}\sup_{B\in\mathcal{F}(V_2)}|P(A\cap B)-P(A)P(B)|\leqq M_2(|V_1|, |V_2|)\alpha(d(V_1, V_2)). \tag{1.3}$$

In what follows we shall agree that the large letter K denotes some absolute constant not depending on V or m (we shall use frequently), not necessarily identical at different occurrences.

We shall consider the following conditions.

(C1). (1a) $\|X_a\|_{2+\delta}\leqq K$ for all $a\in D$ and $\{|X_a|^{2+\delta}; a\in D\}$ is uniformly integrable for some $0\leqq\delta<\infty$.

(1b) $n^{4d/3}g(n)\phi^{(1+\delta)/(2+\delta)}(n)\leqq K$ for some $g\in\mathfrak{G}$.

(C2). (2a) $|X_a|\leqq K$ a. e. for all $a\in D$.

(2b) $n^{4d/3}g(n)\phi(n)\leqq K$ for some $g\in\mathfrak{G}$.

(C3). (3a) $\|X_a\|_{2+\delta}\leqq K$ for some $0<\delta\leqq\infty$ and $\{|X_a|^{2+\delta}; a\in D\}$ $(\delta<\infty)$ is uniformly integrable.

(3b) $M_2(n, m)\leqq K(nm)^k$ for some $k>1/2$.

(3c) $\sum_{n=1}^{\infty}n^{d-1}\alpha^{\delta/(2+\delta)}(n)<\infty$ and $n^{(2k+1)d}g(n)\alpha(n)\leqq K$ for some $g\in\mathfrak{G}$.

Now we state the main results.

Theorem 1. *Suppose that X is ϕ-mixing and satisfies* (C1) *or* (C2). *Then for any sequence $\{V_n\}$ in $\mathfrak{A}$ such that $|V_n|\to\infty$ $(n\to\infty)$ and*

$$\liminf_{n\to\infty} E\{S(V_n)^2\}/|V_n|>0, \tag{1.4}$$

$S(V_n)/\sigma(V_n)$ converges in distribution to the standard normal distribution $N(0, 1)$ as $n\to\infty$, i.e., $S(V_n)/\sigma(V_n)\xrightarrow{\mathfrak{D}}N(0, 1)$ as $n\to\infty$.

Theorem 2. *Suppose that X is strong-mixing and satisfies* (C3). *Then for any sequence $\{V_n\}$ in $\mathfrak{A}$ such that $|V_n|\to\infty$ $(n\to\infty)$ and*

$$\liminf_{n\to\infty} E\{S(V_n)^2\}/|V_n|>0, \tag{1.5}$$

$S(V_n)/\sigma(V_n)\xrightarrow{\mathfrak{D}}N(0, 1)$ as $n\to\infty$.

2. Reviewing of Stein's estimation.

In this section we assume all random variables to be bounded. As in [12] $\mathcal{F}$ and $\mathcal{C}$ are two sub-σ-algebras of $\mathcal{A}$, G and W^* are two random variables measurable with respect to $\mathcal{A}$ and $\mathcal{C}$ respectively. Let $W=E^{\mathcal{F}}G$ be the conditional expectation of G with respect to $\mathcal{F}$. We assume

(2.1) $$E\{G(W-W^*)\}=1.$$

For a fixed $h\in C_b^1$, let $f(x)$ be the function defined in (0.2). Then by [12] we have

(2.2) $$\|f\|_\infty<\infty, \quad \|f'\|_\infty<\infty \quad \text{and} \quad |f''(x)|\leqq K\{|x|+1\} \text{ for some } K.$$

Applying the Taylor expansion formula and putting $\bar{K}=\max\{K, 1\}$, by (0.3) and (2.2) we have

(2.3) $$\begin{aligned}|E\{h(W)\}-Nh|&=|E\{f'(W)-Wf(W)\}|\\ &=|E\{f'(W)-(E^{\mathfrak{F}}G)f(W)\}|=|E\{f'(W)-Gf(W)\}|\\ &=|E\{f'(W)-G[f(W)-f(W^*)]\}-E\{Gf(W^*)\}|\\ &=|E\{f'(W)+G(W^*-W)f'(W)+\frac{G(W^*-W)^2}{2}f''(W+\theta(W^*-W))\}\\ &\quad -E\{Gf(W^*)\}|\end{aligned}$$

where θ is a random variable such that $0<\theta<1$ a.e.,

$$\begin{aligned}&\leqq\bar{K}[|E\{[E^{\mathfrak{F}}G(W-W^*)-EG(W-W^*)]f'(W)\}|+E|G(W-W^*)^2|/2\\ &\quad +E|GW(W-W^*)^2|/2+E|G(W-W^*)^3|/2+|E(Gf(W^*))|]\end{aligned}$$

(cf. (2.1) and (2.2))

$$\begin{aligned}&\leqq\bar{K}[\{E|E^{\mathfrak{F}}G(W-W^*)-EG(W-W^*)|^2\}^{1/2}+E|G(W-W^*)^2|\\ &\quad +E|GW(W-W^*)^2|+E|G(W-W^*)^3|+|EGf(W^*)|].\end{aligned}$$

(cf. pp. 587 [12]).

In the following section we shall apply this estimate to the random fields.

3. General lemmas.

Lemma 3.1 [10]. *Let V_1, V_2 be in $\mathfrak{A}$. If X is $\mathfrak{F}(V_1)$-measurable and Y is $\mathfrak{F}(V_2)$-measurable, then*

$$\|X\|_r<\infty, \quad \|Y\|_s<\infty \quad (\infty\geqq r>1,\ \infty>s\geqq 1 \text{ and } 1/r+1/s=1)$$

implies

(3.1) $$|E(XY)-E(X)E(Y)|\leqq 2M_1^{1/s}(|V_1|)\phi^{1/s}(d(V_1, V_2))\|X\|_r\|Y\|_s.$$

Lemma 3.2 [11]. *Let V_1, V_2 be in $\mathfrak{A}$. If X is $\mathfrak{F}_1(V)$-measurable and Y is $\mathfrak{F}(V_2)$-measurable, then*

$$\|X\|_r<\infty, \quad \|Y\|_s<\infty \quad (\infty\geqq r,\ s>1,\ 1/r+1/s<1)$$

implies

(3.2) $|E(XY)-E(X)E(Y)|\leqq 10M_2^{1/p}(|V_1|,\ |V_2|)\alpha^{1/p}(d(V_1,\ V_2))\|X\|_r\|Y\|_s$

where $1/p=1-(1/r+1/s)$.

Fix an element V $(\neq\phi)$ in $\mathfrak{A}$. Let I be a random variable uniformly distributed over V, independent of $\{X_a\,;\,a\in V\}$. Fix a positive integer m which shall be specified in proofs of theorems. Remark that, in below, we consider only elements in $\{X_a\,;\,a\in V\}$. Set $\mathcal{F}=\sigma\{X_a\,;\,a\in V\}=\mathcal{F}(V)$. And denote by $\mathcal{C}$ the σ-algebra generated by the events $\{I=i,\ X_a\leqq x_a,\ |a-i|>m\}$ where x_a are real numbers. Put

(3.3) $$\partial^2=E\sum_{a\in V}\{X_a\sum_{|b-a|\leq m}X_b\},$$

and assume $\partial>0$.

Now define G and W^* in section 2 as follows

(3.4) $$G=\frac{|V|}{\partial}X_I,\quad W^*=\frac{1}{\partial}\sum_{|a-I|>m}X_a.$$

Then we have

(3.5) $$W=E^{\mathcal{F}}G=\frac{1}{\partial}\sum_{a\in V}X_a=S(V)/\partial,\quad EG(W-W^*)=1.$$

For notational simplicity put

(3.6) $$Z_a=\sum_{|b-a|\leq m}X_b,\quad a\in V.$$

By the result in section 2 we have the following

Lemma 3.3. *Let h be in C_b^1. Then there exists a positive constant K which does not depend on V and m, such that*

(3.7) $$\begin{aligned}&|E\{h(S(V)/\partial)\}-Nh|\\ &\leqq K\Big[\frac{1}{\partial^2}\{E|\sum_{a\in V}X_aZ_a-\sum_{a\in V}EX_aZ_a|^2\}^{1/2}+\frac{1}{\partial^3}\sum_{a\in V}E|X_aZ_a^2|\\ &\qquad+\frac{1}{\partial^4}\sum_{a\in V}E|X_aS(V)Z_a^2|+\frac{1}{\partial^4}\sum E|X_aZ_a^3|+|E(Gf(W^*))|\Big]\\ &=K\{I_1+I_2+I_3+I_4+I_5\},\quad \textit{say}\end{aligned}$$

where f is the function defined by (0.2).

4. Proof of Theorem 1.

In this section we assume that the conditions of Theorem 1 are satisfied.

Lemma 4.1. *If $|X_a|\leqq C$ a.e. for all $a\in D$, then we have*

(4.1) $$E(\sum_{|a|\leq k}X_a)^4\leqq K\,k^{4d-4d/3}$$

for all positive integer k.

Proof. By modifying the proof of Lemma 4 in section 20 of [1] we can obtain the inequality

$$E\Big(\sum_{|a|\leq k} X_a\Big)^4 \leqq K\Big\{\sum_{j=0}^{k} j^{d-1}\phi^{1/2}(j)\Big\}^2 k^{2d}. \tag{4.2}$$

On the other hand, since $\phi^{1/2}(j)\leqq K\, j^{-2d/3}$, we have

$$\sum_{j=0}^{k} j^{d-1}\phi^{1/2}(j)\leqq K\, k^{d-2d/3}, \tag{4.3}$$

hence, by (4.2) we have (4.1).

Lemma 4.2. *If $\|X_a\|_{2+\delta}\leqq M$ for all $a\in D$, then we have*

$$E(S(V)^2)\leqq K|V|. \tag{4.4}$$

Proof is omitted becasue this is obtained by the analogous method to the one dimensional case.

Here we specify the value of m.

$$m=[|V_n|^{3/8d}g(|V_n|)^{-3/8d}] \tag{4.5}$$

where $[x]$ denotes the largest integer contained in x.

By Lemma 3.1 and (4.4) we have the following inequality.

$$\begin{aligned} &|E(S(V_n)^2)-E\{\sum_{a\in V_n} X_a Z_a\}| \\ &=|E\sum_{a\in V_n}(X_a \sum_{|j-a|>m} X_j)| \\ &\leqq K\sum_{a\in V_n}\|\sum_{|j-a|>m} X_j\|_q \phi^{1/q}(m)\|X_a\|_{2+\delta} \qquad (q=(2+\delta)/(1+\delta)) \\ &\leqq K|V_n|\,|V_n|^{1/2}\phi^{1/q}(m). \end{aligned} \tag{4.6}$$

Hence, by (4.5), we may suppose that there exist n_0 and $K>0$ such that

$$E\{\sum_{a\in V_n} X_a Z_a\}/|V_n|\geqq K>0 \tag{4.7}$$

for all $n\geqq n_0$.

Now we truncate each variable X_a. C denotes a positive constant.

$$Y_a=X_a J(|X_a|\leqq C)-EX_a J(|X_a|\leqq C), \quad W_a=\sum_{|b-a|\leqq m} Y_b \quad (a\in D) \tag{4.8}$$

where J denotes the indicator variable such that if the inner condition of the round brackets is satisfied, then it takes one, otherwise zero. By the same argument as (4.6), we may suppose that there exist n_1 and $K>0$ such that

$$E(\sum_{a\in V_n} Y_a W_a)/|V_n|\geqq K>0 \tag{4.9}$$

for all $n \geqq n_1$. For the brevity, in the sequel, $\sum_a$ means $\sum_{a \in V_n}$. Using Y_a in place of X_a in lemma 3.3 we shall estimate the terms I_i $(i=1, \cdots, 5)$.

Lemma 4.3. *For some $K>0$ which depends only on ϕ and C,*

(4.10)
$$\begin{aligned} &(1) \quad I_1 \leqq K\{m^d/|V_n|^{1/2} + m^{(5d-2)/4}/|V_n|^{1/2}\} \\ &(2) \quad I_2 \leqq K\{m^d/|V_n|^{1/2}\} \\ &(3) \quad I_3 \leqq K\{m^{4d/3}/|V_n|^{1/2}\} \\ &(4) \quad I_4 \leqq K\{m^{2d}/|V_n|\} \\ &(5) \quad I_5 \leqq K\{\phi(m)|V_n|^{1/2}\}. \end{aligned}$$

Proof. Put $\hat{\sigma}_n^2 = E(\sum_{a \in V_n} Y_a W_a)$. $\hat{\sigma}_n$ plays the same role as $\hat{\sigma}$ in (3.4).

(I_1)
$$\hat{\sigma}_n^4 I_1^2 = E(\sum_a Y_a W_a - \sum_a E(Y_a W_a))^2$$

(4.11)
$$\begin{aligned} &\leqq K\{\sum_{|a-b| \leqq 2m} E A_a A_b + \sum_{|a-b|>2m} E A_a A_b\} \\ &\leqq K\{\sum_{|a-b| \leqq 2m} \|A_a\|_2 \|A_b\|_2 + \sum_{|a-b|>2m} E A_a A_b\} = J_1 + J_2, \quad \text{say} \end{aligned}$$

where $A_a = Y_a W_a - E(Y_a W_a)$. By Lemma 4.2, we have

(4.12)
$$J_1 \leqq K|V_n| m^{2d}.$$

Next fix a and b such that $|a-b|>2m$. Then

(4.13)
$$\begin{aligned} |E A_a A_b| &= |E A_a \sum_{|j-b| \leqq m} U_{bj}| \quad (U_{bj} = Y_b Y_j - E Y_b Y_j) \\ &\leqq \sum_{|j-b| \leqq m} |E A_a U_{bj}| \\ &\leqq K \sum_{|j-b| \leqq m} \|A_a\|_2 \phi(|a-b|-2m) C^2 \\ &\leqq K \phi(|a-b|-2m) m^d m^{d/2}. \end{aligned}$$

Hence

(4.14)
$$\begin{aligned} J_2 &\leqq K \sum_{|a-b|>m} \phi(|a-b|-2m) m^{3d/2} \\ &\leqq K \sum_a \sum_{j=1}^{\infty} (j+2m)^{d-1} \phi(j) m^{3d/2} \\ &\leqq K \sum_a m^{5d/2-1} = K|V_n| m^{(5d-2)/2}. \end{aligned}$$

Therefore, by (4.9), (4.12) and (4.14), we have

(4.15)
$$I_1 \leqq K\{m^d/|V_n|^{1/2} + m^{(5d-2)/4}/|V_n|^{1/2}\}.$$

(I_2) $$I_2 \leqq K \sum_a E(W_a^2)/|V_n|^{3/2} \leqq K \sum_a m^d/|V_n|^{3/2} \leqq K\{m^d/|V_n|^{1/2}\}.$$

(I_3) $$I_3 \leqq K \sum_a E|(Y_a S_t(V_n) W_a^2)|/|V_n|^2 \quad \text{where } S_t(V_n)=\sum_a Y_a$$

(4.16) $$\leqq K \sum_a \|S_t(V_n)\|_2 \|W_a^2\|_2/|V_n|^2$$
$$\leqq K \sum_a |V_n|^{1/2} m^{2d-2d/3}/|V_n|^2 \quad \text{by Lemma 4.1,}$$
$$\leqq K\{m^{4d/3}/|V_n|^{1/2}\}.$$

(I_4) $$I_4 \leqq K \sum_a E|Y_a W_a^3|/|V_n|^2$$

(4.17) $$\leqq K \sum_a E(W_a^2) m^d/|V_n|^2$$
$$\leqq K \sum_a m^{2d}/|V_n|^2 = K\{m^{2d}/|V_n|\}.$$

(I_5) Since $f(x)$ is bounded, we have

(4.18) $$I_5 = |E(Gf(W^*))| = |E(E^I(Gf(W^*)))| \leqq K\{\phi(m)|V|^{1/2}\}$$

where E^I denotes the conditional expectation with respect to the r.v. I.

Proof of Theorem 1. At first remark that $d<4d/3$ and $(5d-2)/4<4d/3$. Since $m=[|V_n|^{3/8d} g(|V_n|)^{-3/8d}]$, we can easily see that, as $n\to\infty$, I_1, I_2, I_3 and I_4 tend to 0. Next consider I_5. By the definition of the class $\mathfrak{G}$, we have

(4.19) $$I_5 \leqq K\{m^{-3d/4} g^{-1}(m)|V_n|^{1/2}\}$$
$$\leqq K \frac{g^{1/2}(|V_n|)|V_n|^{1/2}}{|V_n|^{1/2} g(|V_n|^{3/8d} g^{-3/8d}(|V_n|))} \to 0 \quad (n\to\infty).$$

Thus we showed that for fixed h and C, as $n\to\infty$,

(4.20) $$E(h(S_t(V_n)/\hat{\sigma}_n)) - Nh \longrightarrow 0$$

which means $S_t(V_n)/\hat{\sigma}_n \xrightarrow{\mathcal{D}} N(0, 1)$ as $n\to\infty$. To complete the proof of theorem 1, we must approximate $S(V_n)/\sigma(V_n)$ by $S_t(V_n)/\hat{\sigma}_n$. But this is well known, and so is omitted. We have proved Theorem 1.

5. Proof of Theorem 2.

We use the truncation technique as well as in section 4. For a fixed C we define Y_a's as in section 4. In order to prove Theorem 2, it is sufficient to prove the C.L.T. for Y_a's. Hence from the start we may assume that $|X_a| \leqq C$ a.e. for all $a\in D$ and that $n^{(2k+1)d} g(n)\alpha(n) \leqq K$ for all n for some $g\in\mathfrak{G}$. We shall estimate I_i $(i=1, \cdots, 5)$ in Lemma 3.3. We use the notations in section 3.

(I_1) $$\sigma^4(V_n) I_1^2 = E\left(\sum_a X_a Z_a - \sum_a E X_a Z_a\right)^2$$

(5.1) $$\leqq KE\{(\sum_{|a-b|\leqq 3m}+\sum_{|a-b|>3m})L_a L_b\}$$

$$\leqq K\{\sum_a \sum_{|a-b|\leqq 3m}\|L_a\|_2\|L_b\|_2+\sum_a \sum_{|a-b|>3m} E(L_a L_b)\}$$

$$=K(J_1+J_2), \qquad \text{(say)}$$

where $L_a=X_aZ_a-E(X_aZ_a)$. Since $E(L_a^2)\leqq KE(Z_a^2)\leqq Km^d$,

(5.2) $$J_1\leqq K|V_n|m^{2d}.$$

For fixed a and b such that $|a-b|>3m$,

(5.3) $$|E(L_aL_b)|=|\sum_{i,j}E(R_{ai}R_{bj})|$$

$$\leqq K\{m^{2d}\alpha(|a-b|-2m)\}$$

where $R_{ci}=X_cX_i-E(X_cX_i)$, $(|i-c|\leqq m)$ $(c=a$ or $b)$. Hence

(5.4) $$J_2\leqq K\sum_a \sum_{|b-a|>3m} m^{2d}\alpha(|a-b|-2m)$$

$$\leqq K\sum_a m^{2d}\sum_{p=3m+1}^{\infty} p^{d-1}\alpha(|a-b|-2m)$$

$$\leqq K\sum m^{3d-1}\sum_{p=m+1}^{\infty} p^{d-1}\alpha(p)$$

$$\leqq K|V_n|m^{2(1-k)d-1}.$$

These imply

(5.5) $$I_1\leqq K\{m^{2d}/|V_n|+m^{2(1-k)d-1}/|V_n|\}^{1/2}.$$

(I_2) $$I_2\leqq K\sum_a E(Z_a^2)/|V_n|^{3/2}\leqq K\{m^d/|V_n|^{1/2}\}.$$

(I_3) Since $n^{(2k+1)d}g(n)\alpha(n)\leqq K$ (remark $k>1/2$) implies $\sum_{i=1}^{\infty} n^{d-1}\alpha^{1/2}(n)<\infty$ we have

(5.6) $$E|Z_a|^4\leqq Km^{2d} \qquad \text{for all } a\in D.$$

From this inequality,

(5.7) $$I_3=E|GW(W-W^*)^2|$$

$$\leqq K\sum_a \|S(V_n)\|_2\|Z_a\|_4^2/|V_n|^2\leqq K\{m^d/|V_n|^{1/2}\}.$$

(I_4) $$I_4=E|G(W-W^*)^3|$$

(5.8) $$\leqq K\sum_a \{E|Z_a|^4\}^{3/4}/|V_n|^2\leqq K\{m^{3d/2}/|V_n|\}.$$

(I_5) Applying Lemma 3.2 with $r=s=+\infty$, we have

(5.9) $$I_5=|E(Gf(W^*))|=|E(E^I(Gf(W^*)))|\leqq K|V_n|^{1/2+k}\alpha(m).$$

Here we specify m as follows;

(5.10) $$m=[|V_n|^{1/2d}g^{-1/(4dk+2d)}(|V_n|)].$$

Then by easy calculations we have that, as $n\to\infty$,

(5.11) $$I_1+I_2+I_3+I_4\longrightarrow 0$$

and

(5.12) $$I_5=O(|V_n|^{1/2+k}\alpha(m))=O\left(\frac{|V_n|^{1/2+k}g^{1/2}(|V_n|)}{|V_n|^{1/2+k}g(|V_n|^{1/2d}g^{-1/(4dk+2d)}(|V_n|))}\right)$$
$$=o(1).$$

In (5.12) we used the property of $\mathfrak{G}$. Thus we have completed the proof of Theorem 2.

6. Concluding remarks.

In the case where the blocking method can be applied, Theorem 2 can be proved under less restrictive condition $n^{(1+k)d}g(n)\alpha(n)\leqq K$ in place of $n^{(1+2k)d}g(n)\alpha(n)\leqq K$.

S. Mase [9] discussed the same problem as ours. He took as the index set a countable subset of a metric space, and proved a theorem analogous to our Theorem 2.

Our results can be applied to the Gibbs phenomena (see, for example, Nahapetian [10] and Neaderhouser [11]).

Added in proof. After this paper being received, the author knew E. Bolthousen's paper "On the central limit theorem for stationary mixing random fields" (Ann. Probab. 10. 1047-1050 (1982)). Remark that in the proof of Theorem 2 in the present paper we made use of only the mixing condition introduced in Bolthousen's paper.

References

[1] P. Billingsley: *Convergence of probability measures.* New York: Wiley: 1968.
[2] A. V. Bulinskii and I. G. Zurbenko: *The central limit theorem for random fields.* Soviet Math. Dokl. 17 (1976), 14-17.
[3] Louis, H. Y. Chen: *On the convergence of Poisson binomial to Poisson distributions.* Ann. Probab. 2 (1974), 178-180.
[4] Louis H. Y. Chen: *Two central limit problems for dependent random variables.* Z. Wahrscheinlichkeitstheorie verw. Gebiete. 43 (1978), 223-243.
[5] Louis H. Y. Chen: *Stein's method in limit theorems for dependent random variables.* Sea Bull. Math. (special issue), (1979), 36-50.
[6] R. V. Erickson: *L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique.* Ann. Probab. 2 (1974), 522-529.
[7] V. V. Gorodeckii: *The invariance principle for stationary random fields satisfying*

the strong mixing condition. Theory Probab. Appl. **27** (1982), 358-364. (In Russian)

[8] V. A. Malysev: *The central limit theorem for Gibbsian random fields.* Soviet Math. Dokl. **16** (1975), 1141-1154.

[9] S. Mase: *A central limit theorem for random fields and its application to random set theory.* Technical Report Series of Hiroshima Univ. Statist. Res. Group. No. 8 (1979).

[10] B. S. Nahapetian: *The central limit theorem for random fields.* In *Multicomponent Random Systems* (R. L. Dobrushin and Ya. G. Sinai, ed), 531-542. New York: Marcel Dekker: 1980.

[11] C. C. Neaderhouser: *Limit theorems for multiply indexed mixing random variables, with application to Giobs random fields.* Ann. Probab. **6** (1978), 207-215.

[12] C. Stein: *A bound for the error in the normal approximation to the distribution of a sum of dependent random variables.* Proc. Sixth Berkeley Symp. Math. Statist. Prob. **2** (1972), 583-603.

[13] H. Takahata: *L_∞-bound for asymptotic normality of weakly dependent summands using Stein's result.* Ann. Probab. **9** (1981), 676-683.

Department of Mathematics
Tokyo Gakugei University
4-1-1 Nukuikita-machi
Koganei-shi, Tokyo, Japan