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ON THE SPECTRA OF TOPOLOGICAL A-TENSOR PRODUCT A-ALGEBRAS

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Introduction

Complex tensor products of commutative Banach algebras and canonical decompositions of their spectra (: Gel'fand spaces) have already been examined in several instances by B. R. Gelbaum [2], J. Tomiyama [22]. In a more general situation (complex) topological (not normed) tensor products with respect to tensorial "compatible" topologies as well as similar decompositions of their spectra have defined and systematically studied by A. Mallios [11-15].

Now, due to recent considerations, one is led to study analogous results in case of topological A-algebras, that is topological (not normed) algebras with "coefficients" not necessarily the complex number field $\boldsymbol{C}$, but a general topological algebra A. This kind of algebras find numerous applications in several branches of Mathematics (cf., e.g. [16, 17]).

More precisely, we first define the A-tensor product A-algebra of two given A-algebras as a solution of a universal problem (cf. (1.1)) realizing it into two equivalent ways (cf. § 1, in particular, (1.4)). Then, on this (algebraic) A-tensor product of two topological A-algebras, we define tensorial topologies as A-compatible topologies (Definition 1.1), within an entirely general context. This constitutes an extension of the classical case of complex tensor products of locally convex spaces [4] or topological algebras [11, 12], as well as a generalization of the corresponding situation in case of Banach A-modules [19] or commutative A-algebras [3, 7].

Moreover, by examining the (numerical) spectrum of the above topological A-tensor product algebra and its completion, in connection with the spectra of the factor algebras (Theorem 2.1), we get an extension of the corresponding results of [7] to a more general case than that of Banach A-algebras, such that the previous results of [2, 3, 11, 12, 13, 22] are natural consequences. On the other hand, in the theory of topological A-algebras a basic notion is that of the generalized A-spectrum (cf. § 3), an extension of the notion of the (numerical) spectrum as well as of the generalized spectrum due to [14]. In this respect, by considering locally convex A-algebras, we are concerned with a decomposition of the generalized A-spectrum of a topological A-tensor product A-algebra and its completion, in terms of the spectra of factor algebras; we still suppose the algebras having bounded approximate identities (Proposition 3.1, Theorem 3.1), extending thus [11, 14, 15] (cf. also [23]) where in the latter case the algebras involved have identity elements. Furthermore, by considering continuous central A-morphisms (cf. § 4; a useful notion, particularly, in the theory of representations of (*-) algebras, cf. [18]) of topological A-tensor product algebras we get analogous relations connecting these sets of morphisms with those of generalized A-spectra (Propositions 4.1, 4.2).

Now, by still specializing to locally convex A-algebras one has the posibility of taking more convenient forms, concerning the previous A-tensor products. Thus, on the one hand, we define the projective (topological) A-tensor product A-algebra via a universal property (Definition 5.1) and on the other hand we introduce the "topological tensor product over A" characterized by Proposition 5.2 and constituting an extension of the corresponding tensor products of commutative Banach A-algebras introduced by B. R. Gelbaum [3]. The relation (5.10) (cf. also (5.11), (5.14), (5.15)) allows us to take the form of the continuous (A-) morphisms defined on the latter tensor product in terms of the corresponding morphisms of the factor algebras (Proposition 5.3, relations (5.12), (5.13), (5.18)).

Applications of the preceding to the case of topological geometric $\mathcal{A}$-spaces as well as an extension to infinite tensor product A-algebra will be considered elsewhere (cf. [8, 9, 9a]).

1. Topological A-tensor products of topological A-algebras

The topological vector spaces and topological algebras (with separately continuous multiplication) considered in the sequel are over the complex field $\boldsymbol{C}$ and will be assumed Hausdorff. Throughout the paper A will denote a commutative algebra.

Thus, let E, F, D be A-algebras and φ an A-bilinear ($:\varphi(ax, y)=\varphi(x, ay)=a\varphi(x, y)$, with $a\in A$, $(x, y)\in E\times F$) multiplicative ($:\varphi(xx', yy')=\varphi(x, y)\varphi(x', y')$ x, x' in E, y, y' in F) map of $E\times F$ into D. The pair (D, φ) is said to be an (A-algebra) *A-tensor product* of E, F, if the following universal property is satisfied:

(1.1) *For every pair (M, u), where M is an A-algebra and $u: E\times F \longrightarrow M$ an A-bilinear multiplicative map, there exists a unique A-morphism $\tilde{u}: D \longrightarrow M$ such that $u=\tilde{u}\circ\varphi$.*

(1.1) implies, of course, the uniqueness of (D, φ) within an isomorphism of A-algebras.

On the other hand, one gets an A-tensor product as follows: Let E, F be A-algebras and $(E\otimes F, \varphi_1)$ the usual (complex) tensor product algebra of E, F (cf.

[1: Chap. 8, § 1] and also [11: Chapt. VI, § 1]) being an A-algebra by defining $a\cdot(x\otimes y):=ax\otimes y$, $a\in A$, $x\otimes y\in E\otimes F$. If $I:=[S]$ is the vector subspace of $E\otimes F$ generated by

$$S=\{ax\otimes y-x\otimes ay\,;\, a\in A,\ x\in E,\ y\in F\}\,, \tag{1.2}$$

then I is a 2-sided A-ideal of $E\otimes F$ such that $E\otimes F/I$ is (naturally) an A-algebra. The pair $(E\otimes F/I,\ \varphi:=\rho\circ\varphi_1)$, where ρ is the canonical quotient A-morphism, is the *A-tensor product* of E, F.

One gets another realization of the preceding A-tensor product of E, F, by considering the free vector space generated by $E\times F$, $\mathfrak{F}(E, F)_\circ\equiv\{f\in C^{E\times F}; f$ has finite support$\}$ (cf. [1: Chap. 8]). This is a "convolution algebra" (cf. [10: p. 107]), and moreover an A-algebra by $a\cdot f:=\sum_{(x,y)} f(x, y)\delta_{(ax,y)}$, for $a\in A$, $f\in\mathfrak{F}(E, F)_\circ$, where $\delta_{(x,y)}\equiv\delta(x, y)\in\mathfrak{F}(E, F)_\circ$, with $(x, y)\in E\times F$, such that for $x=y=0$, $\delta_{(x,y)}=0$ and for $(x, y)\neq 0$, $\delta_{(x,y)}$ is the characteristic function of $\{(x, y)\}$. If H is the vector subspace of $\mathfrak{F}(E, F)_\circ$ generated by the functions

$$\begin{aligned} &\text{i)} && \delta_{(\lambda x_1+\mu x_2,\, y)}-\lambda\delta_{(x_1,\, y)}-\mu\delta_{(x_2,\, y)}\\ &\text{ii)} && \delta_{(x,\, \lambda y_1+\mu y_2)}-\lambda\delta_{(x,\, y_1)}-\mu\delta_{(x,\, y_2)}\\ &\text{iii)} && \delta_{(ax,\, y)}-\delta_{(x,\, ay)} \end{aligned} \tag{1.3}$$

for all x, x_1, $x_2\in E$, y, y_1, $y_2\in F$, $a\in A$, λ, $\mu\in C$, then H is a 2-sided A-ideal of $\mathfrak{F}(E, F)_\circ$, in such a way that $(\mathfrak{F}(E, F)_\circ/H,\ \varphi:=\rho\circ\delta)$ is the *A-tensor product of* E, F, where ρ is the canonical quotient A-morphism.

Given two A-algebras E, F there always exists a unique (within an isomorphism of A-algebras) *pair (D, φ), "the" A-tensor product of E, F, which will be denoted by $E\otimes_A F$,* that is

$$E\otimes_A F=E\otimes F/I=\mathfrak{F}(E, F)_\circ/H\,, \tag{1.4}$$

within isomorphisms of A-algebras. The elements $\varphi(x, y)\equiv x\otimes_A y$, $x\in E$, $y\in F$ (: decomposable A-tensors) generate $E\otimes_A F$ as a vector space, so that every element $z\in E\otimes_A F$ is of the form $z=\sum_{i=1}^{n} x_i\otimes_A y_i$.

Now, given a (commutative) topological algebra A, an A-algebra E is called a *topological A-algebra* whenever E is, a topological algebra and a topological A-module. A *locally (m-) convex A-algebra* E is a topological A-algebra such that E, A are locally (m-) convex algebras.

Definition 1.1. Let E, F be topological A-algebras. By an *A-compatible topology* on the corresponding A-tensor product A-algebra $E\otimes_A F$ we mean a (Hausdorff) topology $\mathfrak{T}$ such that the pair $(E\otimes_A F, \mathfrak{T})\equiv E\otimes_A^{\mathfrak{T}} F$ is a topological A-algebra of the same type with E, F.

For $A=C$ we get the usual compatible topology on $E\otimes F$, cf. [13: Def. 3.1].

Now, let E, F be topological A-algebras and $E \underset{\tau}{\otimes} F$ the usual (complex) tensor product algebra of E, F, equipped with a compatible topology τ on $E\otimes F$ [13: Definition 3.1], such that the bilinear map

$$A\times(E\underset{\tau}{\otimes}F)\longrightarrow E\underset{\tau}{\otimes}F:(a,\ x\otimes y)\longmapsto a\cdot(x\otimes y)\equiv ax\otimes y \tag{1.5}$$

is continuous, i.e. $E\underset{\tau}{\otimes}F$ is a topological A-algebra. If I is the closed vector subspace of $E\underset{\tau}{\otimes}F$ generated by (1.2) being, in particular, a closed 2-sided A-ideal of $E\underset{\tau}{\otimes}F$, then the corresponding (Hausdorff) quotient topology on $E\underset{\tau}{\otimes}F/I$ is an *A-compatible topology on $E\otimes_A F$*, which will be called an *A-compatible topology on $E\otimes_A F$ associated with the topology τ on $E\otimes F$* (denote it also by $\mathfrak{T}$), such that one sets by definition

$$E\otimes_A^{\mathfrak{T}}F=E\underset{\tau}{\otimes}F/I \tag{1.6}$$

as topological A-algebras.

In this respect, we remark that if A is a locally bounded algebra with continuous multiplication (cf. [13: § 2]), then every locally bounded A-algebra with continuous multiplication is a topological A-algebra (not necessarily locally convex). Thus, given a pair (E, F) of such algebras, the corresponding quotient topology of the compatible algebra topology on $E\otimes F$ (cf. [11: Chapt. VI, Theorem 3.1]) is, in fact, an A-compatible topology on $E\otimes_A F$ (not necessarily locally convex); cf., Definition 1.1 and also relation (1.6).

2. The (numerical) spectrum of a topological A-tensor product A-algebra

By the (numerical) *spectrum* of a topological algebra E we mean the set of continuous characters on E (denoted by $\mathfrak{M}(E)$) endowed with the relative topology induced on it by the weak topological dual space E'_s, viz., the dual E' of E in the weak topology $\sigma(E', E)$ (cf. [11, 12, 13]).

Throughout this Section we suppose that given two topological A-algebras E, F the corresponding topological A-tensor product (A-algebra) is endowed with an A-compatible topology $\mathfrak{T}$ associated with a compatible topology τ on $E\otimes F$ (cf. (1.6) and also [13: Definition 3.1]).

Now, we remark that *the above topology $\mathfrak{T}$ is of the same type with τ*, in the sense that, whenever τ satisfies the following three conditions (in this respect, cf. also [13]), the corresponding ones, concerning $E\otimes_A^{\mathfrak{T}}F$, are also valid for $\mathfrak{T}$:

(2.1) *The canonical map of $E\times F$ into $E\underset{\tau}{\otimes}F$ is separately continuous*

(2.2) *For every pair* $(f, g)\in\mathfrak{M}(E)\times\mathfrak{M}(F)$, $f\otimes g\in(E\underset{\tau}{\otimes}F)'_s$, *where*
$(f\otimes g)(x\otimes y):=f(x)\cdot g(y)$, $x\in E$, $y\in F$.

(2.3) *For any equicontinuous subsets* M, N *of* $\mathfrak{M}(E)$, $\mathfrak{M}(F)$ *respectively,* $M\otimes N=\{f\otimes g : f\in M,\ g\in N\}$ *is an equicontinuous subset of* $(E\underset{\tau}{\otimes}F)'_s$.

So, (2.1), (2.2) follow immediately by (1.6). Concerning (2.3), let M, N be equicontinuous subsets of $\mathfrak{M}(E)$, $\mathfrak{M}(F)$ respectively. Then, $M\otimes_A N$ defined by $(M\otimes_A N)\circ\rho=(M\otimes N)_I:=\{f\otimes g\in M\otimes N : I\subseteq ker(f\otimes g)\}\subseteq M\otimes N$, ($\rho$ is the canonical quotient map, cf. (1.6)) is an equicontinuous subset of $(E\otimes_A^{\mathfrak{T}}F)'_s$ since $M\otimes N\subseteq(E\underset{\tau}{\otimes}F)'_s$ is equicontinuous (cf. [5: p. 199, Proposition 5] and (2.3)).

Thus, given $E\otimes_A^{\mathfrak{T}}F$, consider the closed subsets $h(I)=\{\omega\in\mathfrak{M}(E\underset{\tau}{\otimes}F) : I\subseteq ker(\omega)\}$ (: *hull of* $I\subseteq E\underset{\tau}{\otimes}F$, cf. (1.6) and [11: Chapt. V, Definition 1.1]) and $R=\{(f, g) : f(ax)g(y)=f(x)g(ay)\,;\ a\in A,\ x\in E,\ y\in F\}$ of $\mathfrak{M}(E\underset{\tau}{\otimes}F)$, $\mathfrak{M}(E)\times\mathfrak{M}(F)$, respectively. Then,

$$\mathfrak{M}(E\otimes_A^{\mathfrak{T}}F)=h(I)=R\subseteq\mathfrak{M}(E)\times\mathfrak{M}(F)=\mathfrak{M}(E\underset{\tau}{\otimes}F) \tag{2.4}$$

within homeomorphisms (cf. (1.6), as well as [11: Chapt. V, Lemma 2.4; Chapt. VI, Lemma 5.1]).

Another expression of (2.4) is provided by considering the continuous maps

$$\begin{aligned}&\mu:\mathfrak{M}(E)\longrightarrow\mathfrak{M}(A)^+\Big(: f\longmapsto\mu(f)(a):=\frac{f(ax)}{f(x)}\Big)\\ &\nu:\mathfrak{M}(F)\longrightarrow\mathfrak{M}(A)^+\Big(: g\longmapsto\nu(g)(a):=\frac{g(ay)}{g(y)}\Big)\end{aligned} \tag{2.5}$$

for $x\in E$, $y\in F$ such that $f(x)\neq 0$, $g(y)\neq 0$ and $\mathfrak{M}(A)^+\equiv\mathfrak{M}(A)\cup\{0\}$. Then, if Δ^+ is the diagonal of $\mathfrak{M}(A)^+\times\mathfrak{M}(A)^+$, one gets

$$(\mu\times\nu)^{-1}(\Delta^+)=R\,. \tag{2.6}$$

Throughout the sequel equalities or inclusion relations (: $\subseteq$) *between topological spaces will always be understood within homeomorphisms.*

The following theorem specializes to [7: Theorem 3.3] for commutative Banach A-algebras, as well as to [11: Chap. VI, Theorems 5.1, 5.2] for $A=C$.

Theorem 2.1. *Let* E, F *be topological* A*-algebras and* $E\otimes_A^{\mathfrak{T}}F$ *the respective* A*-tensor product* A*-algebra equipped with an* A*-compatible topology* $\mathfrak{T}$ *associated with a compatible tensorial topology* τ *on* $E\otimes F$ *satisfying* (2.1), (2.2). *Then, one gets*

$$\mathfrak{M}(E\otimes_A^{\mathfrak{T}}F)=h(I)=(\mu\times\nu)^{-1}(\Delta^+)\subseteq\mathfrak{M}(E)\times\mathfrak{M}(F)=\mathfrak{M}(E\underset{\tau}{\otimes}F)\,. \tag{2.7}$$

Moreover, if E, F have continuous multiplications and locally equicontinuous spectra and the topology τ on $E\otimes F$ satisfies also (2.3), *then one has*

$$\mathfrak{M}(E\hat{\otimes}_A^\tau F)=h(I)=(\mu\times\nu)^{-1}(\Delta^+)\subseteqq\mathfrak{M}(E)\times\mathfrak{M}(F)=\mathfrak{M}(E\underset{\tau}{\hat{\otimes}}F) \tag{2.8}$$

("$\wedge$" means completion).

Proof. By (2.4), (2.6) and [13: Theorem 4.1] one gets immediately (2.7). Now, $E\otimes_A^\tau F$ is a topological A-algebra with continuous multiplication, since E, F have continuous multiplications (cf. (1.6)). Moreover, the local equicontinuity of $\mathfrak{M}(E)$, $\mathfrak{M}(F)$ implies the local equicontinuity of $\mathfrak{M}(E\otimes_A^\tau F)$ cf. (2.3) and also [11: Chapt. V, Lemma 5.2]) such that $\mathfrak{M}(E\hat{\otimes}_A^\tau F)=\mathfrak{M}(E\otimes_A^\tau F)$. Hence, (2.8) is a consequence of (2.7) and [13: Theorem 4.2]. ■

Equivalently, Theorem 2.1 says that $\mathfrak{M}(E\otimes_A^\tau F)$ (resp. $\mathfrak{M}(E\hat{\otimes}_A^\tau F)$) is the "*pullback*" (: *fiber product*) of $\mathfrak{M}(E)\times\mathfrak{M}(F)$ w.r.t. $\mathfrak{M}(A)^+$ (cf. also [16]).

Scholium 2.1. Concerning the set of continuous morphisms on $E\otimes_A^\tau F$ with values in a given unital topological algebra G with continuous multiplication (: *generalized spectrum of* $E\otimes_A^\tau F$ *w.r.t.* G denoted by $\mathcal{M}(E\otimes_A^\tau F, G)$; cf. [14: § 2] and/or [15]), we have analogous results to that of Theorem 2.1: So, if E, F are unital topological A-algebras and the topology τ of Theorem 2.1 satisfies the corresponding conditions to (2.1), (2.2) (cf. [15: (2.1), (2.2)]), one gets

$$\mathcal{M}(E\otimes_A^\tau F, G)=h(I)=R\subseteqq\mathcal{M}(E, G)\times\mathcal{M}(F, G), \tag{2.9}$$

where $h(I)=\{\omega\in\mathcal{M}(E\underset{\tau}{\otimes}F, G): I\subseteqq ker(\omega)\}$, (cf. (1.6)), $R=\{(f, g): f(x)g(y)=g(y)f(x),\ f(ax)g(y)=f(x)g(ay);\ a\in A,\ x\in E,\ y\in F\}\subseteqq\mathcal{M}(E, G)\times\mathcal{M}(F, G)$. Besides, by considering completions (: $E\hat{\otimes}_A^\tau F$), one has an analogous decomposition to (2.9), under the additional hypothesis that E, F have continuous multiplications, τ satisfies the corresponding condition to (2.3) (cf. 15: (2.3)]), G is complete and $\mathcal{M}(E, F)$, $\mathcal{M}(F, G)$ are locally equicontinuous (cf. also [15: Lemma 2.1, Theorem 2.2]).

Regarding (2.9), we do not have equalities everywhere, even if G is commutative or still the complexes (Theorem 2.1), unless $A=\boldsymbol{C}$ (cf. [14: Theorems 2.1, 3.1]). However, we get better estimations by considering A-morphisms (cf. Proposition 3.1, Theorem 3.1 below).

3. The generalized A-spectrum of $E\otimes_A^\tau F$

If E, F are topological A-algebras, the *generalized A-spectrum of E* (w.r.t. F) is the set $\mathcal{M}_A(E, F)$ of non-zero continuous A-morphisms of E into F, equipped with the topology induced on it by $\mathcal{L}_A(E, F)_s$ (: the space of continuous A-linear maps between the corresponding modules with the relative topology from $\mathcal{L}_s(E, F)$, cf. [5]). In particular, if E has a *bounded approximate identity* (*b.a.i*) $(u_i)_{i\in K}$ (i.e. a bounded directed net $(u_i)_{i\in K}$ of elements of E such that $\lim_i u_i x=\lim_i x u_i=x$, $x\in E$; cf. [11: Chapt. VI, Definition 11.2]), then we suppose for every $f\in\mathcal{M}_A(E, F)$, *$(f(u_i))_{i\in K}$ is a (bounded) approximate identity of F.*

Now, let E, F, G be topological A-algebras and $f\in\mathcal{M}_A(E, G)$, $g\in\mathcal{M}_A(F, G)$. The map $f\times g: E\times F\to G:(x, y)\mapsto(f\times g)(x, y):=f(x)g(y)$ is a separately continuous A-bilinear map, such that there exists the corresponding tensor product A-linear map $f\otimes_A g: E\otimes_A F\to G$, being also an A-morphism whenever G is commutative.

In the sequel, we are interested in A-compatible topologies $\mathfrak{T}$ (Definition 1.1) satisfying the following conditions:

(3.1) *The canonical map of $E\times F$ into $E\otimes_A^{\mathfrak{T}}F$ is (jointly) continuous.*

(3.2) *For any pair $(f, g)\in\mathcal{M}_A(E, G)\times\mathcal{M}_A(F, G)$, $f\otimes_A g\in\mathcal{L}_A(E\otimes_A^{\mathfrak{T}}F, G)_s$.*

A stronger version of (3.2) is applied when completed A-tensor product A-algebras have to be considered. That is,

(3.3) *For any equicontinous subsets $M\subseteq\mathcal{M}_A(E, G)$, $N\subseteq\mathcal{M}_A(F, G)$, $M\otimes_A N:=\{f\otimes_A g: f\in M, g\in N\}$ is an equicontinuous subset of $\mathcal{L}_A(E\otimes_A^{\mathfrak{T}}F, G)_s$.*

In this respect, one has an analogous situation with that described by §2 (cf. relations (2.1), (2.2), (2.3)), when considering instead generalized A-spectra. This is for instance the case for locally (m-) convex A-algebras and the projective A-tensor product topology π on $E\otimes_A F$ (cf. Appendix).

Now, let E, F be locally convex A-algebras with b.a.i.'s $(u_i)_{i\in K}$, $(v_j)_{j\in J}$ respectively, and $h\in\mathcal{M}_A(E\otimes_A^{\mathfrak{T}}F, G)$ where $\mathfrak{T}$ is an A-compatible topology on $E\otimes_A F$ satisfying (3.1) and G a locally convex A-algebra. Then, the maps

$$(3.4)\qquad x\longmapsto\lim_j h(x\otimes_A v_j),\ x\in E \quad\text{and}\quad y\longmapsto\lim_i h(u_i\otimes_A y),\ y\in F,$$

whenever there exist (cf., for instance, [11: Chapt. VI, Theorem 11.1]), define a pair $(f, g)\in\mathcal{M}_A(E, G)\times\mathcal{M}_A(F, G)$ such that we get the map

$$(3.5)\qquad \mathcal{M}_A(E\otimes_A^{\mathfrak{T}}F, G)\longrightarrow\mathcal{M}_A(E, G)\times\mathcal{M}_A(F, G): h\longmapsto(f, g).$$

Proposition 3.1. *Let E, F be locally convex A-algebras with b.a.i's, and G a locally convex A-algebra with continuous multiplication. Moreover, let $\mathfrak{T}$ be an A-compatible topology on $E\otimes_A F$ satisfying* (3.1), (3.2) *and let the closed set*

$$(3.6)\quad Q=\{(f, g): f(x)g(y)=g(y)f(x);\ x\in E,\ y\in F\}\subseteq\mathcal{M}_A(E, G)\times\mathcal{M}_A(F, G).$$

Then one gets

$$\mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G) = Q\,, \tag{3.7}$$

and for G commutative

$$\mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G) = \mathcal{M}_A(E,\ G) \times \mathcal{M}_A(F,\ G)\,. \tag{3.8}$$

Proof. Given an $h \in \mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G)$, by (3.4) one gets

$$h = f \otimes_A g \tag{3.9}$$

(cf. also [11: Chapt. VI, condition (10.57)]) and thus (3.5) is an injection. Now, given $(f,\ g) \in Q$, one has $h \equiv f \otimes_A g \in \mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G)$ (cf. (3.2)) and every element thus defined yields according to (3.9) the initial pair (f, g), that is (3.5) is a bijection onto Q. The bicontinuity of (3.5) can be proved analogously to [15: Theorem 2.1] and thus one gets (3.7). Moreover, (3.8) is an immediate consequence of (3.6), (3.7). ■

Now, let E, G be topological $\boldsymbol{A}$-algebras where E has continuous multiplication and G is complete. The map

$$f \longmapsto \bar{f} : \mathcal{M}_A(E,\ G) \longrightarrow \mathcal{M}_A(\hat{E},\ G) \tag{3.10}$$

where $\bar{f}$ is the (continuous) extension of f to the completion $\hat{E}$ of E, is a *continuous bijection.* In particular,

(3.11) *The bijection* (3.10) *is a homeomorphism if and only if either one of the sets* $\mathcal{M}_A(E,\ G)$, $\mathcal{M}_A(\hat{E},\ G)$ *is locally equicontinuous*

(cf. [11: Chapt. III, Theorem 8.3] in case $\boldsymbol{A} = \boldsymbol{C}$).

Proposition 3.1 and relation (3.11) yield the following lemma, by using the arguments of [11: Chapt. VI, Lemma 6.2] for $\boldsymbol{A} = \boldsymbol{C}$.

Lemma 3.1. *Let E, F, G be locally convex* $\boldsymbol{A}$*-algebras as in Proposition* 3.1 *and* $\mathfrak{T}$ *an* $\boldsymbol{A}$*-compatible topology satisfying* (3.1), (3.3). *Moreover, consider the following assertions:*

i) $\mathcal{M}_A(E,\ G)$, $\mathcal{M}_A(F,\ G)$ *are both locally equicontinuous.*

ii) $\mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G)$ *is locally equicontinuous.*

Then i)⇒ii). *Besides, if E, F, G are unital, for every* $(f,\ g) \in Q$ *there exist an equicontinuous neighbourhood U of f in* $\mathcal{M}_A(E,\ G)$ *and V of g in* $\mathcal{M}_A(F,\ G)$ *such that* $U \otimes_A V$ *is an equicontinuous neighbourhood of* $h \equiv f \otimes_A g$ *in* $\mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F,\ G)$. *In particular,* ii)⇒i) *as well, whenever G is commutative. In the latter case, it suffices the topology* $\mathfrak{T}$ *to satisfy* (3.13) *below* (*not necessarily* (3.1)). ■

Concerning the generalized $\boldsymbol{A}$-spectrum of the (complete) locally convex $\boldsymbol{A}$-algebra $E \hat{\otimes}_A^{\mathfrak{T}} F$, we have

Theorem 3.1. *Let E, F be locally convex* $\boldsymbol{A}$*-algebras with b.a.i.'s and con-*

tinuous multiplications. Moreover, suppose that $\mathcal{M}_A(E, G)$, $\mathcal{M}_A(F, G)$ are locally equicontinuous, where G is a complete locally convex A-algebra with continuous multiplication and let $\mathfrak{T}$ be an A-compatible topology on $E \otimes_A F$ satisfying (3.1), (3.3). *If Q is the set* (3.6), *then one has*

$$\begin{aligned} \mathcal{M}_A(E \hat{\otimes}_A^{\mathfrak{T}} F, G) = Q = \mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F, G) \underset{\rightarrow}{\subseteq} \mathcal{M}_A(\hat{E}, G) \times \mathcal{M}_A(\hat{F}, G) \\ = \mathcal{M}_A(E, G) \times \mathcal{M}_A(F, G). \end{aligned} \tag{3.12}$$

In case G is commutative, the "inclusion sign" in (3.12) *may be replaced by an equality.*

Proof. By Lemma 3.1, $\mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F, G)$ is locally equicontinuous so that $\mathcal{M}_A(E \hat{\otimes}_A^{\mathfrak{T}} F, G) = \mathcal{M}_A(E \otimes_A^{\mathfrak{T}} F, G)$ (cf. (3.10), (3.11)). Now, the assertion is a straightforward consequence of Proposition 3.1 (cf. also (3.11). ■

Remark 3.1. One gets, of course, the same conclusions with those of Proposition 3.1 (resp. Theorem 3.1), in case of unital topological A-algebras, where the topology $\mathfrak{T}$ satisfies instead of (3.1) the condition:

(3.13) *The canonical map of $E \times F$ into $E \otimes_A^{\mathfrak{T}} F$ is separately continuous,*

the rest hypotheses of Proposition 3.1 (resp. Theorem 3.1) remaining unchanged. The above specializes to [11: Chapt. VI, Lemma 6.1, Theorem 6.2] (cf. also [14: Theorems 2.2, 3.1]).

In this respect, we remark that, if H is a complete locally bounded A-module, whose topology is defined by an a-norm (cf. [13]), $\mathcal{L}_A(H)$ (: continuous linear A-endomorphisms of H) is, w.r.t. "operator norm", a complete locally bounded (not necessarily locally convex) A-algebra with continuous multiplication and identity id_H. Thus, by analogous considerations to [11: Chapt. VI, Theorem 11.1, Lemma 11.2], the preceding results are fulfilled in case of locally bounded A-algebras.

4. Continuous central A-morphisms

Let E, F be topological A-algebras with identities 1_E, 1_F. Then, a *continuous central A-morphism* is an element $h \in \mathcal{M}_A(E, F)$ such that $Im(h)$ has *trivial center in F over A* (equivalently, *$Im(h)$ is a central A-subalgebra of F*),

$$\mathfrak{C}(Im(h)) = A \cdot 1_F, \tag{4.1}$$

where $\mathfrak{C}(Im(h))$ denotes the *center of $Im(h)$*, i.e. $\mathfrak{C}(Im(h)) = Im(h) \cap (Im(h))'$, with $(Im(h))' \equiv \{y \in F : yh(x) = h(x)y,\ x \in E\}$ the *commutant* of $Im(h)$ in F. The set of continuous central A-morphisms of E into F endowed with the relative topology induced on it by $\mathcal{M}_A(E, F)$ will be denoted by $\mathcal{M}_A^0(E, F)$.

Under the conditions of Remark 3.1, for each $h\in\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)$ of the form $h=f\otimes_A g$ (cf. (3.9)), one gets

$$\mathfrak{C}(Im(g))\subseteq\mathfrak{C}(Im(h)), \qquad \mathfrak{C}(Im(f))=Im(f)\subseteq\mathfrak{C}(Im(h)), \tag{4.2}$$

whenever E is commutative (cf. also [11: Chapt. VI, Theorem 7.2]). Thus, *$h=f\otimes_A g$ is a continuous central A-morphism if, and only if, this is true for f, g*: Indeed, if $h\in\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)$, then by (4.2) $(f, g)\in\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G)$. Conversely, if $(f, g)\in\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G)$, then $f(x)=a_x\cdot 1_G$, $a_x\in A$ (cf. (4.2)) such that $f(x)g(y)=a_x\cdot g(y)=g(y)f(x)$, for all $(x, y)\in E\times F$, so that $h\equiv f\otimes_A g\in\mathcal{M}_A(E\otimes_A^{\mathfrak{T}}F, G)$ (cf. Proposition 3.2). Moreover, $Im(h)\subseteq Im(g)$ (cf. (3.9)), hence $A\cdot 1_G=\mathfrak{C}(Im(f))\subseteq\mathfrak{C}(Im(h))\subseteq\mathfrak{C}(Im(g))=A\cdot 1_G$, i.e. $h\in\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)$.

Remark 3.1 and the preceding comments yield the next proposition (cf. also Proposition 3.1).

Proposition 4.1. *Let F, F, G be unital topological A-algebras, where one of E, F is commutative and G has continuous multiplication. Moreover, let $E\otimes_A^{\mathfrak{T}}F$ be the topological A-tensor product A-algebra of E, F equipped with an A-compatible topology $\mathfrak{T}$ satisfying* (3.2), (3.13). *Then,*

$$\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)=\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G). \quad \blacksquare \tag{4.3}$$

Now, let E be a topological A-algebra. An element $0\neq x\in E$ is said to be a *topologically torsion free element if the canonical map $\omega_x: A\to E: a\mapsto\omega_x(a):=ax$ is an isomorphism "into" of the corresponding topological A-algebras ($: A\underset{\overrightarrow{\omega_x}}{\subset} E$), that is E is a topological extension of the topological algebra A.* (cf. [6: Chapt. IV, p. 220], for the "discrete" case).

Lemma 4.1. *Let E, F be topological A-algebras with identities 1_E, 1_F respectively, where E is also commutative and 1_F topologically torsion free. Then, every $f\in\mathcal{M}_A^0(E, F)$ takes the form*

$$f=\chi\otimes_A 1_F, \tag{4.4}$$

where $\chi\in\mathfrak{M}_A(E)$ ($:=\mathcal{M}_A(E, A)$, A-spectrum of E), such that $(\chi\otimes_A 1_F)(x):=\chi(x)\cdot 1_F$, $x\in E$. In particular, one gets

$$\mathfrak{M}_A(E)=\mathcal{M}_A^0(E, F). \tag{4.5}$$

Proof. By (4.2), $f(x)=a_x\cdot 1_F$, $a_x\in A$ for every $x\in E$; hence, by defining $\chi: E\to A: x\mapsto\chi(x):=a_x$, such that $f(x)=\chi(x)\cdot 1_F\equiv(\chi\otimes_A 1_F)(x)$, one gets $\chi\in\mathfrak{M}_A(E)$. On the other hand, by the previous comments, one obtains an isomorphism of topological A-algebras defined by ω_{1_F}, that is $A\underset{\omega_{1_F}}{\cong} A\cdot 1_F\subseteq F$, in such a way that (4.5) follows immediately. $\blacksquare$

Now, let the conditions of Proposition 4.1 be valid, whereas A is unital and

1_G is a topologically torsion free element. If, moreover, E is commutative then each $h \in \mathcal{M}_A^0(E \otimes_A^\tau F, G)$ is of the form $h = \chi \otimes_A g$, with $\chi \in \mathfrak{M}_A(E)$, $g \in \mathcal{M}_A^0(F, G)$, such that

$$\mathcal{M}_A^0(E \otimes_A^\tau F, G) = \mathfrak{M}_A(E) \times \mathcal{M}_A^0(F, G) \tag{4.6}$$

(cf. (4.3), (4.5)). Furthermore, if E, F are both commutative and A has continuous multiplication, then for each $h \in \mathcal{M}_A^0(E \otimes_A^\tau F, G)$ we have $h = \chi \otimes_A \psi \otimes_A 1_G$, where $(\chi, \psi) \in \mathfrak{M}_A(E) \times \mathfrak{M}_A(F)$, such that

$$\mathcal{M}_A^0(E \otimes_A^\tau F, G) = \mathfrak{M}_A(E) \times \mathfrak{M}_A(F) = \mathfrak{M}_A(E \otimes_A^\tau F) \tag{4.7}$$

(cf. (4.3), (4.6) and (3.8) for $G = A$).

Examining continuous central A-morphisms defined on $E \hat{\otimes}_A^\tau F$ we need also the following: Let E, F be topological A-algebras where E has continuous multiplication and F is complete. Let also f be a continuous A-morphism of E into F. Then (cf. also (3.10))

$$(Im(\bar{f}))' = (Im(f))' = (\overline{Im(f)})' \tag{4.8}$$

($\overline{Im(f)}$ is the closure of $Im(f)$ in F) and

$$\mathfrak{C}(Im(f)) = Im(f) \cap (Im(\bar{f}))' \subseteq \mathfrak{C}(Im(\bar{f})) \tag{4.9}$$

that is f *is a continuous central* A*-morphism whenever* $\bar{f}$ $(: \hat{E} \to F)$ *is such a morphism.* Thus, the map

$$\theta : \mathcal{M}_A^0(\hat{E}, F) \longrightarrow \mathcal{M}_A^0(E, F) : \bar{f} \longmapsto \bar{f}|_E = f \tag{4.10}$$

is, of course, a *continuous bijection onto* $Im(\theta) \equiv H \subseteq \mathcal{M}_A^0(E, F)$, *by considering* H *equipped with the relative topology induced on it by* $\mathcal{M}_A(E, F)$. In particular, $\mathcal{M}_A^0(\hat{E}, F)$ *is locally equicontinuous if and only if* H *is locally equicontinuous,* so that *the hypothesis of local equicontinuity of one of the last sets implies*

$$\mathcal{M}_A^0(\hat{E}, F) = H \subseteq \mathcal{M}_A^0(E, F) \tag{4.11}$$

(cf. (4.10), (3.10), (3.11) and also [11: Chapt. III, Remark 8.1]).

On the other hand, with E, F as above, let $\mathcal{M}_A^*(E, F)$ be the set of continuous A-morphisms of E into F with *trivial commutant* (i.e., $(Im(f))' = A \cdot 1_F$, $f \in \mathcal{M}_A(E, F)$) equipped with the relative topology from $\mathcal{M}_A(E, F)$ (§ 3). Then, by (4.9) one gets

$$\mathcal{M}_A^*(\hat{E}, F) = \mathcal{M}_A^*(E, F) \tag{4.12}$$

if, and only if, one of the sets of (4.12) *is locally equicontinuous.*

Furthermore, let E, F be unital topological A-algebras with continuous multiplications, E being also commutative and A, F complete. If 1_F is topologically torsion free, then *the local equicontinuity of each one of the sets* $\mathfrak{M}_A(E)$, H,

$\mathcal{M}_A^0(E, F)$ implies the local equicontinuity of the other two (cf. (4.5), (4.11), (3.11)). Besides, under the hypotheses of Proposition 4.1 one obtains that *$\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)$ is locally equicontinuous if, and only if, $\mathcal{M}_A^0(E, G)$, $\mathcal{M}_A^0(F, G)$ are locally equicontinuous* (cf. Lemma 3.1).

Proposition 4.2. *Let E, F, G be unital topological A-algebras with continuous multiplications where G is complete and one of E, F commutative. Moreover, suppose that $\mathcal{M}_A^0(E, G)$, $\mathcal{M}_A^0(F, G)$ are locally equicontinuous and the A-compatible topology $\mathfrak{T}$ on $E\otimes_A F$ satisfies* (3.3), (3.13). *Then, one gets*

$$\mathcal{M}_A^0(E\hat{\otimes}_A^{\mathfrak{T}}F, G)\underset{\rightarrow}{\subset}\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)=\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G). \tag{4.13}$$

In particular, if $\mathfrak{T}$ satisfies, moreover, (3.1), *then*

$$\begin{aligned}\mathcal{M}_A^0(E\hat{\otimes}_A^{\mathfrak{T}}F, G)&\underset{\rightarrow}{\subset}\mathcal{M}_A^0(\hat{E}, G)\times\mathcal{M}_A^0(\hat{F}, G)\\ &\underset{\rightarrow}{\subset}\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G)=\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G).\end{aligned} \tag{4.14}$$

Proof. By (4.11) we have $\mathcal{M}_A^0(E\hat{\otimes}_A^{\mathfrak{T}}F, G)\underset{\rightarrow}{\subset}\mathcal{M}_A^0(E\otimes_A^{\mathfrak{T}}F, G)$, so that (4.3) implies (4.13). Furthermore, by the continuity of $\varphi: E\times F\rightarrow E\otimes_A^{\mathfrak{T}}F$ (cf. (3.1)) one gets $\hat{E}\otimes_A^{\mathfrak{T}}\hat{F}=\varphi(\hat{E}\times\hat{F})\subseteq\widehat{\varphi(E\times F)}=E\hat{\otimes}_A^{\mathfrak{T}}F$, which implies $\mathcal{M}_A^0(E\hat{\otimes}_A^{\mathfrak{T}}F, G)\underset{\rightarrow}{\subset}\mathcal{M}_A^0(\hat{E}\otimes_A^{\mathfrak{T}}\hat{F}, G)$ (cf. (4.11)), hence (4.14) immediately follows by (4.3). ∎

By combining Proposition 4.2 and relations (4.6), (4.7) (cf. also remarks before Proposition 4.2) one gets more informative decompositions than (4.13), (4.14), as it is, for instance, the validity of (4.13), (4.14) with equalities.

5. Appendix

In this Section, on the one hand we introduce the notion of projective (topological) A-tensor product A-algebra, and on the other hand we define the topological tensor product over A of (locally convex) A-algebras in a similar way to that defined by B. R. Gelbaum [3] for Banach A-algebras. Besides, the connection between the previous two A-tensor products is also considered together with the corresponding "spectra" of the topological algebras involved.

Thus, we first set the following definition.

Definition 5.1. Let E, F be locally convex A-algebras. The pair (D, φ) consisting of a locally convex A-algebra D and a continuous A-bilinear multiplicative map φ of $E\times F$ into D is said to be *projective* (*topological*) *A-tensor product* (*A-algebra*) *of* E, F if the following universal property is valid:

(5.1) *For any pair (M, u), where M is a locally convex A-algebra and u a continuous A-bilinear multiplicative map of $E\times F$ into M, there exists a uniquely defined continuous A-morphism $\tilde{u}: D\rightarrow M$, such that $u=\tilde{u}\circ\varphi$.*

The uniqueness of the projective A-tensor product is, of course, easily realized.

The existence of the projective A-tensor product of E, F is provided by the *projective A-tensorial locally convex topology on $E\otimes_A F$* (denoted by π). Thus, let $(E, (p_a)_{a\in K})$, $(F, (q_\lambda)_{\lambda\in L})$ be locally convex A-algebras, where (p_a), (q_λ) are the families of (continuous) semi-norms defining the topologies of E, F respectively. Then the relation

$$r_{(a,\lambda)}(z)=\inf\sum_{i=1}^{n}p_a(x_i)q_\lambda(y_i)\,, \tag{5.2}$$

where "inf" is taken over all expression of $z=\sum_{i=1}^{n}x_i\otimes_A y_i\in E\otimes_A F$, defines $r_{(a,\lambda)}$ as a semi-norm on $E\otimes_A F$, for every (a, λ), so that the respective locally convex topology defined by the family $(r_{(a,\lambda)})$ is an A-compatible topology on $E\otimes_A F$ (Definition 1.1).

By applying (1.1), a continuous A-bilinear multiplicative map u of $E\times F$ into a locally convex A-algebra M gives rise to a uniquely defined continuous A-morphism $\tilde{u}$ of $(E\otimes_A F, (r_{(a,\lambda)}))$ into M, since, for each continuous semi-norm ν on M, $\nu(\tilde{u}(z))\leqq k\cdot\sum_i\nu(u(x_i, y_i))\leqq l\cdot\sum_i p_a(x_i)q_\lambda(y_i)$ with $l>0$ so that $\nu(\tilde{u}(z))\leqq l\cdot r_{(a,\lambda)}(z)$, for $z=\sum_i x_i\otimes_A y_i\in E\otimes_A F$. Thus, the *projective A-tensor product topology π is the finest locally convex topology on $E\otimes_A F$ making the canonical A-bilinear multiplicative map φ continuous.* On the other hand, *the locally convex A-algebra* $(E\otimes_A F, (r_{(a,\lambda)}))\equiv E\otimes_A^\pi F$ (with the canonical map φ) is "*the*" *projective A-tensor product of* E, F (cf. Definition 5.1).

In the sequel we give another realization of the projective A-tensor product of E, F: Thus, let $(E, (p_a))$, $(F, (q_\lambda))$ be locally convex A-algebras and $E\underset{\pi}{\otimes}F$ the corresponding projective tensor product locally convex algebra of E, F, cf. [12: Proposition 3.2], being, in fact, a locally convex A-algebra. If I is the closed 2-sided A-ideal of $E\underset{\pi}{\otimes}F$ as in (1.6), then $E\underset{\pi}{\otimes}F/I$ endowed with the corresponding quotient topology is a locally convex A-algebra such that *the pair* $(E\underset{\pi}{\otimes}F/I,\ \varphi)$, with $\varphi=\rho\circ\varphi_1$, where ρ is the canonical quotient map and $\varphi_1: E\times F\to E\underset{\pi}{\otimes}F$ the canonical continuous bilinear map, *is the projective A-tensor product A-algebra of E, F.* Moreover, if $(p_a\otimes q_\lambda)$ is the family of continuous semi-norms defining the projective topology on $E\otimes F$ (cf. [12: Lemma 3.1]), then $(p_a\otimes_A q_\lambda)$ with

$$(p_a\otimes_A q_\lambda)(\dot{z})=\inf_{z\in\dot{z}}(p_a\otimes q_\lambda)(z),\ z\in E\otimes F,\ \dot{z}\in E\otimes F/I \tag{5.3}$$

makes $E\otimes F/I$ into a locally convex A-algebra, so that $(E\otimes F/I,\ (p_a\otimes_A q_\lambda))\equiv E\underset{\pi}{\otimes}F/I$ (with the continuous map $\varphi=\rho\circ\varphi_1$) is the *projective A-tensor product locally convex A-algebra of E, F* (Definition 5.1).

We examine now another expression of the projective $\boldsymbol{A}$-tensor product (cf. [2, 7] for Banach $\boldsymbol{A}$-algebras). If $(E, (p_\alpha))$, $(F, (q_\lambda))$ are locally convex $\boldsymbol{A}$-algebras, the relation

$$\begin{aligned} r_{(\alpha,\lambda)} \equiv (p_\alpha, q_\lambda) : \mathcal{F}(E, F)_0 \longrightarrow \boldsymbol{R}_+ : f \longmapsto r_{(\alpha,\lambda)}(f) : \\ := \sum_{(x,y)} |f(x, y)| p_\alpha(x) q_\lambda(y) \end{aligned} \tag{5.4}$$

defines a semi-norm on $\mathcal{F}(E, F)_0$ for every (α, λ), so that $(\mathcal{F}(E, F)_0, (r_{(\alpha,\lambda)}))$ is actually a locally convex $\boldsymbol{A}$-algebra. Moreover, if H is the closed vector subspace of $\mathcal{F}(E, F)_0$ generated by elements of the form (1.3), being also a closed 2-sided $\boldsymbol{A}$-ideal of $(\mathcal{F}(E, F)_0, (r_{(\alpha,\lambda)}))$, then $(\mathcal{F}(E, F)_0/I, (r_{(\alpha,\lambda)})) \equiv \mathcal{F}(E, F)_0/H$ is a *locally convex $\boldsymbol{A}$-algebra*, where $(\dot{r}_{(\alpha,\lambda)})$ is the corresponding family of quotient semi-norms of (5.4), such that the pair $(\mathcal{F}(E, F)_0/H, \varphi = \rho \circ \delta)$ (cf. § 1) is the *projective $\boldsymbol{A}$-tensor product locally convex $\boldsymbol{A}$-algebra of E, F* (cf. Definition 5.1).

Summarizing the previous results, one has

Proposition 5.1. *Let E, F be locally convex $\boldsymbol{A}$-algebras. Then,*

$$E \otimes_A^\pi F = E \underset{\pi}{\otimes} F/I = \mathcal{F}(E, F)_0/H \tag{5.5}$$

within isomorphisms of locally convex $\boldsymbol{A}$-algebras. In particular, if E, F have continuous multiplications, then

$$E \hat{\otimes}_A^\pi F = \widehat{E \underset{\pi}{\otimes} F/I} = \widehat{\mathcal{F}(E, F)_0/H}, \tag{5.6}$$

within isomorphisms of locally convex $\boldsymbol{A}$-algebras. ■

Analogous results are also obtained in case of locally m-convex $\boldsymbol{A}$-algebras (cf. also [12: Proposition 3.1]). Besides, *the projective locally (m-) convex $\boldsymbol{A}$-tensor product topology on $E \otimes_A F$ is Hausdorff if and only if this is the case for the topologies of each one of the algebras E and F* (cf. (5.5) and also [11: Chapt. VI, § 3], [5: p. 105, Proposition 5]).

Remark 5.1. One can also topologize $E \otimes_A F$ in different manners than the previous one, either by means of a universal property analogous to (5.1) (cf., e.g. [9a]) or directly by the relation (1.6), using the known tensorial topologies on the (usual) complex tensor product, setting, e.g., $\tau = \varepsilon$ (: *biprojective tensorial topology*; [4: Chap. I, Definition 5, p. 89] and/or [11: Chapt. VI, Definition 2.3]), or $\tau = i$ (: *inductive tensorial topology*; [4: Chap. I, Proposition 13]) with the analogous arrangements. On the other hand, all the preceding results as well as those of Section 1 are also valid (under suitable modifications) not only for locally (m-) convex $\boldsymbol{A}$-algebras or locally convex $\boldsymbol{A}$-modules, but, more generally, by defining in an obvious way *topological $(\boldsymbol{A}, \boldsymbol{B})$-tensor product algebras* (or *modules*; denoted by $E_A \otimes_B F$) for a given locally convex left $\boldsymbol{A}$-algebra E and a locally convex right $\boldsymbol{B}$-algebra F (cf. also [20: p. 157ff] for relevant reports).

Now, we introduce the concept of topological tensor products over topological algebras in a similar way to [3], considering first the algebraic construction. Thus, let E, F be $\boldsymbol{A}$-algebras and $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ the set of the maps $f: E\times F\to \boldsymbol{A}$ with finite support being an algebra under a convolution (cf. [10: p. 107]) and, in particular, an $\boldsymbol{A}$-algebra by pointwise "multiplication of coefficients". Furthermore, defining

$$\chi_\alpha: E\times F \longrightarrow \mathscr{F}_{\boldsymbol{A}}(E, F)_0:(x, y)\longmapsto \chi_\alpha(x, y):=\alpha\cdot\delta_{(x,y)},\ \alpha\in \boldsymbol{A} \tag{5.7}$$

(cf. § 1), we obtain that every $f\in\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ is of the form $f=\sum_{(x,y)}\chi_{f(x,y)}(x, y)$. Thus, if $I_{\boldsymbol{A}}$ is the vector subspace of $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ generated by the functions

$$\begin{array}{ll} \chi_\alpha(x+x', y)-\chi_\alpha(x, y)-\chi_\alpha(x', y) & \chi_\alpha(\lambda x, y)-\chi_\alpha(x, \lambda y) \\ \chi_\alpha(x, y+y')-\chi_\alpha(x, y)-\chi_\alpha(x, y') & \chi_\alpha(\beta x, y)-\chi_\alpha(x, \beta y) \\ \lambda\chi_\alpha(x, y)-\chi_\alpha(\lambda x, y) & \beta\chi_\alpha(x, y)-\chi_\alpha(\beta x, y) \end{array} \tag{5.8}$$

with $x, x'\in E$, $y, y'\in F$, $\alpha, \beta\in \boldsymbol{A}$, $\lambda\in \boldsymbol{C}$, $I_{\boldsymbol{A}}$ is a 2-sided $\boldsymbol{A}$-ideal of $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ such that the quotient $\boldsymbol{A}$-algebra $\mathscr{F}_{\boldsymbol{A}}(E, F)_0/I_{\boldsymbol{A}}$ is called the *tensor product of* E, F *over* $\boldsymbol{A}$.

Now, if $(E, (p_\alpha))$, $(F, (q_\lambda))$ are locally convex $(\boldsymbol{A}, (\nu_\mu))$-algebras, then

$$\begin{aligned} \gamma_{(\mu,\alpha,\lambda)}\equiv(\nu_\mu, p_\alpha, q_\lambda): \mathscr{F}_{\boldsymbol{A}}(E, F)_0 &\longrightarrow \boldsymbol{R}_+: f\longmapsto \gamma_{(\mu,\alpha,\lambda)}(f) \\ &:=\sum_{(x,y)}\nu_\mu(f(x, y))p_\alpha(x)q_\lambda(y) \end{aligned} \tag{5.9}$$

defines a semi-norm on $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$, such that $(\mathscr{F}_{\boldsymbol{A}}(E, F)_0, \gamma_{(\mu,\alpha,\lambda)})$ is, of course, a locally convex $\boldsymbol{A}$-algebra.

In case $\boldsymbol{A}$, E, F have continuous multiplications (resp. are locally m-convex algebras), $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ is an algebra of the same type. If E, F have continuous multiplications but $\boldsymbol{A}$ has separately continuous multiplication, then $\mathscr{F}_{\boldsymbol{A}}(E, F)_0$ is a locally convex $\boldsymbol{A}$-algebra with separately continuous multiplication.

Thus, if $I_{\boldsymbol{A}}$ is the previous closed 2-sided $\boldsymbol{A}$-ideal of $(\mathscr{F}_{\boldsymbol{A}}(E, F)_0, \gamma_{(\mu,\alpha,\lambda)})$ the quotient locally convex $\boldsymbol{A}$-algebra $\mathscr{F}_{\boldsymbol{A}}(E, F)_0/I_{\boldsymbol{A}}$ is said to be the *topological tensor product of* E, F *over* $\boldsymbol{A}$. The next proposition provides a universal property characterizing the previous tensor product (for Banach A-algebras cf. [7: Theorem 2.1]).

Throughout the sequel we suppose that $\boldsymbol{A}$ *is a locally convex algebra with continuous multiplication.*

Proposition 5.2. *Given two locally convex* $\boldsymbol{A}$*-algebras* E, F, *the topological tensor product of* E, F *over* $\boldsymbol{A}$ *is the projective* $\boldsymbol{A}$*-tensor product of* $\boldsymbol{A}$, E, F, *that is*

(5.10) $$\mathcal{T}_A(E,\ F)_0/I_A = A\otimes_A^\pi E\otimes_A^\pi F,$$

within an isomorphism of locally convex A-algebras.

Proof. The continuous maps $\chi: A\times E\times F\to\mathcal{T}_A(E,\ F)_0 : (\alpha,\ x,\ y)\mapsto\chi(\alpha,\ x,\ y) := \chi_\alpha(x,\ y)$ (cf. (5.7)) and $\rho: \mathcal{T}_A(E,\ F)_0\to\mathcal{T}_A(E,\ F)_0/I_A$ (: canonical quotient A-morphism) define the continuous A-trilinear multiplicative map $\varphi: =\rho\circ\chi$, such that if u is a continuous A-trilinear multiplicative map of $A\times E\times F$ into a locally convex A-algebra M, then one defines a map $h: \mathcal{T}_A(E,\ F)_0\to M: f\mapsto h(f(:=\sum_{(x,y)} u(f)x,\ y),\ x,\ y)$ with $u=h\circ\chi$, being a continuous A-morphism with $I_A\subseteq ker(h)$. Thus, there exists a cintinuous A-morphism $\tilde{u}: \mathcal{T}_A(E,\ F)_0/I_A\to M$ with $h=\tilde{u}\circ\rho$, so that $u=\tilde{u}\circ\varphi$. Now, $(\mathcal{T}_A(E,\ F)_0/I_A,\ \varphi)$ satisfies the analogous of Definition 5.1, and this completes the proof. ■

In particular, if the algebras E, F have continuous multiplications, then

(5.11) $$\widehat{\mathcal{T}_A(E,F)_0/I_A} = A\hat{\otimes}_A^\pi E\hat{\otimes}_A^\pi F,$$

within an isomorphism of (complete) locally convex A-algebras.

Concerning the (numerical) spectrum of $\mathcal{T}_A(E,\ F)_0/I_A$, we remark that if Δ is the diagonal of $\mathfrak{M}(A)\times\mathfrak{M}(A)$ and μ, ν, Δ^+ as in Theorem 2.1, then by the canonical injection $\mathfrak{M}(A)\underset{i}{\overrightarrow{\subset}}\mathfrak{M}(A)^+$, one gets $(i\times\mu\times\nu)^{-1}(\Delta^+)=(\mu\times\nu)^{-1}(\Delta)$ ($\subseteq\mathfrak{M}(E)\times\mathfrak{M}(F)$ closed), such that

(5.12) $$\mathfrak{M}(\mathcal{T}_A(E,\ F)_0/I_A)=(\mu\times\nu)^{-1}(\Delta)\subseteq\mathfrak{M}(E)\times\mathfrak{M}(F)$$

(cf. (5.10) and Theorem 2.1). Moreover, if E, F have continuous multiplications and A, E, F have locally equicontinuous spectra, then

(5.13) $$\mathfrak{M}(\widehat{\mathcal{T}_A(E,\ F)_0/I_A})=(\mu\times\nu)^{-1}(\Delta)\subseteq\mathfrak{M}(E)\times\mathfrak{M}(F)$$

(cf. (5.11), (5.12) and also [11: Chapt. III, Theorem 2.1]).

In case E, F are Banach A-algebras the above specializes to [3: Theorem 1].

Considering now the generalized A-spectrum of $\mathcal{T}_A(E,\ F)_0/I_A$ and its completion the following comments are necessary: By the *essential part* of an A-module E we mean the vector subspace $[AE]$ of E generated by the set $AE=\{ax;\ a\in A,\ x\in E\}$, being in particular an A-module. If E is a topological A-algebra, then $[AE]$ is also a topological A-algebra endowed with the relative topology of E. Moreover, E is said to be an *essential* (A-algebra) if $[AE]$ is a dense subset of E (cf. also [21]).

Lemma 5.1. *Let E be a locally convex A-algebra, where A has a b.a.i. $(e_\delta)_{\delta\in D}$. Then,*

(5.14) $$A\otimes_A^\pi E=[AE],$$

within an isomorphism of locally convex A-algebras.

Proof. The continuous A-bilinear multiplicative map $\omega: A\times E\to E:(a, x)\mapsto \omega(a, x):=ax$ defines a unique continuous A-morphism $\tilde{\omega}: A\otimes_A^\pi E\to E$ such that $\omega=\tilde{\omega}\circ\varphi$ (φ is the canonical map of $A\times E$ into $A\otimes_A^\pi E$, cf. Definition 5.1), hence $\tilde{\omega}(A\otimes_A^\pi E)=[AE]$. Now, for each $u\in(A\otimes_A^\pi E)'_s$, there exists $f\in\mathcal{L}_A(E, A'_s)$ such that $u(\sum_i a_i\otimes_A x_i)=\sum_i f(x_i)(a_i)$, and hence there exists a linear map $v:[AE]\to \boldsymbol{C}$ with $v\circ\tilde{\omega}=u$, which yields that $\tilde{\omega}$ is injection, i.e. a continuous A-isomorphism. Concerning the continuity of $\tilde{\omega}^{-1}$, let r be a continuous semi-norm on $A\otimes_A^\pi E$ (cf. (5.2)). Then, for each $\varepsilon>0$, the relation $\lim_\delta(\sum_i(e_\delta a_i)\otimes_A x_i)=\sum_i a_i\otimes_A x_i$, implies $r(\sum_i(e_\delta a_i)\otimes_A x_i-\sum_i a_i\otimes_A x_i)\leqq\varepsilon$, for every $\delta>\delta_0$, that is there exist $\lambda>0$ and a continuous semi-norm p on E such that $r(\tilde{\omega}^{-1}(\sum_i a_i x_i))\leqq\lambda\cdot p(\sum_i a_i x_i)$, which completes the proof. ■

Under the conditions of Lemma 5.1 and the supposition that E is essential with continuous multiplication, one gets

$$A\hat{\otimes}_A^\pi E=\widehat{[AE]}=\hat{E}, \tag{5.15}$$

within an isomorphism of locally convex A-algebras.

Now, if A is unital, (5.14) implies $A\otimes_A^\pi E=E$ within an isomorphism of locally convex A-algebras, and so (5.15) is valid without E being an essential algebra. In this case, the projective A-tensor product A-algebra of E, F (Definition 5.1) and the topological tensor product of E, F over A coincide, within an isomorphism of locally convex A-algebras.

Proposition 5.3. *Let E, F be locally convex A-algebras with b.a.i.'s (A has also b.a.i.) and continuous multiplications, where E, F are essential. Furthermore, let $\mathcal{M}_A(E, G)$, $\mathcal{M}_A(F, G)$ be locally equicontinuous, where G is a complete locally convex A-algebra with continuous multiplication and let Q be the set* (3.6). *Then, one has*

$$\begin{aligned}\mathcal{M}_A(\widehat{\mathfrak{T}_A(E, F)_0/I_A}, G)=Q=\mathcal{M}_A(\mathfrak{T}_A(E, F)_0/I_A, G)&\underset{\rightarrow}{\subset}\mathcal{M}_A(\hat{E}, G)\times\mathcal{M}_A(\hat{F}, G)\\ &=\mathcal{M}_A(E, G)\times\mathcal{M}_A(F, G)\,.\end{aligned} \tag{5.16}$$

In case G is commutative the "inclusion sign" in (5.16) *may be replaced by an equality.*

Proof. $E\otimes_A^\pi F$ is an essential algebra since E, F are essential, thus

$$\widehat{\mathfrak{T}_A(E, F)_0/I_A}=(5.11)\,A\hat{\otimes}_A^\pi E\hat{\otimes}_A^\pi F=(5.15)\,\widehat{[A(E\otimes_A^\pi F)]}=E\hat{\otimes}_A^\pi F, \tag{5.17}$$

within isomorphisms of topological A-algebras. On the other hand, $\mathcal{M}_A(E\otimes_A^\pi F, G)$ is locally equicontinuous (Lemma 3.1), so that $\mathcal{M}_A(\mathfrak{T}_A(E, F)_0/I_A, G)=\mathcal{M}_A(\widehat{\mathfrak{T}_A(E, F)_0/I_A}, G)=(5.17)\,\mathcal{M}_A(E\hat{\otimes}_A^\pi F, G)$. The assertion now follows by Theorem 3.1. ■

Under the conditions of Proposition 5.3, where now the algebras E, F, G are unital and moreover, $\mathcal{M}_A^0(E, G)$, $\mathcal{M}_A^0(F, G)$ are locally equicontinuous, concerning the continuous central A-morphisms defined on (5.11), one gets

$$\begin{aligned}(5.18)\qquad \mathcal{M}_A^0(\widehat{\mathcal{T}_A(E, F)_0/I_A}, G)&=\mathcal{M}_A^0(E\hat{\otimes}_A^{\pi}F, G)\underset{\rightarrow}{\subset}\mathcal{M}_A^0(\hat{E}, G)\times\mathcal{M}_A^0(\hat{F}, G)\\ &\underset{\rightarrow}{\subset}\mathcal{M}_A^0(E, G)\times\mathcal{M}_A^0(F, G)=\mathcal{M}_A^0(E\otimes_A^{\pi}F, G)\end{aligned}$$

(cf. (4.14), (5.18)). Besides, analogous thoughts to those after Proposition 4.2 can be formulated for the present case (cf. (4.5), (5.18)).

Remark 5.2. Given a finite family of locally convex A-algebras (resp. A-modules), we define the *(projective) finite A-tensor product locally convex A-algebra* (resp. *A-module*) $\bigotimes_{\alpha\in J}{}_A^{\pi}E_{\alpha}$ in a similar way to Definition 5.1. (The uniqueness and existence of the last tensor product follow also as before). In this case we still get results analogous to those of the preceding Sections. Moreover, one defines the *infinite topological A-tensor product of an arbitrary family of locally convex A-algebras* $(E_{\alpha})_{\alpha\in K}$, with results analogous to those of the finite case. A more detailed analysis thereon will be given elsewhere (cf. [9], [9a]).

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