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AFFINE AND PROJECTIVE VECTOR FIELDS ON COMPLETE NON-COMPACT RIEMANNIAN MANIFOLDS

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1. Every affine vector field on a compact orientable Riemannian manifold is a Killing vector field, and every projective vector field on a compact orientable Riemannian manifold having non-positive Ricci curvature is a parallel (Killing) vector field (cf. [3], [4]).

The purpose of this note is that of extending the above results to the non-compact case. We know a related result, that is, if the length of an affine vector field on a complete Riemannian manifold is bounded, then its affine vector field is a Killing vector field [2].

Our results are as follows:

Theorem A. *Let M be a complete non-compact Riemannian manifold. Every affine vector field on M with finite global norm is a Killing vector field.*

Theorem B. *Let M be a complete non-compact Reemannian manifold having non-positive Ricci cuuvature. Every projective vector field on M with finite global norm is a parallel (Killing) vector field.*

A related result for affine vector field appears in [1]. But G. Gigante's view in [1] differs from our point of view.

The discussions of conformal and Killing vector fields with finite global norms appear in [6, 8].

We shall be in C^∞-category and deal only with connected and orientable manifolds. We use the Einstein summation convention.

2. Let M be a complete non-compact Riemannian manifold (without boundary) of dimension m. Let $(x^1, \cdots, x^m)$ be a local coordinate system in M. We denote the Riemannian metric (resp. the Levi-Civita connection) on M by g (resp. ∇). We set $\nabla_i = \nabla_{\partial/\partial x^i}$ and $\nabla^i = g^{ij}\nabla_j$.

For two tensor fields T and S on M (of same type), we define the global scalar product $\langle\langle T, S\rangle\rangle$ by

$$\langle\langle T, S\rangle\rangle=\int_M \langle T, S\rangle\, dvol$$

(cf. [4], [8]). We set $\|T\|^2=\langle\langle T, T\rangle\rangle$ and we remark that $\|T\|^2\leqq+\infty$.

We denote the space of all s-forms on M by $\wedge^s(M)$, and let $\wedge_0^s(M)$ denote the subspace of $\wedge^s(M)$ composed of forms with compact supports. Let $L_2^s(M)$ be the completion of $\wedge_0^s(M)$ with respect to the global scalar product $\langle\langle\,,\rangle\rangle$. The operator d denotes the exterior derivative and δ is defined by

$$\delta=(-1)^{sm+m+1}*d*$$

acting on $\wedge^s(M)$. Then we have

$$\langle\langle d\xi, \eta\rangle\rangle=\langle\langle\xi, \delta\eta\rangle\rangle$$

for any $\xi\in\wedge_0^s(M)$ and $\eta\in\wedge_0^{s+1}(M)$. The Laplacian operator Δ is defined by

(1) $$\Delta=d\delta+\delta d\,.$$

For a 1-form ξ, we have

(2) $$(d\xi)_{ij}=\nabla_i\xi_j-\nabla_j\xi_i$$

(3) $$(\delta\xi)=-\nabla^i\xi_i$$

(4) $$(\Delta\xi)_i=-\nabla^j\nabla_j\xi_i+R^j{}_i\xi_j$$

where

$$R(\partial/\partial x^i, \partial/\partial x^j)\partial/\partial x^h=\nabla_i\nabla_j\partial/\partial x^h-\nabla_j\nabla_i\partial/\partial x^h$$
$$=R^k{}_{hij}\partial/\partial x^k\,,$$
$$R_{hj}=R^k{}_{hkj}\,,\qquad R^j{}_i=g^{jk}R_{ki}$$

and R_{ki} denote the components of the Ricci tensor of ∇.

Through this note, we identify vector fields and its dual 1-forms with respect to g and they are represented by the same letters. For a vector field $\xi=\xi^i\partial/\partial x^i$, we have its dual 1-form $\xi=\xi_j dx^j=g_{ij}\xi^i dx^j$.

Definition 1. A vector field ξ on M is called a vector field with *finite global norm* if its dual 1-form with respect to g belongs in $L_2^1(M)\cap\wedge^1(M)$, that is, $\xi\in L_2^1(M)\cap\wedge^1(M)$.

Definition 2. A vector field ξ on M is called an *affine vector field* if ξ satisfies

(5) $$\nabla_j\nabla_k\xi^i+R^i{}_{khj}\xi^h=0\,.$$

We remark that (5) implies

(6) $$\nabla^k\nabla_k\xi^i+R^i{}_k\xi^k=0$$

(7) $$d\delta\xi=0\,.$$

Difinition 3. A vector field ξ on M is called a *projective vector field* if there exists a tensor field ψ on M such that

(8) $$\nabla_j\nabla_k\xi^i+R^i{}_{khj}\xi^h=\psi_j\delta^i_k+\psi_k\delta^i_j .$$

We remark that (8) implies

(9) $$\nabla^k\nabla_k\xi_i+R^k{}_i\xi_k=2\psi_i$$

(10) $$\psi=-\frac{1}{m+1}d\delta\xi .$$

Definition 4. A vector field ξ on M is called a *Killing vector field* if ξ satisfies

(11) $$\nabla_j\xi_i+\nabla_i\xi_j=0 .$$

We remark that (11) implies

(12) $$\nabla_i\xi^i=0 .$$

3. Let o be a point of M and fix it. For each point $p\in M$, we denote by $\rho(p)$ the geodesic distance from o to p. We set

$$B(r)=\{p\in M\,|\,\rho(p)<r\}$$

for any $r>0$. Then there exists a Lipschitz continuous function w_r on M satisfying the following properties:

$$0\leqq w_r(p)\leqq 1 \quad \text{for any} \quad p\in M$$

$$\operatorname{supp} w_r\subset B(2r)$$

$$w_r(p)=1 \quad \text{for any} \quad p\in B(r)$$

$$\lim_{r\to\infty} w_r=1$$

$$|dw_r|\leqq\frac{C}{r} \quad \text{almost everywhere on } M$$

where $C(>0)$ is a constant independent of r (cf. [5], [8]). Then we have

Lemma 1 (cf. [6]). *For any $\xi\in\wedge^s(M)$, there exists a positive constant A independent of r such that*

$$\|dw_r\wedge\xi\|^2_{B(2r)}\leqq\frac{A}{r^2}\|\xi\|^2_{B(2r)}$$

$$\|dw_r\wedge{*\xi}\|^2_{B(2r)}\leqq\frac{A}{r^2}\|\xi\|^2_{B(2r)}$$

where

$$\|\xi\|^2_{B(2r)} = \langle\!\langle \xi, \xi \rangle\!\rangle_{B(2r)} = \int_{B(2r)} \langle \xi, \xi \rangle dvol .$$

Now, we remark that, for $\xi \in L^s_2(M) \cap \wedge^s(M)$, $w_r\xi$ has compact support and $w_r\xi \to \xi$ $(r \to +\infty)$ in the strong sense.

For any $\xi \in \wedge^1(M)$, we have

$$d(w_r^2\xi) = w_r^2 d\xi + 2w_r dw_r \wedge \xi \qquad \text{(a.e. on } M) \tag{13}$$

$$\delta(w_r^2)\xi = w_r^2\delta\xi - *(2w_r dw_r \wedge *\xi) \quad \text{(a.e. on } M) . \tag{14}$$

4. In previous notes [6, 8], we have

Theorem C (cf. [6, 8]). *Let M be a complete non-compact Riemannian manifold having non-positive Ricci curvature. Every Killing vector field on M with finite global norm is a parallel vector field.*

Theorem D (cf. [8]). *Let M be a complete non-compact Riemannian manifold and ξ a vector field on M with finite global norm. ξ is a Killing vector field if and only if ξ satisfies*

$$\nabla^k \nabla_k \xi^i + R^i{}_k \xi^k = 0 \quad \textit{and} \quad \nabla_i \xi^i = 0 .$$

Now, we prove Theorem A and B.

Proof of Theorem A. Let ξ be an affine vector field on M with finite global norm. By (7), (14), Lemma 1 and the Schwarz inequality, we have

$$\begin{aligned} 0 &= \langle\!\langle d\delta\xi, w_r^2\xi \rangle\!\rangle_{B(2r)} \\ &= \langle\!\langle \delta\xi, \delta(w_r^2\xi) \rangle\!\rangle_{B(2r)} \\ &= \langle\!\langle \delta\xi, w_r^2\delta\xi \rangle\!\rangle_{B(2r)} - 2\langle\!\langle w_r\delta\xi, *(dw_r \wedge *\xi) \rangle\!\rangle_{B(2r)} \\ &\geqq \|w_r\delta\xi\|^2_{B(2r)} - 2\|w_r\delta\xi\|_{B(2r)} \|dw_r \wedge *\xi\|_{B(2r)} \\ &\geqq \|w_r\delta\xi\|^2_{B(2r)} - \left\{ \frac{1}{4} \|w_r\delta\xi\|^2_{B(2r)} + 4\|dw_r \wedge *\xi\|^2_{B(2r)} \right\} \\ &\geqq \frac{3}{4} \|w_r\delta\xi\|^2_{B(2r)} - \frac{4A}{r^2} \|\xi\|^2_{B(2r)} . \end{aligned}$$

We have, letting $r \to +\infty$,

$$\|\delta\xi\|^2 = 0 ,$$

that is, $\delta\xi = 0$. Thus ξ satisfies (6) and $\nabla_i\xi^i = 0$. Therefore, by Theorem D, ξ is a Killing vector field.

Proof of Theorem B. Let ξ be a projective vector field on M with finite global norm. By (4), (9) and (10), we have

$$(15) \qquad (\Delta\xi)_i = 2R^k{}_i\xi_k + \frac{2}{m+1}(d\delta\xi)_i\,.$$

Let $\mathfrak{R}$ denote the Ricci transformation on $\wedge^1(M)$ defined by $(\mathfrak{R}\xi)_i = R^k{}_i\xi_k$. Then we have, by (13), (14), (15), Lemma 1 and the Schwarz inequality,

$$\begin{aligned}
&2\langle\!\langle \mathfrak{R}\xi,\ w_r{}^2\xi\rangle\!\rangle_{B(2r)}\\
&=\langle\!\langle \Delta\xi,\ w_r{}^2\xi\rangle\!\rangle_{B(2r)} - \frac{2}{m+1}\langle\!\langle d\delta\xi,\ w_r{}^2\xi\rangle\!\rangle_{B(2r)}\\
&=\frac{m-1}{m+1}\langle\!\langle \delta\xi,\ \delta(w_r{}^2\xi)\rangle\!\rangle_{B(2r)} + \langle\!\langle d\xi,\ d(w_r{}^2\xi)\rangle\!\rangle_{B(2r)}\\
&=\frac{m-1}{m+1}\langle\!\langle w_r\delta\xi,\ w_r\delta\xi\rangle\!\rangle_{B(2r)} + \langle\!\langle w_r d\xi,\ w_r d\xi\rangle\!\rangle_{B(2r)}\\
&\quad -\frac{2(m-1)}{m+1}\langle\!\langle w_r\delta\xi,\ *(dw_r\wedge *\xi)\rangle\!\rangle_{B(2r)} + \langle\!\langle w_r d\xi,\ dw_r\wedge\xi\rangle\!\rangle_{B(2r)}\\
&\geqq \frac{m-1}{m+1}\|w_r\delta\xi\|^2_{B(2r)} + \|w_r d\xi\|^2_{B(2r)}\\
&\quad -\frac{2(m-1)}{m+1}\|w_r\delta\xi\|_{B(2r)}\|dw_r\wedge *\xi\|_{B(2r)} - \|w_r d\xi\|_{B(2r)}\|dw_r\wedge\xi\|_{B(2r)}\\
&\geqq \frac{m-1}{m+1}\|w_r\delta\xi\|^2_{B(2r)} + \|w_r d\xi\|^2_{B(2r)}\\
&\quad -\frac{m-1}{m+1}\left\{\frac{1}{4}\|w_r\delta\xi\|^2_{B(2r)} + 4\|dw_r\wedge *\xi\|^2_{B(2r)}\right\}\\
&\quad -\frac{1}{2}\left\{\frac{1}{4}\|w_r d\xi\|^2_{B(2r)} + 4\|dw_r\wedge\xi\|^2_{B(2r)}\right\}\\
&=\frac{3(m-1)}{4(m+1)}\|w_r\delta w\|^2_{B(2r)} + \frac{7}{8}\|w_r d\xi\|^2_{B(2r)}\\
&\quad -\left(\frac{4(m-1)}{m+1}\frac{A}{r^2} + \frac{2A}{r^2}\right)\|\xi\|^2_{B(2r)}\,.
\end{aligned}$$

Thus, if $\limsup_{r\to+\infty}\langle\!\langle w_r\mathfrak{R}\xi,\ w_r\xi\rangle\!\rangle_{B(2r)} < +\infty$ then we have

$$(16) \qquad \begin{aligned}
&\limsup_{r\to+\infty}\langle\!\langle w_r\mathfrak{R}\xi,\ w_r\xi\rangle\!\rangle_{B(2r)}\\
&\qquad \geqq \frac{3(m-1)}{8(m+1)}\|\delta\xi\|^2 + \frac{7}{16}\|d\xi\|^2\,.
\end{aligned}$$

By the non-positivity of Ricci curvature, we have

$$\limsup_{r\to+\infty}\langle\!\langle w_r\mathfrak{R}\xi,\ w_r\xi\rangle\!\rangle_{B(2r)} \leqq 0\,.$$

Thus we have, from (16),

$$\frac{3(m-1)}{8(m+1)}\|\delta\xi\|^2+\frac{7}{16}\|d\xi\|^2\leqq 0,$$

and so we have

$$\delta\xi=0 \quad \text{and} \quad \phi=0.$$

Therefore, by Theorem C and D, ξ is a parallel (Killing) vector field.

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