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HOLOMORPHIC LOCAL FRAME FIELDS OF A CERTAIN NORMAL BUNDLE

Dedicated to Professor Shigeo Sàsaki on his 70th birthday

By

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0. Introduction.

In [2] Ogiue studied Kaehler submanifolds of complex space forms. Let $P_N(C)$ be an N-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 2 and M a Kaehler submanifold of complex dimension m in $P_N(C)$. Let $\tau(P_N(C))$ (resp. $\tau(M)$) be a holomorphic tangent bundle of $P_N(C)$ (resp. M) and $\tau(P_N(C))|_M$ a restriction of $\tau(P_N(C))$ to M. Then $\tau(M)$ is a holomorphic subbundle of it. Let $\nu(M)$ be an orthogonal complement of $\tau(M)$ in $\tau(P_N(C))|_M$. The main purpose of this note is to show that $\nu(M)$ is a holomorphic hermitian vector bundle with local trivialities (holomorphic local frame fields) canonically induced by "distinguished local charts" of M, by which locally M is regarded as a graph of a certain holomorphic mapping (Theorem 2.1). However $\nu(M)$ is not a holomorphic subbundle of $\tau(P_N(C))|_M$, but holomorphically isomorphic to a quotient bundle $\tau(P_N(C))|_M/\tau(M)$. Next by means of the above holomorphic local frame fields we obtain the concretely computable formulas of the connection and curvature matrices of $\nu(M)$, second fundamental form and scalar normal curvature. Last we apply the results to the computations of the normal curvature tensor, its length and scalar normal curvature of the Segre embedding.

1. Preliminaries.

Let C^n be the complex vector space of all n-dimensional complex column vectors and $M(n, m; C)$ the complex vector space of all complex matrices of type (n, m). Then $C^n = M(n, 1; C)$. For $A \in M(n, m; C)$, A^* is the transposed-conjugate matrix of A. Let $H(n, C)$ be the real vector space of all hermitian matrices of degree n and $U(n)$ the n-dimensional unitary group. For $A=(a_{ij}) \in M(m, p; C)$ and $B=(b_{\mu\nu}) \in M(n, q; C)$, $A \otimes B$ means the matrix

$$\begin{pmatrix} Ab_{11} & \cdots & \cdots & Ab_{1q} \\ \vdots & & \vdots & \vdots \\ \vdots & \cdots & Ab_{\mu\nu} & \cdots \\ Ab_{n1} & \cdots & \cdots & Ab_{nq} \end{pmatrix} \in M(mn,\ pq\,;\ C),$$

where $a_{ij}b_{\mu\nu}$ is the $(m(\mu-1)+i,\ p(\nu-1)+j)$-component of $A\otimes B$. Hereafter we use the following convention. If $X=(\xi_{ij})$ and $Y=(\eta_{jk})$ are a matrix of (p, q)-forms and a matrix of (p', q')-forms respectively, then $XY=(\zeta_{ik})$ means the matrix of $(p+p', q+q')$-forms obtained by multiplying X and Y in the usual way, where $\zeta_{ik}=\sum \xi_{ij}\wedge\eta_{jk}$. dX (resp. ∂X, $\bar\partial X$) means the matrix $(d\xi_{ij})$ (resp. $(\partial\xi_{ij})$, $(\bar\partial\xi_{ij})$).

Let M be a complex manifold of complex dimension m and ξ a holomorphic hermitian vector bundle over M with fibre C^k. Suppose that $(U\,;z^1, \cdots, z^m)$ is a local chart of M and that $(e)=(e_1, \cdots, e_k)$ is a local holomorphic frame field on U for ξ, that is, $(e)=(e_1, \cdots, e_k)$ induces a local triviality of ξ on U. Then the hermitian metric of ξ gives a $H(k\,;C)$-valued function h on U by the rule $h_{\alpha\beta}=(e_\alpha, e_\beta)$, i.e., h is a Gramian matrix for frame field $(e)=(e_1, \cdots, e_k)$. It is well known that a hermitian metric on a holomorphic vector bundle induces uniquely the metric connection compatible with the complex structure which is called a canonical connection of a holomorphic hermitian vector bundle. Let θ be the connection matrix associated with the canonical connection of ξ and the frame field $(e)=(e_1, \cdots, e_k)$ and Θ the curvature matrix associated with the connection matrix θ. By [1; (2.3) and (2.4)] we have

$$\theta=h^{-1}\partial h\,, \tag{1.1}$$

$$\Theta=\bar\partial\theta\,. \tag{1.2}$$

Let $G_{N,k}(C)$ be a complex Grassmann manifold of all linear subspaces of complex dimension $k+1$ in C^{N+1} $(0\leqq k<N)$. For each matrix $X\in M(N+1, k+1\,;C)$ of rank $k+1$, we denote by $[X]$ a point of $G_{N,k}(C)$, i.e., a linear subspace of complex dimension $k+1$ in C^{N+1} spanned by all column vectors of X. For each point $[X]\in G_{N,k}(C)$, there exists a unitary matrix $A\in U(N+1)$ such that

$$[X]=\left[A\begin{pmatrix} E_{k+1} \\ 0 \end{pmatrix}\right]=A\begin{bmatrix} E_{k+1} \\ 0 \end{bmatrix},$$

where E_{k+1} is the $(k+1)$-dimensional unit matrix and $0\in M(N-k,\ k+1\,;C)$. Then the mapping $\psi_A: M(N-k,\ k+1\,;C)\to G_{N,k}(C)$ defined by

$$\psi_A(Z)=\left[A\begin{pmatrix} E_{k+1} \\ Z \end{pmatrix}\right]=A\begin{bmatrix} E_{k+1} \\ Z \end{bmatrix}$$

is a holomorphic local parametrization near $[X]$. Put $U_A=\psi_A(M(N-k,\ k+1\,;C))$ and $\phi_A=\psi_A^{-1}: U_A\to M(N-k,\ k+1\,;C)$. Then $\{(U_A, \phi_A)\,;A\in U(N+1)\}$ is the system of holomorphic local charts of $G_{N,k}(C)$. The coordinate transformation

$\phi_B\circ\phi_A^{-1}:\phi_A(U_A\cap U_B)\to\phi_B(U_A\cap U_B)$ for $U_A\cap U_B\neq\phi$ is given by

$$\phi_B\circ\phi_A^{-1}(Z)=(b+dZ)(a+cZ)^{-1}=-(d^*-Zb^*)^{-1}(c^*-Za^*),$$

where

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}=B^{-1}A\in U(N+1),\quad a\in M(k+1,\ k+1;\ \boldsymbol{C}).$$

Put

$$\gamma_{N,k}(\boldsymbol{C})=\{(W,\ u)\in G_{N,k}(\boldsymbol{C})\times\boldsymbol{C}^{N+1};\ u\in W\},$$

$$\gamma_{N,k}^{\perp}(\boldsymbol{C})=\{(W,\ v)\in G_{N,k}(\boldsymbol{C})\times\boldsymbol{C}^{N+1};\ v\perp W\},$$

where $v\perp W$ means that v is orthogonal to W with respect to the usual hermitian innner product of $\boldsymbol{C}^{N+1}$. Define the mapping $\pi:\gamma_{N,k}(\boldsymbol{C})\to G_{N,k}(\boldsymbol{C})$, $\pi_{\perp}:\gamma_{N,k}^{\perp}(\boldsymbol{C})\to G_{N,k}(\boldsymbol{C})$ by $\pi(W,\ u)=W$, $\pi_{\perp}(W,\ v)=W$ respectively. Then $\gamma_{N,k}(\boldsymbol{C})$ and $\gamma_{N,k}^{\perp}(\boldsymbol{C})$ are holomorphic hermitian vector bundles over $G_{N,k}(\boldsymbol{C})$ with the following local trivialities: $F_A:U_A\times\boldsymbol{C}^{k+1}\to\pi^{-1}(U_A)$, $F_A^{\perp}:U_A\times\boldsymbol{C}^{N-k}\to\pi_{\perp}^{-1}(U_A)$,

$$F_A(\phi_A(Z),\ x)=\left(\phi_A(Z),\ A\begin{pmatrix} E_{k+1} \\ Z \end{pmatrix}x\right),$$

$$F_A^{\perp}(\phi_A(Z),\ y)=\left(\phi_A(Z),\ A\begin{pmatrix} -Z^* \\ E_{N-k} \end{pmatrix}M_Z^{-1}y\right),$$

where $M_Z=E_{N-k}+ZZ^*$. $\gamma_{N,k}(\boldsymbol{C})$ and $\gamma_{N,k}^{\perp}(\boldsymbol{C})$ are called the universal subbundle and the universal quotient bundle respectively. The transition functions $g_{BA}:U_A\cap U_B\to GL(k+1;\ \boldsymbol{C})$, $g_{BA}^{\perp}:U_A\cap U_B\to GL(N-k;\ \boldsymbol{C})$ of $\gamma_{N,k}(\boldsymbol{C})$, $\gamma_{N,k}^{\perp}(\boldsymbol{C})$ are given by

$$g_{BA}(\phi_A(Z))=a+cZ, \tag{1.3}$$

$$g_{BA}^{\perp}(\phi_A(Z))=(d^*-Zb^*)^{-1} \tag{1.4}$$

respectively, where

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}=B^{-1}A\in U(N+1),\quad a\in M(k+1,\ k+1;\ \boldsymbol{C}).$$

Now we regard $G_{N,0}(\boldsymbol{C})$ as the N-dimensional complex projective space $P_N(\boldsymbol{C})$ with the Fubini-Study metric of constant holomorphic sectional curvature 2. Let p be an arbitrarily fixed point of $P_N(\boldsymbol{C})$. For $A\in U(N+1)$, the holomorphic local parametrization $\phi_A:\boldsymbol{C}^N\to P_N(\boldsymbol{C})$ near p is given by

$$\phi_A(\zeta)=\left[A\begin{pmatrix} 1 \\ \zeta \end{pmatrix}\right]=A\begin{bmatrix} 1 \\ \zeta \end{bmatrix}\qquad(\phi_A(0)=p).$$

Then the Gramian matrix of the natural frame $(\partial_A)=(\partial/\partial\zeta^1,\ \partial/\partial\zeta^2,\ \cdots,\ \partial/\partial\zeta^N)$ is

$$((1+\zeta^*\zeta)(E_N+\zeta\zeta^*))^{-1}, \tag{1.5}$$

where E_N is the N-dimensional unit matrix, i.e., the Fubini-Study metric of $P_N(\boldsymbol{C})$

of constant holomorphic sectional curvature 2 is

$$ds^2=\frac{2}{1+\zeta^*\zeta}\sum\left(\delta_{KL}-\frac{\zeta^K\bar{\zeta}^L}{1+\zeta^*\zeta}\right)d\bar{\zeta}^K d\zeta^L,$$

where $1\leqq K,\ L\leqq N$.

Let M be a Kaehler submanifold of complex dimension m in $P_N(\boldsymbol{C})$ (i.e., complex submanifold of $P_N(\boldsymbol{C})$ with the induced Kaehler structure) and U be a non-empty open set in M. Hereafter unless otherwise stated, we put $n=N-m$ and use the following convention on the range of indices:

$$1\leqq a,\ b,\ c,\ d,\ i,\ j\leqq m;$$

$$m+1\leqq \alpha,\ \beta,\ \mu,\ \nu,\ \sigma,\ \tau\leqq m+n\,.$$

Suppose that $(c_1, \cdots, c_m)$ (resp. $(d_{m+1}, \cdots, d_{m+n})$) is a unitary tangent (resp. normal) frame field on U and that $(\xi^1, \cdots, \xi^m)$ (resp. $(\eta^{m+1}, \cdots, \eta^{m+n})$) is the dual frame field of $(c_1, \cdots, c_m)$ (resp. $(d_{m+1}, \cdots, d_{m+n})$) on U. Let σ be the second fundamental form of M in $P_N(\boldsymbol{C})$ and put

$$\sigma' :=\sum k^{\alpha}_{ab}\xi^a\otimes\xi^b\otimes d_{\alpha} \tag{1.6}$$

where σ' is the holomorphic part of σ and k^{α}_{ab} is a complex valued C^{∞}-function on U. Denote by K the normal curvature tensor of M in $P_N(\boldsymbol{C})$, then

$$K=\sum (d_{\alpha}\otimes\eta^{\beta})L^{\alpha}_{\beta a\bar{b}}(\xi^a\wedge\bar{\xi}^b)\,, \tag{1.7}$$

where $L^{\alpha}_{\beta a\bar{b}}$ is a complex valued C^{∞}-function on U. Let $\|\sigma\|$ be the length of σ and K_N the scalar normal curvature of the embedding of M in $P_N(\boldsymbol{C})$. Then by [2; p. 80, p. 85 and p. 94], we have

$$\|\sigma\|^2=4\sum \bar{k}^{\alpha}_{ab}k^{\alpha}_{ab}\,, \tag{1.8}$$

$$K_N=16\sum \bar{k}^{\alpha}_{ia}k^{\alpha}_{ib}k^{\beta}_{ja}\bar{k}^{\beta}_{jb}\,. \tag{1.9}$$

We define the square of the length of the normal curvature tensor K by

$$\|K\|^2 :=\sum \bar{L}^{\alpha}_{\beta ab}L^{\alpha}_{\beta ab}\,. \tag{1.10}$$

We now study the tangent projective space and Gauss mapping of a Kaehler submanifold of the complex projective space. Let M be a Kaehler submanifold of complex dimension m in $P_N(\boldsymbol{C})$ and p an arbitrarily fixed point of M. Then there exist a unitary matrix $A\in U(N+1)$ and a holomorphic mapping $f: D\to \boldsymbol{C}^n$ such that the mapping $\psi_{Af}: D\to M$ defined by

$$\psi_{Af}(z) :=\psi_A\begin{pmatrix} z \\ w \end{pmatrix}=A\begin{bmatrix} 1 \\ z \\ w \end{bmatrix} \qquad (w=f(z),\ f(0)=0)$$

is a holomorphic local parametrization of M near p, where D is a certain open neighborhood of 0 in C^m and $\psi_{Af}(0)=p$. Put $U_{Af}:=\psi_{Af}(D)$ and $\phi_{Af}:=\psi_{Af}^{-1}$. Then (U_{Af}, ϕ_{Af}) is a holomorphic local chart of M near p. We call (U_{Af}, ϕ_{Af}) a distinguished local chart in this note. Let z^i be the i-th component of ϕ_{Af} and $w^\alpha=f^{\alpha-m}$ the $(\alpha-m)$-th component of $f\circ\phi_{Af}$. Then z^i (resp. w^α) is the i-th coordinate (resp. holomorphic) function on U_{Af}. Put

$$J:=J(f)=\left(\frac{\partial w}{\partial z}\right) \quad \text{(Jacobian matrix of } f\text{)}.$$

We may regard J (resp. z, w) as a $M(n, m; C)$-valued (resp. $M(m, 1; C)$-valued, $M(n, 1; C)$-valued) holomorphic function on U_{Af}. The tangent projective space $P(M)_p$ of M at p is

$$P(M)_p=\left\{\left[A\begin{pmatrix}1 & 0\\ z & E_m\\ w & J\end{pmatrix}x\right]\in P_N(C)\,;\ x\in C^{m+1}-\{0\}\right\},$$

where $p=\psi_{Af}(z)$. Hence the Gauss mapping $\Gamma_M: M\to G_{N,m}(C)$ of the submanifold M in $P_N(C)$ is given by

$$\Gamma_M(\psi_{Af}(z))=A\begin{bmatrix}1 & 0\\ z & E_m\\ w & J\end{bmatrix}=\psi_A((w-Jz, J)),$$

where ψ_A is the local parametrization of the complex Grassmann manifold $G_{N,m}(C)$ near a point

$$\psi_A(0)=A\begin{bmatrix}E_{m+1}\\ 0\end{bmatrix}.$$

Therefore the local representation on U_{Af} of the Gauss mapping Γ_M is

$$\phi_A\circ\Gamma_M\circ\phi_{Af}^{-1}(z)=(w-Jz, J)\,, \tag{1.11}$$

where ϕ_A is the coordinate mapping of the local chart (U_A, ϕ_A) of $G_{N,m}(C)$ near $\psi_A(0)$.

2. Holomorphic local frame fields of the normal bundle.

Let M be a Kaehler submanifold of complex dimension m in $P_N(C)$. Let $\tau((P_N(C))$ (resp. $\tau(M)$) be a holomorphic tangent bundle of $P_N(C)$ (resp. M) and $\nu(M)$ a normal bundle of M in $P_N(C)$. We regard the underlying set of $\nu(M)$ as an orthogonal complement of $\tau(M)$ in $\tau(P_N(C))|_M$ (a restriction of $\tau(P_N(C))$ to M). However $\nu(M)$ is not a holomorphic subbundle of $\tau(P_N(C))|_M$, but holomorphically equivalent to the quotient $\tau(P_N(C))|_M/\tau(M)$. We use the same notations as in Section 1. We define a scalar-valued C^∞-function and some matrix-valued C^∞-functions on U_{Af} by

$$\lambda:=\lambda_f=1+z^*z+w^*w\,, \tag{2.1}$$

(2.2) $$\Lambda := \Lambda_f = E_{m+n} + \begin{pmatrix} z \\ w \end{pmatrix}(z^*, w^*),$$

(2.3) $$G := G_f = (E_m, J^*)(\lambda\Lambda)^{-1}\begin{pmatrix} E_m \\ J \end{pmatrix},$$

(2.4) $$H := H_f = \left\{(-J, E_n)(\lambda\Lambda)\begin{pmatrix} -J^* \\ E_n \end{pmatrix}\right\}^{-1},$$

(2.5) $$L := L_f = \lambda G,$$

(2.6) $$M := M_f = (\lambda H)^{-1}.$$

Put $(\partial_{Af}) := (\partial/\partial z^1, \cdots, \partial/\partial z^m) = (\partial_1, \cdots, \partial_m)$. Then (∂_{Af}) (resp. (∂_A)) is a holomorphic tangent frame field of M (resp. $P_N(\boldsymbol{C})$) on U_{Af} (resp. U_A) and gives a holomorphic local triviality of $\tau(M)$ (resp. $\tau(P_N(\boldsymbol{C}))$). We denote by $(\tilde{\partial}_A)$ the restriction of (∂_A) to U_{Af}. Then we have

$$(\partial_{Af}) = (\tilde{\partial}_A)\begin{pmatrix} E_m \\ J \end{pmatrix}.$$

Hence the Gramian matrix for the natural frame field (∂_{Af}) is equal to G. Put

(2.7) $$(e_{Af}) := (e_{m+1}, \cdots, e_{m+n}) = (\tilde{\partial}_A)(\lambda\Lambda)\begin{pmatrix} -J^* \\ E_n \end{pmatrix}H.$$

Then (e_{Af}) is a normal frame field and of class C^∞. In fact, since it follows from (1.5) that the Gramian matrix of (∂_{Af}) is equal to $(\lambda\Lambda)^{-1}$, the (n, m)-matrix

$$H(-J, E_n)(\lambda\Lambda)(\lambda\Lambda)^{-1}\begin{pmatrix} E_m \\ J \end{pmatrix}$$

whose (α, a)-component is the inner product of e_α and ∂_a is zero, i.e., each member of (e_{Af}) is perpendicular to any member of (∂_{Af}). The Gramian matrix for (e_{Af}) is equal to H, which is non-singular at any point of U_{Af}, i.e., (e_{Af}) is linearly independent at any point of U_{Af}. Then we may prove the following theorem.

Theorem 2.1. *The local triviality of $\nu(M)$ induced by*

$$(e_{Af}) = (\tilde{\partial}_A)(\lambda\Lambda)\begin{pmatrix} -J^* \\ E_n \end{pmatrix}H$$

is holomorphic, i.e., the normal frame field (e_{Af}) on U_{Af} is holomorphic and its Gramian matrix is equal to H.

Proof. We calculate the transition function between such two distinct local trivialities. Set

(i) $$\varphi_{Af}(z) = \varphi_A\begin{pmatrix} z \\ f(z) \end{pmatrix} = A\begin{bmatrix} 1 \\ z \\ f(z) \end{bmatrix} = B\begin{bmatrix} 1 \\ w \\ g(w) \end{bmatrix} = \varphi_B\begin{pmatrix} w \\ g(w) \end{pmatrix} = \varphi_{Bg}(w),$$

(ii) $(e_{Af})u=(\tilde{\partial}_A)\Lambda_f\begin{pmatrix}-J^*\\ E_n\end{pmatrix}M_f^{-1}u=(\tilde{\partial}_B)\Lambda_g\begin{pmatrix}-K^*\\ E_n\end{pmatrix}M_g^{-1}v=(e_{Bg})v$,

where $u,\ v\in \boldsymbol{C}^n$ and

$$J=\left(\frac{\partial f}{\partial z}\right),\quad K=\left(\frac{\partial g}{\partial w}\right).$$

Put

$$B^{-1}A=\begin{pmatrix} a & c^* & e^* \\ b & S & D^* \\ d & C & T\end{pmatrix}\in U(N+1),$$

where $a\in \boldsymbol{C}$; $b,\ c\in \boldsymbol{C}^m$; $d,\ e\in \boldsymbol{C}^n$; $C,\ D\in M(n,\ m;\ \boldsymbol{C})$; $S\in M(m,\ m;\ \boldsymbol{C})$ and $T\in M(n,\ n;\ \boldsymbol{C})$. Then from (i) we obtain

$$\begin{pmatrix} w\\ g\end{pmatrix}=-\left(\begin{pmatrix} S^* & C^*\\ D & T^*\end{pmatrix}-\begin{pmatrix} z\\ f\end{pmatrix}(b^*,\ d^*)\right)^{-1}\left(\begin{pmatrix} c\\ e\end{pmatrix}-\begin{pmatrix} z\\ f\end{pmatrix}\bar{a}\right)$$

$$=\left(\begin{pmatrix} b\\ d\end{pmatrix}+\begin{pmatrix} S & D^*\\ C & T\end{pmatrix}\begin{pmatrix} z\\ f\end{pmatrix}\right)(a+c^*z+e^*f)^{-1}=\phi_B\circ\phi_A^{-1}\begin{pmatrix} z\\ f\end{pmatrix},$$

$$J(\phi_B\circ\phi_A^{-1})=\left(\begin{pmatrix} S^* & C^*\\ D & T^*\end{pmatrix}-\begin{pmatrix} z\\ f\end{pmatrix}(b^*,\ d^*)\right)^{-1}(a+c^*z+e^*f)^{-1},$$

$$(g-Kw,\ K)=-\left(T^*-(f-Jz,\ J)\begin{pmatrix} d^*\\ C^*\end{pmatrix}\right)^{-1}$$

$$\times\left((e,\ D)-(f-Jz,\ J)\begin{pmatrix} \bar{a} & b^*\\ c & S^*\end{pmatrix}\right),$$

where $J(\phi_B\circ\phi_A^{-1})$ is the Jacobian matrix of the coordinate transformation $\phi_B\circ\phi_A^{-1}$ in $P_N(\boldsymbol{C})$, $f=f(z)$ and $g=g(z)$. Moreover we have

$$(\tilde{\partial}_B)=(\tilde{\partial}_A)Qr,\quad \Lambda_g=Q^{-1}\Lambda_f Q^{*-1},$$

$$M_g^{-1}=P^*M_f^{-1}P,\quad \begin{pmatrix}-K^*\\ E_n\end{pmatrix}=Q^*\begin{pmatrix}-J^*\\ E_n\end{pmatrix}P^{*-1},$$

where

$$P=\left(T^*-(f-Jz,\ J)\begin{pmatrix} d^*\\ C^*\end{pmatrix}\right),\quad Q=\left(\begin{pmatrix} S^* & C^*\\ D & T^*\end{pmatrix}-\begin{pmatrix} z\\ f\end{pmatrix}(b^*,\ d^*)\right),$$

$$r=a+c^*z+e^*f.$$

Consequently it follows from (ii) that $u=Pvr$. Hence such a transition function $g_{Bg,Af}: U_{Af}\cap U_{Bg}\rightarrow GL(n\ ;\ \boldsymbol{C})$ is given by

$$g_{Bg,Af}(\phi_{Af}(z))=\left(T^*-(f-Jz,\ J)\begin{pmatrix} d^*\\ C^*\end{pmatrix}\right)^{-1}(a+c^*z+e^*f)^{-1}. \tag{2.8}$$

Thus $g_{Bg,Af}$ is holomorphic on $U_{Af}\cap U_{Bg}$. This fact completes the proof.

q. e. d.

Let $\gamma_{N,0}(C)|_M$ be a restriction of the Hopf bundle $\gamma_{N,0}(C)$ to M and $(\gamma_{N,0}(C)|_M)^*$ a dual bundle of $\gamma_{N,0}(C)|_M$. $(\gamma_{N,0}(C)|_M)^*$ is called a hyperplane bundle of M. It follows from (1.3) that the transition function $h_{Bg,Af}: U_{Af}\cap U_{Bg} \to GL(1; C)$ of $\gamma_{N,0}(C)|_M$ is given by

$$h_{Bg,Af}(\psi_{Af}(z))=a+c^*z+e^*f.$$

Hence the transition function ${}^t h_{Bg,Af}^{-1}$ of $(\gamma_{N,0}(C)|_M)^*$ is given by

$${}^t h_{Bg,Af}^{-1}(\psi_{Af}(z))=(a+c^*z+e^*f)^{-1}.$$

Let $\Gamma_M^{-1}(\gamma_{N,m}^{\perp}(C))$ be a pull-back of $\gamma_{N,m}^{\perp}(C)$ by the Gauss mapping $\Gamma_M: M\to G_{N,m}(C)$. By (1.4) and (1.11) it is easily seen that its transition function $q_{Bg,Af}: U_{Af}\cap U_{Bg}\to GL(n; C)$ is given by

$$q_{Bg,Af}(\psi_{Af}(z))=\left(T^*-(f-Jz,\ J)\begin{pmatrix} d^* \\ C^* \end{pmatrix}\right)^{-1}.$$

Therefore by (2.8) we have the following corollary.

Corollary 2.2. *$\nu(M)$ is holomorphically isomorphic to a tensor product*

$$\Gamma_M^{-1}(\gamma_{N,m}^{\perp}(C))\otimes(\gamma_{N,0}(C)|_M)^*.$$

3. Hermitian differential geometry of the submanifold M.

We use the same notations as in Section 2. Let θ_f (resp. $\theta^{\perp}_f$) be the connection matrix associated with the canonical connection of $\tau(M)$ (resp. $\nu(M)$) and the holomorphic local frame field (∂_{Af}) (resp. (e_{Af})). Let Θ_f (resp. $\Theta^{\perp}_f$) be the curvature matrix associated with θ_f (resp. $\theta^{\perp}_f$). We prove the following proposition.

Proposition 3.1.

$$\theta_f=-\frac{1}{\lambda}dz(z^*+w^*J)-\frac{1}{\lambda}(z^*+w^*J)dzE_m+G^{-1}(E_m,\ J^*)(\lambda\Lambda)^{-1}\begin{pmatrix} 0 \\ E_n \end{pmatrix}dJ,$$

$$\Theta_f=(dzdz^*)G-dz^*GdzE_m+G^{-1}dJ^*HdJ,$$

$$\theta^{\perp}_f=-\frac{1}{\lambda}(z^*+w^*J)dzE_n+dJ(E_m,\ 0)(\lambda\Lambda)\begin{pmatrix} -J^* \\ E_n \end{pmatrix}H,$$

$$\Theta^{\perp}_f=-dz^*GdzE_n+dJG^{-1}dJ^*H.$$

Proof. It can be shown that

$$\Lambda=\begin{pmatrix} E_m \\ J \end{pmatrix}L^{-1}(E_m,\ J^*)+\Lambda\begin{pmatrix} -J^* \\ E_n \end{pmatrix}M^{-1}(-J,\ E_n)\Lambda, \qquad \partial\lambda=(z^*+w^*J)dz,$$

$$\partial(\lambda^{-1})=-\lambda^{-2}\partial\lambda,\quad \partial\Lambda=\begin{pmatrix} E_m \\ J \end{pmatrix}dz(z^*,\ w^*),\quad \partial(\Lambda^{-1})=-(\lambda\Lambda)^{-1}\partial\Lambda,$$

$$\partial L=-\frac{L}{\lambda}dz(z^*+w^*J)+(E_m,\ J^*)\Lambda^{-1}\binom{0}{E_n}dJ,\quad \partial(L^{-1})=-L^{-1}(\partial L)L^{-1},$$

$$\partial M=-dJ(E_m,\ 0)\Lambda\binom{-J^*}{E_n},\quad \partial(M^{-1})=-M^{-1}(\partial M)M^{-1},$$

$$\bar\partial\lambda=(\partial\lambda)^*,\quad \bar\partial(\lambda^{-1})=(\partial(\lambda^{-1}))^*,\quad \bar\partial\Lambda=(\partial\Lambda)^*,\quad \bar\partial(\Lambda^{-1})=(\partial(\Lambda^{-1}))^*,$$

$$\bar\partial L=(\partial L)^*,\quad \bar\partial(L^{-1})=(\partial(L^{-1}))^*,\quad \bar\partial M=(\partial M)^*,\quad \bar\partial(M^{-1})=(\partial(M^{-1}))^*.$$

By these formulas, the proposition follows from $\theta_f=G^{-1}\partial G$, $\Theta_f=\bar\partial\theta_f$, $\theta^\perp{}_f=H^{-1}\partial H$ and $\Theta^\perp{}_f=\bar\partial\theta^\perp{}_f$. q. e. d.

Denote by Θ_{ab} (resp. $\Theta_{\alpha\beta}$) the (a, b)-component (resp. (α, β)-component) of Θ_f (resp. $\Theta^\perp{}_f$) and set

$$\Theta_{ab}:=\sum R^a_{bc\bar d}dz^c\wedge d\bar z^d,\quad \Theta_{\alpha\beta}:=\sum K^\alpha_{\beta c\bar d}dz^c\wedge d\bar z^d,$$

$$R_{\bar a bc\bar d}:=\sum g_{\bar a i}R^i_{bc\bar d}\,,\qquad K_{\bar\alpha\beta c\bar d}:=\sum h_{\bar\alpha\mu}K^\mu_{\beta c\bar d}\,,$$

where $G=(g_{\bar a b})$ and $H=(h_{\bar\alpha\beta})$. Then we have the following corollary.

Corollary 3.2.

$$R_{\bar a bc\bar d}=g_{\bar a b}g_{c\bar d}+g_{\bar a c}g_{b\bar d}-\sum h_{\bar\alpha\beta}\partial^2_{\bar d\bar a}w^{\bar\alpha}\partial^2_{cb}w^\beta,$$

$$K^\alpha_{\beta c\bar d}=\delta^\alpha_\beta g_{c\bar d}+\sum g^{i\bar j}h_{\beta\bar\nu}\partial^2_{ic}w^\alpha\partial^2_{\bar j\bar d}\,w^{\bar\nu}$$

$$K_{\bar\alpha\beta c\bar d}=h_{\bar\alpha\beta}g_{c\bar d}+\sum g^{i\bar j}h_{\bar\alpha\mu}h_{\beta\bar\nu}\partial^2_{ic}w^\mu\partial^2_{\bar j\bar d}\,w^{\bar\nu},$$

where

$$\partial^2_{ci}w^\mu=\frac{\partial^2w^\mu}{\partial z^c\partial z^i},\quad \partial^2_{\bar d\bar j}\,w^{\bar\nu}=\overline{\partial^2_{dj}w^\nu},\quad G^{-1}=(g^{i\bar j}).$$

Next we calculate the second fundamental form and scalar normal curvature of the embedding of M in $P_N(\boldsymbol{C})$. Denote by $\hat\nabla'$ (resp. ∇') the (1, 0)-part of the covariant differential operator $\hat\nabla$ (resp. ∇) on $P_N(\boldsymbol{C})$ (resp. M) with respect to the Fubini-Study (resp. induced Kaehler) metric. Let ξ, η be two C^∞-vector fields on $P_N(\boldsymbol{C})$ of type (1, 0). Then on the coordinate neighborhood U_A, we may put

$$\xi=(\partial_A)x\,,\quad \eta=(\partial_A)y\,,$$

where x, y are two smooth $\boldsymbol{C}^N$-valued functions on U_A and we have

$$(*)\qquad \hat\nabla'_\xi\eta=(\partial_A)\Big((dy)x-\frac{1}{1+\|\zeta\|^2}(x\zeta^*y+y\zeta^*x)\Big),$$

where (dy) is the Jacobian matrix of y. Let X, Y be two C^∞-vector fields on M of type (1, 0). Then on the coordinate neighborhood U_{Af}, we can set

$$X=(\tilde\partial_A)\binom{E_m}{J}u\,,\quad Y=(\tilde\partial_A)\binom{E_m}{J}v\,,$$

where both of u and v are smooth C^m-valued functions on U_{Af} by Proposition 3.1, we have

$$\nabla'_X Y=(\tilde{\partial}_A)\begin{pmatrix} E_m \\ J \end{pmatrix}\Big((dv)u-\frac{1}{\lambda}(u(z^*+w^*J)v+v(z^*+w^*J)u)$$
$$+G^{-1}(E_m,\ J^*)(\lambda\Lambda)^{-1}\begin{pmatrix} 0 \\ E_n \end{pmatrix}(d^2w)(u,\ v)\Big),$$

where $u={}^t(u^1,\ \cdots,\ u^m)$, $v={}^t(v^1,\ \cdots,\ v^m)$,

$$(dv)u=\sum\frac{\partial v}{\partial z^a}u^a,\quad (d^2w)(u,\ v)=\sum\frac{\partial^2 w}{\partial z^a\partial z^b}u^a v^b.$$

We denote by $\tilde{\nabla}'$ the restriction of $\hat{\nabla}'$ to $\tau(P_N(C))|_M$. By (*), we have

$$\tilde{\nabla}'_X Y=(\tilde{\partial}_A)\Big(\begin{pmatrix} 0 \\ E_n \end{pmatrix}(d^2w)(u,\ v)+\begin{pmatrix} E_m \\ J \end{pmatrix}\Big((dv)u-\frac{1}{\lambda}(u(z^*+w^*J)v+v(z^*+w^*J)u)\Big)\Big).$$

If we denote by σ' the holomorphic part of the second fundamental form σ of M in $P_N(C)$, then it is easily shown that

$$\sigma'(X,\ Y)=\tilde{\nabla}'_X Y-\nabla'_X Y=(\tilde{\partial}_A)(\lambda\Lambda)\begin{pmatrix} -J^* \\ E_n \end{pmatrix}H(d^2w)(u,\ v).$$

Consequently we have the following proposition.

Proposition 3.3.

$$\sigma'=(e_{Af})\sum\frac{\partial^2 w}{\partial z^a\partial z^b}dz^a\otimes dz^b=\sum\partial^2_{ab}w^\alpha dz^a\otimes dz^b\otimes e_\alpha,$$

where

$$\partial^2_{ab}w^\alpha=\frac{\partial^2 w^\alpha}{\partial z^a\partial z^b}.$$

Put

$$c:=(c_1,\ \cdots,\ c_m)=(\partial_{Af})G^{-1/2},$$
$$d:=(d_{m+1},\ \cdots,\ d_{m+n})=(e_{Af})H^{-1/2},$$
$$\xi:={}^t(\xi^1,\ \cdots,\ \xi^m)=G^{1/2}dz,$$
$$\eta:={}^t(\eta^{m+1},\ \cdots,\ \eta^{m+n})=H^{1/2}(e_{Af})^*,$$

where $(e_{Af})^*={}^t(e^{m+1},\ \cdots,\ e^{m+n})$ is a dual frame field of (e_{Af}) on U_{Af}. Then c (resp. d) is a tangent (resp. normal) unitary frame field on U_{Af} and ξ (resp. η) is a dual frame field of c (resp. d) on U_{Af}. Put

$$G^{-1}=(g^{ij}),\quad G^{-1/2}=(\gamma^i_j),\quad H^{1/2}=(\rho^\alpha_\beta),\quad H^{-1/2}=(\sigma^\beta_\alpha).$$

Then it follows that

$$\bar{\gamma}^i_j=\gamma^j_i,\quad \bar{\rho}^\alpha_\beta=\rho^\beta_\alpha,\quad g^{ij}=\sum\gamma^i_a\gamma^a_j,\quad h_{\bar{\alpha}\beta}=\sum\rho^\alpha_\mu\rho^\mu_\beta.$$

By $\sigma'=\sum k^\alpha_{ab}\xi^a\otimes\xi^b\otimes d_\alpha=\sum\partial^2_{ab}w^\alpha dz^a\otimes dz^b\otimes e_\alpha$, we have

$$k^{\alpha}_{\bar{a}b}=\sum \gamma^{i}_{a}\gamma^{j}_{b}\rho^{\alpha}_{\beta}\partial^{2}_{ij}w^{\beta}.$$

Hence by (1.8) and (1.9), we have the following proposition.

Proposition 3.4.

$$\|\sigma\|^{2}=4\mathrm{tr}(G^{-1}Q), \quad K_{N}=16\mathrm{tr}(G^{-1}Q)^{2},$$

where $\bar{Q}=(q_{\bar{a}b})$, $q_{\bar{a}b}=\sum g^{\bar{i}j}h_{\bar{\alpha}\beta}\partial^{2}_{\bar{i}\bar{a}}\,w^{\alpha}\partial^{2}_{jb}w^{\beta}$.

Remark. The Kaehler form Φ of the Grassmann manifold $G_{N,m}(\boldsymbol{C})$ is given by

$$\Phi=\sqrt{-1}\,\mathrm{tr}(dZL_{Z}^{-1}dZ^{*}M_{Z}^{-1}),$$

where $L_{Z}=E_{m+1}+Z^{*}Z$, $M_{Z}=E_{n}+ZZ^{*}$. The pull-back of Φ by the Gauss mapping $\Gamma_{M}: M\to G_{N,m}(\boldsymbol{C})$ is equal to

$$-\sqrt{-1}\,dz^{*}Qdz=\sqrt{-1}\,\mathrm{tr}(dJL^{-1}dJ^{*}M^{-1}).$$

Moreover the curvature matrix $\Omega^{\perp}$ of the universal quotient bundle $\gamma^{\perp}_{N,m}(\boldsymbol{C})$ associated with its canonical connection and frame field (q_{A}) on U_{A} is given by

(3.1) $$\Omega^{\perp}=dZL_{Z}^{-1}dZ^{*}M_{Z}^{-1}=\bar{\partial}(M_{Z}\partial M_{Z}^{-1}),$$

where

$$(q_{A})=A\begin{pmatrix}-Z^{*}\\ E_{n}\end{pmatrix}M_{Z}^{-1}.$$

Now we compute the length of the normal curvature tensor K. By Corollary 3.2, we have

$$K=\sum(e_{\alpha}\otimes e^{\beta})K^{\alpha}_{\beta a\bar{b}}(dz^{a}\wedge d\bar{z}^{b}),$$

$$K^{\alpha}_{\beta a\bar{b}}=\delta^{\alpha}_{\beta}g_{a\bar{b}}+\sum g^{i\bar{j}}h_{\beta\bar{\mu}}\partial^{2}_{ai}w^{\alpha}\partial^{2}_{\bar{b}\bar{j}}w^{\bar{\mu}}.$$

Hence it follows from (1.7) that

$$L^{\alpha}_{\beta a\bar{b}}=\sum \gamma^{i}_{a}\gamma^{\bar{j}}_{\bar{b}}(g_{i\bar{j}}\delta^{\alpha}_{\beta}+\sum \rho^{\alpha}_{\sigma}\sigma^{\mu}_{\beta}g^{c\bar{d}}h_{\mu\bar{\tau}}\partial^{2}_{ci}w^{\sigma}\partial^{2}_{\bar{d}\bar{j}}w^{\bar{\tau}}),$$

where $\gamma^{\bar{j}}_{\bar{b}}=\bar{\gamma}^{b}_{j}$. By (1.10), we have the following proposition.

Proposition 3.5.

$$\|K\|^{2}=mn+2\mathrm{tr}(G^{-1}Q)+\mathrm{tr}(G^{-1}Q)^{2}.$$

4. Applications to the Segre embedding.

Let e be a mapping of $P_{m}(\boldsymbol{C})\times P_{n}(\boldsymbol{C})$ into $P_{m+n+mn}(\boldsymbol{C})$ defined by

$$e([\xi],[\eta]):=[\xi\otimes\eta].$$

Let $P_{m}(\boldsymbol{C})\times P_{n}(\boldsymbol{C})$ be equipped with a product metric of the Fubini-Study metrics. Then e is an isometric embedding which is called the Segre embedding. For an arbitrarily fixed point $[\xi_0\otimes\eta_0]\in e(P_m(\boldsymbol{C})\times P_n(\boldsymbol{C}))$, there exists a unitary matrix $A\in U((m+1)(n+1))$ such that the mapping $\psi_{Af}: \boldsymbol{C}^{m+n}\to P_{m+n+mn}(\boldsymbol{C})$ defined by

$$\psi_{Af}\begin{pmatrix} u \\ v \end{pmatrix}=A\begin{bmatrix} 1 \\ u \\ v \\ u\otimes v \end{bmatrix}, \quad \left(f\begin{pmatrix} u \\ v \end{pmatrix}=u\otimes v\right)$$

gives a holomorphic local parametrization of $e(P_m(\boldsymbol{C})\times P_n(\boldsymbol{C}))$ around

$$[\xi_0\otimes\eta_0]=A\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $u\in\boldsymbol{C}^m$ and $v\in\boldsymbol{C}^n$. The Jacobian matrix $J(f)$ of f is given by

$$J(f)_{\binom{u}{v}}=(E_m\otimes v,\ u\otimes E_n)\in M(mn,\ m+n\ ;\ \boldsymbol{C}),$$

where $u\in\boldsymbol{C}^m$ and $v\in\boldsymbol{C}^n$. Put

$$\sigma=1+u^*u\,, \quad S=E_m+uu^*,$$

$$\tau=1+v^*v\,, \quad T=E_n+vv^*.$$

Then the following theorem on the Segre embedding e is easily proved.

Theorem 4.1.

$$G=(\sigma S)^{-1}\oplus(\tau T)^{-1}=\begin{pmatrix} (\sigma S)^{-1} & 0 \\ 0 & (\tau T)^{-1} \end{pmatrix},$$

$$H=(\sigma S)^{-1}\otimes(\tau T)^{-1}.$$

Put

$$(\sigma S)^{-1}=(s_{\bar{a}b})\,, \quad (\tau T)^{-1}=(t_{\bar{\alpha}\beta})\,.$$

Then we have

$$s_{\bar{a}b}=\frac{1}{\sigma}\left(\delta_{ab}-\frac{u^a\bar{u}^b}{\sigma}\right), \quad t_{\bar{\alpha}\beta}=\frac{1}{\tau}\left(\delta_{\alpha\beta}-\frac{v^\alpha\bar{v}^\beta}{\tau}\right).$$

Denote by $w^{a\alpha}$ the $(a+m(\alpha-m-1))$-th component of $w\in\boldsymbol{C}^{mn}$. For the mapping $f: \boldsymbol{C}^{m+n}\to\boldsymbol{C}^{mn}$ defined by

$$f(z)=f\begin{pmatrix} u \\ v \end{pmatrix}=u\otimes v=w\,,$$

we have

$$\partial^2_{ab}w^{c\beta}=0=\partial^2_{\alpha\beta}w^{a\mu}, \tag{4.1}$$

$$\partial^2_{a\alpha}w^{b\beta}=\delta^a_b\delta^\alpha_\beta=\partial^2_{\alpha a}w^{b\beta}. \tag{4.2}$$

Then the following corollary on the normal curvature tensor of the Segre embedding e is easily shown by the above theorem and Corollary 3.2.

Corollary 4.2.

$$K_{\overline{a\alpha},b\beta,c,\bar{d}}=(s_{\bar{a}b}s_{c\bar{d}}+s_{\bar{a}c}s_{b\bar{d}})t_{\bar{\alpha}\beta}\,,$$

$$K_{\overline{a\alpha}, b\beta, \mu, \bar{i}} = 0 = K_{\overline{a\alpha}, b\beta, j, \bar{\nu}},$$

$$K_{\overline{a\alpha}, b\beta, \mu, \bar{\nu}} = s_{\bar{a}b}(t_{\bar{\alpha}\beta} t_{\mu\bar{\nu}} + t_{\bar{\alpha}\mu} t_{\beta\bar{\nu}}).$$

By the above theorem, (4.1) and (4.2), we have

$$Q = \begin{pmatrix} n(\sigma S)^{-1} & 0 \\ 0 & m(\tau T)^{-1} \end{pmatrix}.$$

Therefore the following corollary follows from Proposition 3.4 and Proposition 3.5.

Corollary 4.3.

$$\|\sigma\|^2 = 8mn, \quad K_N = 16mn(m+n),$$

$$\|K\|^2 = 2mn(m+n+2).$$

References

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