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CERTAIN GENERALIZATIONS OF THE ROBERTSON FUNCTIONS

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1. Introduction. Let

$$f(z)=z+a_2z^2+\cdots \tag{1.1}$$

denote a function analytic in the unit disk $E=\{z:|z|<1\}$ satisfying the condition

$$\mathrm{Re}\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0 \tag{1.2}$$

for some real $\lambda(|\lambda|<\pi/2)$ and for all z in E. The class C^{λ} of the functions f of the form (1.1) and satisfying the condition (1.2) has been studied by Robertson [6], Libera and Ziegler [4], Bajpai and Mehrok [1], and Kulshrestha [3]. If f is in C^{λ}, we say that f is a λ-Robertson function. In [2], Chichra has defined the class $C^{\lambda}(\alpha)$ of functions f of the form (1.1) and satisfying the condition

$$\mathrm{Re}\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha\cos\lambda \tag{1.3}$$

for some α, $\lambda(0\leqq\alpha<1,\ -\pi/2<\lambda<\pi/2)$ and for all z in E. We say, a function f of the form (1.1) is a λ-Robertson function of order α if and only if f is a member of $C^{\lambda}(\alpha)$. The class $C^{\lambda}(\alpha)$ has also been studied by Sizuk [8], who called $zf'(z)$ λ-spiral-shaped of order α.

In [5], Mogra and the author have introduced the class, $S^{\lambda}(\alpha,\beta)$, of λ-spiral-like functions of order α and type β. Accordingly, a function f of the form (1.1) belongs to $S^{\lambda}(\alpha,\beta)$ if and only if for all z in E, the inequality

$$\left|\frac{zf'(z)/f(z)-1}{2\beta(zf'(z)/f(z)-1+(1-\alpha)e^{-i\lambda}\cos\lambda)-(zf'(z)/f(z)-1)}\right|<1 \tag{1.4}$$

holds for some α, β, $\lambda(0\leqq\alpha<1,\ 0<\beta\leqq1,\ -\pi/2<\lambda<\pi/2)$, and for all z in E. Motivated from our class $S^{\lambda}(\alpha,\beta)$, we, in the present paper, introduce the concept of 'type' for the class, $C^{\lambda}(\alpha)$, of λ-Robertson functions of order α as follows:

A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, is said to be a λ-Robertson function of order α and type β if and only if for all z in the unit disk E, the inequality

$$(1.5) \qquad \left| \frac{zf''(z)/f'(z)}{2\beta(zf''(z)/f'(z)+(1-\alpha)e^{-i\lambda}\cos\lambda)-zf''(z)/f'(z)} \right| <1$$

holds for some α, β, λ as above. We shall denote the class of all such functions by $C^\lambda(\alpha, \beta)$.

It follows immediately from the definitions of $S^\lambda(\alpha, \beta)$ and $C^\lambda(\alpha, \beta)$ that a function f is in the class $C^\lambda(\alpha, \beta)$ if and only if $zf'(z)$ is in the class $S^\lambda(\alpha, \beta)$.

We observe that by taking the suitable values of the parameters α, β, λ our class $C^\lambda(\alpha, \beta)$ gives rise to a number of classes previously studied; for instance, $C^\lambda(0, 1)\equiv C^\lambda$ and $C^\lambda(\alpha, 1)\equiv C^\lambda(\alpha)$. Further, the classes $C^\lambda(0, (2M-1)/2M)\equiv C_{\lambda,M}$ $C^0(\alpha, 1)\equiv C(\alpha)$ have been introduced and investigated, respectively, by Kulshrestha [3] and Robertson [7]. The class $C(\alpha)$ is known as the class of functions which are convex of order α.

It is to be noted that for general values of α, β, $\lambda(0\leqq\alpha<1,\ 0<\beta\leqq1,\ -\pi/2<\lambda<\pi/2)$, a function in $C^\lambda(\alpha, \beta)$ may not be univalent in E. For example, the function

$$f(z)=i(1-z)^i-i=z+\cdots$$

belongs to $C^{\pi/4}(0, 1)$, but it has a zero at each of the points $1-e^{-2n\pi}(n=0, 1, 2, \cdots)$.

In the present paper, using the results proved in [5], we establish a representation formula, distortion properties, and coefficient estimates for a λ-Robertson function of order α and type β. Finally, the radius of convexity for $C^\lambda(\alpha, \beta)$ have been obtained. For the appropriate choices of the parameters, our theorems in this paper not only give rise to a number of results previously known, but can also yield many new interesting results for a number of classes earlier studied.

2. The Representation Formula. Let A denote the class of functions which are analytic in the unit disk E and which satisfy $|\phi(z)|\leqq1$ for all z in E. Since f is in $C^\lambda(\alpha, \beta)$ if and only if $zf'(z)$ is in $S^\lambda(\alpha, \beta)$, the following theorem follows immediately from [5, Theorem 1].

Theorem 1. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in E. Then f is in the class $C^\lambda(\alpha, \beta)$ if and only if*

$$(2.1) \qquad f'(z)=\exp\left\{-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot\int_0^z\frac{\phi(t)}{1+(2\beta-1)t\phi(t)}dt\right\}$$

for some ϕ in A.

For $\alpha=0$, $\beta=1$ we deduue the following result:

Corollary 1. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in the unit disk E. Then f is a λ-Robertson function if and only if*

(2.2)
$$f'(z)=\exp\left\{-2e^{-i\lambda}\cos\lambda\int_0^z \phi(t)/(1+t\phi(t))dt\right\}$$

for some ϕ *in* A.

For $\alpha=0=\lambda$, $\beta=1$, the above theorem yields the following result.

Corollary 2. *Let* $f(z)=z+a_2z^2+\cdots$ *be analytic in* E. *Then* f *is a convex function if and only if*

(2.3)
$$f'(z)=\exp\left\{-2\int_0^z \phi(t)/(1+t\phi(t))dt\right\}$$

for some ϕ *in* A.

3. The Sufficient Condition.

Theorem 2. *Let* $f(z)=z+a_2z^2+\cdots$ *be analytic in* E. *Then* f *is in the class* $C^\lambda(\alpha, \beta)$, *if for some* $\alpha(0\leqq\alpha<1)$ *and* $\lambda(-\pi/2<\lambda<\pi/2)$,

(3.1)
$$\sum_{n=2}^{\infty} n\{2(1-\beta)n-1+|1-2\beta+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|$$
$$\leqq 2\beta(1-\alpha)\cos\lambda, \quad \textit{whenever } 0<\beta\leqq 1/2,$$

(3.2)
$$\sum_{n=2}^{\infty} n\{n-1+|(2\beta-1)(n-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|$$
$$\leqq 2\beta(1-\alpha)\cos\lambda, \quad \textit{whenever } 1/2\leqq\beta\leqq 1$$

holds.

Proof. The proof follows from [5, Theorem 2] on using the fact that $f\in C^\lambda(\alpha, \beta) \Leftrightarrow zf'(z)\in S^\lambda(\alpha, \beta)$, $z\in E$.

Corollary 3. *A function* $f(z)=z+a_2z^2+\cdots$, *analytic in* E, *is a* λ*-Robertson function if*

$$\sum_{n=2}^{\infty} n\{n-1+\sqrt{1+2n\cos(2\lambda)+n^2}\}|a_n|\leqq 2\cos\lambda$$

holds for some $\lambda(-\pi/2<\lambda<\pi/2)$.

Remark. By fixing the parameters α, β and λ in Theorem 2, we can obtain sufficient conditions for a function to be in the classes $C^\lambda(\alpha)$, $C_{\lambda,M}$, $C(\alpha)$, and many others.

4. Distortion Theorem. Theorem 3 in [5] together with the fact that 'f is in $C^\lambda(\alpha, \beta)$ if and only if $zf'(z)$ is in $S^\lambda(\alpha, \beta)$' yields the following distortion properties for the class $C^\lambda(\alpha, \beta)$.

Theorem 3. *Let $f(z)=z+a_2z^2+\cdots$ be analytic in the unit disk E. If f is in $C^\lambda(\alpha, \beta)$, then for $|z|=r<1$ and for all $\alpha\in[0, 1)$, $\beta\in(0, 1/2)\cup(1/2, 1]$, $\lambda\in(-\pi/2, \pi/2)$,*

$$|f'(z)|\leqq\left\{\frac{(1+(2\beta-1)r)^{(1-\cos\lambda)}}{(1-(2\beta-1)r)^{(1+\cos\lambda)}}\right\}^{\beta(1-\alpha)\cos\lambda/(2\beta-1)}, \tag{4.1}$$

$$|f'(z)|\geqq\left\{\frac{(1-(2\beta-1)r)^{(1-\cos\lambda)}}{(1+(2\beta-1)r)^{(1+\cos\lambda)}}\right\}^{\beta(1-\alpha)\cos\lambda/(2\beta-1)}; \tag{4.2}$$

whereas for $\alpha\in[0, 1)$, $\beta=1/2$, $\lambda\in(-\pi/2, \pi/2)$,

$$|f'(z)|\leqq\exp((1-\alpha)\cos\lambda\cdot r), \tag{4.3}$$

and

$$|f'(z)|\geqq\exp(-(1-\alpha)\cos\lambda\cdot r). \tag{4.4}$$

The function given by

$$f_0'(z)=\begin{cases}1/\{1-(2\beta-1)e^{i\theta}z\}^{2\beta(1-\alpha)\cos\lambda\cdot e^{-\lambda}/(2\beta-1)}, & \beta\neq1/2\\ \exp\{(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}z\}, & \beta=1/2\end{cases} \tag{4.5}$$

provides equality in (4.1) *and* (4.3) *when θ is given by*

$$\tan\theta/2=\frac{1-(2\beta-1)r}{1+(2\beta-1)r}\cot(\pi/2-\lambda/2). \tag{4.6}$$

Further, the above function gives equality in (4.2) *and* (4.4) *when θ is given by the equation*

$$\tan\theta/2=\frac{1-(2\beta-1)r}{1+(2\beta-1)r}\cot(-\lambda/2). \tag{4.7}$$

Corollary 4. *If $f(z)=z+a_2z^2+\cdots$, analytic in E, is a λ-Robertson function of order α, then for $|z|=r<1$,*

$$\left\{\frac{(1-r)^{(1-\cos\lambda)}}{(1+r)^{(1+\cos\lambda)}}\right\}^{(1-\alpha)\cos\lambda}\leqq|f'(z)|\leqq\left\{\frac{(1+r)^{(1-\cos\lambda)}}{(1-r)^{(1+\cos\lambda)}}\right\}^{(1-\alpha)\cos\lambda}.$$

The estimates are sharp.

Corollary 5. *If $f(z)=z+a_2z^2+\cdots$, analytic in E, is a convex function of order α, then for $|z|=r$,*

$$\frac{1}{(1+r)^{2(1-\alpha)}}\leqq|f'(z)|\leqq\frac{1}{(1-r)^{2(1-\alpha)}}.$$

The estimates are sharp.

Remark. For suitable values of α, β, λ the above theorem can yield the distortion properties for the classes C^λ, $C_{\lambda,M}$, and many others.

5. Coefficient Estimates.

Theorem 4. *Let* $f(z)=z+a_2z^2+\cdots$ *be in* $C^{\lambda}(\alpha, \beta)$.

(a) *If* $\beta(1-\alpha)(2-\alpha)\cos^2\lambda>(1-\beta)(1+(1-\alpha)\cos^2\lambda)$, *let*

$$N=\left[\frac{2\beta(1-\alpha)(2-\alpha)\cos^2\lambda}{(1-\beta)(1+(1-\alpha)\cos^2\lambda)}\right],$$

where N *is the greatest integer of the expression within the square bracket. Then*

(5.1) $$|a_n|\leqq\frac{1}{n!}\prod_{k=2}^{n}|(2\beta-1)(k-2)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|,$$

for $n=2, 3, \cdots, N+2$; *and*

(5.2) $$|a_n|\leqq\frac{1}{n(n-1)(N+1)!}\prod_{k=2}^{N+3}|(2\beta-1)(k-2)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|, \quad n>N+2.$$

(b) *If* $\beta(1-\alpha)(2-\alpha)\cos^2\lambda\leqq(1-\beta)(1+(1-\alpha)\cos^2\lambda)$, *then*

(5.3) $$|a_n|\leqq\frac{2\beta(1-\alpha)\cos\lambda}{n(n-1)}, \qquad n\geqq2.$$

The estimates in (5.1) *are sharp for the function given by*

(5.4) $$1+z\frac{f''(z)}{f'(z)}=\frac{1-((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)z}{1-(2\beta-1)z},$$

where

$$\beta(1-\alpha)(2-\alpha)\cos^2\lambda>(1-\beta)(1+(1-\alpha)\cos^2\lambda),$$

while the estimates in (5.3) *are sharp for the functions given by*

(5.5) $$f_n'(z)=\{1-(2\beta-1)z^{n-1}\}^{-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda/(2\beta-1)(n-1)}$$

for $\beta\neq1/2$; *whereas for* $\beta=1/2$

(5.6) $$f_n'(z)=\exp\left\{\left(\frac{(1-\alpha)e^{-i\lambda}\cos\lambda}{n-1}\right)z^{n-1}\right\}, \qquad (n\geqq2).$$

Proof. Since

$$zf'(z)=z+2a_2z^2+\cdots$$

is a λ-spiral-like function of order α and type β, this theorem is an immediate consequence of [5, Theorem 4].

Corollary 6. *If* $f(z)=z+a_2z^2+\cdots$ *is a* λ*-Robertson function of order* α, *then*

$$|a_n|\leqq\frac{1}{n!}\prod_{k=0}^{n-2}|2(1-\alpha)\cos\lambda\cdot e^{-i\lambda}+k|, \qquad (n\geqq2)$$

and these bounds are sharp.

Corollary 7. *Let* $f(z)=z+a_2z^2+\cdots$ *be in* $C_{\lambda,M}$.

(a) *If* $M>(4+\tan^2\lambda)/4$, *let*

$$N=\left[\frac{2(2M-1)}{2+\tan^2\lambda}\right].$$

Then

$$|a_n|\leqq\frac{1}{n!}\prod_{k=0}^{n-2}|l(c+k-1)|,\qquad n=2,3,\cdots,N+2;$$

$$|a_n|\leqq\frac{1}{(N+1)!\,n(n-1)}\prod_{k=0}^{N+3}|l(c+k+1)|,\qquad n>N+2,$$

where $c=(1+l)\cos\lambda\cdot e^{-i\lambda}/l-1$, $l=1-1/M$.

(b) *If* $1/2<M<(4+\tan^2\lambda)/4$, *then*

$$|a_n|\leqq\frac{(1+l)\cos\lambda}{n(n-1)},\qquad (n\geqq 2).$$

The bounds are sharp.

6. Radius of convexity. In [5], Mogra and the author have determined the radius of starlikeness for the class $S^\lambda(\alpha,\beta)$. Making use of the relationship between $C^\lambda(\alpha,\beta)$ and $S^\lambda(\alpha,\beta)$, we may write the following as a consequence of [5, Theorem 5].

Theorem 5. *Let* $f(z)=z+a_2z^2+\cdots$ *be analytic in* E, *and* f *is a member of* $C^\lambda(\alpha,\beta)$. *Then* f *is convex in*

$$(6.1)\qquad |z|<\frac{1}{\beta(1-\alpha)\cos\lambda+\sqrt{\beta^2(1-\alpha)^2\cos^2\lambda+(2\beta-1)^2-2\beta(1-\alpha)(2\beta-1)\cos^2\lambda}}.$$

The estimate for $|z|$ *is sharp for the function given by* (4.5), *where* θ *is given by* (4.7).

Corollary 8. *The radius of convexity of* C^λ *is* $\{|\sin\lambda|+\cos\lambda\}^{-1}$. *The result is sharp.*

The above result has also been determined, by using different method, by Libera and Ziegler [4]. Fixing $\beta=1$, the last theorem gives the following result due to Chichra [2].

Corollary 9. *Let* f *be in* $C^\lambda(\alpha)$. *Then* f *is convex in*

$$|z|<\frac{1}{(1-\alpha)\cos\lambda+\sqrt{\sin^2\lambda+\alpha^2\cos^2\lambda}}.$$

The estimate is sharp.

On taking the appropriate values of the parameters α,β,λ the above theorem can give the corresponding radius of convexity for the functions in the classes C^λ, $C_{\lambda,M}$, and others.

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References

[1] S.K. Bajpai and T.J.S. Mehrok: *On the coefficient structure and growth theorems for the functions $f(z)$ for which $zf'(z)$ is spirallike*, Publ. Inst. Math. (Beograd), **16** (1975), 5-12.

[2] P.N. Chichra: *Regular functions $f(z)$ for which $zf'(z)$ is α-spirallike*, Proc. Amer. Math. Soc. **49** (1975), 151-160.

[3] P.K. Kulshrestha: *Bounded Robertson functions*, Rend. Mat. (6) **9** (1976), 137-150.

[4] R.J. Libera and M.R. Ziegler: *Regular functions $f(z)$ for which $zf'(z)$ is α-spiral*, Trans. Amer. Math. Soc. **166** (1972), 361-370.

[5] M.L. Mogra and O.P. Ahuja: *On spiral-like functions of order α and type β*, Yokohama Mathematical Journal, **29** (1981), 145-156.

[6] M.S. Robertson: *Univalent functions $f(z)$ for which $zf'(z)$ is spirallike*, Michigan Math. J., **16** (1969), 97-101.

[7] M.S. Robertson: *On the theory of univalent functions*, Ann. Math. **37** (1936), 374-408.

[8] P.I. Sizuk: *Regular functions $f(z)$ for which $zf'(z)$ is θ-spiral shaped of order α*, Sibirsk. Mat. Z., **16** (1975), 1286-1290, 1371.

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