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A SUPPLEMENT TO CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS

By

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1. Introduction

Let $\{X_n\}_{n\geqq 1}$ be a sequence of i.i.d. random variables with common distribution F and let $S_n=X_1+\cdots+X_n$, $S_0=0$. Throughout this paper we suppose that the following weak law of large numbers holds for some α, $0<\alpha<2$,

$$\lim_n P(|S_n|>n^{1/\alpha}\varepsilon)=0 \qquad \text{for every} \quad \varepsilon>0\,. \tag{1}$$

As was shown in [3] and [5], (1) holds if and only if the following two conditions are satisfied:

$$P(|X|>n^{1/\alpha})=o(n^{-1})\,, \tag{2}$$

where $X=X_1$, and

$$\int_{|x|>n^{1/\alpha}} xF(dx)=o(n^{1/\alpha-1})\,. \tag{3}$$

Note that if $0<\alpha<1$ then (2) implies (3), if $1\leqq\alpha<2$ then (2) and $E(X)=0$ are sufficient for (3).

It is well-known ([1], [5], [6]) that

$$\sum_{n=1}^{\infty} P(|S_n|>n^{1/\alpha}\varepsilon)<\infty \qquad \text{for every} \quad \varepsilon>0\,, \tag{4}$$

if and only if $E|X|^{2\alpha}<\infty$ and (3) hold. If (4) holds then

$$\lim_{x\to\infty} \sum_{0\leqq n\leqq (x/\varepsilon)^\alpha} P(x\leqq S_n\leqq x+h)=0 \tag{5}$$

for every $\varepsilon>0$ and $h>0$. However as will be shown later (1) does not imply (5). In this note we give a sufficient condition for (5), and show that it is close to the best possible one. Note that (5) can be written as follows:

$$\lim_{x\to\infty} EN((x/\varepsilon)^\alpha, [x, x+h])=0\,, \tag{6}$$

where $N(x, I)$ is the number of S_k, $0\leqq k\leqq x$, falling within an interval I.

This paper stemmed from an attempt to improve Bickel and Yahav's renewal theorem in the plane [2]. Let $\|\cdot\|$ be a norm in R^2 such that its unit ball $B=\{\mathbf{x}; \|\mathbf{x}\| \leqq 1\}$ is a convex polygon. Let $\{\mathbf{X}_n\}$ be a sequence of i.i.d. two-dimensional random vectors having finite non-zero expectation, and let $\mathbf{S}_n=\mathbf{X}_1+\cdots+\mathbf{X}_n$. Suppose that the distribution of $\mathbf{X}_1$ is not supported by a proper closed subgroup of R^2. Bickel and Yahav showed that if either $E|\mathbf{X}_1|^2<\infty$ or $E(\mathbf{X}_1)$ is not parallel to sides of B then the following Blackwell type renewal theorem:

$$\lim_{x\to\infty}\sum_{n=1}^{\infty} P(x\leqq \|\mathbf{S}_n\|\leqq x+h)=h/\|E(\mathbf{X}_1)\|$$

holds for $h>0$. Our theorem enables us to show that when $E(\mathbf{X}_1)$ is parallel to a side of B the condition $E|\mathbf{X}_1|^2<\infty$ cannot be weakened to $E|\mathbf{X}_1|^r<\infty$, $r<3/2$.

2. Results

The main result we are going to prove is the following:

Theorem. (i) *If* (3) *and*

$$P(|X|>n)=o(n^{-3\alpha/2}) \tag{7}$$

are satisfied then (5) *holds for every* $\varepsilon>0$ *and* $h>0$.

(ii) *In the preceding statement condition* (7) *cannot be replaced by*

$$P(|X|>n)=o(n^{-3\alpha/2}\psi(n)) . \tag{8}$$

where $\psi(x)$ *is an arbitrary nonnegative function on* $[0,\infty)$ *which increases to* $+\infty$.

Note that a random walk which satisfies (1) and fails to satisfy (5) must be recurrent. In fact suppose $\{S_n\}$ is transient. If $E|X|=\infty$ then

$$\lim_{x\to\infty}\sum_{n=1}^{\infty} P(x\leqq S_n\leqq x+h)=0$$

(see [4], p. 368) and therefore $\{S_n\}$ satisfies (5). If $E|X|<\infty$ then we must have $E(X)\neq 0$ since $\{S_n\}$ is transient. In this case (1) does not hold $1\leqq\alpha<2$, and both (1) and (5) hold if $0<\alpha<1$.

As is well-known $E|X|^{3\alpha/2}<\infty$ implies (7). Furthermore if

$$P(|X|>n)\sim n^{-3\alpha/2}\log n ,$$

then $E|X|^{3\alpha/2}=\infty$, while $E|X|^r<\infty$ for $r<3\alpha/2$. Therefore we obtain the following:

Corollary. (i) *If* $0<\alpha<1$ *then* $E|X|^{3\alpha/2}<\infty$ *implies* (5). *If* $1\leqq\alpha<2$ *then* $E|X|^{3\alpha/2}<\infty$ *and* $E(X)=0$ *imply* (5).

(ii) *In the above statements the condition* $E|X|^{3\alpha/2}<\infty$ *cannot be replaced by the following*: $E|X|^r<\infty$ *for every* $r<3\alpha/2$.

Proof of Theorem.

(i) It is obvious that if (5) holds for $h=1$ then it holds every finite $h>0$. Our method of proof does not depend on the choice of $\varepsilon>0$. Therefore we may assume $h=1$ and $\varepsilon=1$. Furthermore it suffices to prove (5) for nondegenerate X, since the statement (i) is obvious for degenerate random variables. For nondegenerate case we have (see [7], p. 49)

$$\sup_x P(x \leqq S_n \leqq x+1) \leqq Cn^{-1/2}, \tag{9}$$

where C is a constant depending only on the distribution of X. Therefore we obtain

$$\sup_y EN(x^\alpha, [y, y+1]) \leqq C \sum_{n \leqq x^\alpha}^{\infty} n^{-1/2} = O(x^{\alpha/2}) \tag{10}$$

as $x \to \infty$. The following inequality is also well-known ([7], p. 50)

$$P\left(\max_{1 \leqq k \leqq n} S_k - m(S_k - S_n) \geqq x \right) \leqq 2P(S_n \geqq x), \tag{11}$$

where $m(X)$ is a median of a random variable X. From (1) we have $m(S_n)=o(n^{1/\alpha})$, and therefore for large n

$$P\left(\max_{1 \leqq k \leqq n} S_k \geqq n^{1/\alpha} \right) \leqq 2P(S_n > 2^{-1} n^{1/\alpha}). \tag{12}$$

It follows from a theorem of Heyde-Rohatgi [5] that (3) and (7) implies

$$P(|S_n| > 2^{-1} n^{1/\alpha}) = o(n^{-1/2}). \tag{13}$$

From (12) and (13) we obtain

$$P\left(\max_{1 \leqq k \leqq n^\alpha} S_k \geqq n \right) = o(n^{-\alpha/2}). \tag{14}$$

Let T_K denote the first time the random walk $\{S_n\}$ hits a set K:

$$T_K = \inf\{k \geqq 0;\ S_k \in K\},$$

and $T_K=\infty$ if $S_k \notin K$ for every k. In what follows we write for simplicity $I=[0,1]$ and $J=[-1,1]$. Then we have

$$\begin{aligned}
EN(x^\alpha, I+x) &= \sum_{k=0}^{[x^\alpha]} E[N(x^\alpha, I+x),\ T_{I+x}=k] \\
&= \sum_{k=0}^{[x^\alpha]} P(T_{I+x}=k)\cdot E[N(x^\alpha, I+x) \mid T_{I+x}=k] \\
&\leqq \sum_{k=0}^{[x^\alpha]} P(T_{I+x}=k)\cdot EN(x^\alpha - k, J) \\
&\leqq EN(x^\alpha, J)\cdot P(S_k \in I+x \text{ for some } k \leqq x^\alpha) \\
&\leqq EN(x^\alpha, J)\cdot P\left(\max_{1 \leqq k \leqq n^\alpha} S_k \geqq x \right).
\end{aligned}$$

It follows from (10) and (14) that this tends to zero as $x\to\infty$.

(ii) We shall construct a sequence $\{X_n\}$ of i.i.d. symmetric random variables which satisfies (8) and fails to satisfy (6). Let $0<\psi(x)\uparrow\infty$. Choose $\phi(x)$ satisfying $0<\phi(x)\uparrow\infty$, $x^{-\alpha/2}\phi(x)\downarrow 0$, and $\phi(x)/\psi(x)\to 0$ as $x\to\infty$. Let $\{X_n^{(j)}\}_{n\geqq 1, j\geqq 0}$ be a double sequence of mutually independent random variables such that $X_1^{(j)}, X_2^{(j)}, \cdots$ have a common distribution for every $j\geqq 0$. Let $X_n^{(0)}$ be uniformly distributed over $[-1,1]$, and for $j\geqq 1$ let

$$P(X_n^{(j)}=a_j)=P(X_n^{(j)}=-a_j)=p_j\,,$$
$$P(X_n^{(j)}=0)=1-2p_j\,,$$

where $a_j>0$ and $p_j>0$ are determined later.

For every $k\geqq 1$,

$$S_n^{(k-1)}=\sum_{m=1}^{n}\sum_{j=0}^{k-1}X_m^{(j)}$$

is a sum of n i.i.d. random variables, each having an absolutely continuous distribution with zero mean and finite variance. Hence by Shepp's local limit theorem (see [7], p. 214) we can find for every $k\geqq 1$ two constants $c_k>0$ and $n_k>0$ such that

$$P(0\leqq S_n^{(k-1)}\leqq 1)\geqq 2c_k n^{-1/2} \qquad \text{for} \quad n\geqq n_k\,. \tag{15}$$

By (15) we can choose an increasing sequence $\{a_n\}$, $a_n>0$, such that

$$\sum_{j=1}^{\infty} a_j^{-\alpha/2}\phi(a_j)<\infty\,, \tag{16}$$

$$2^{3\alpha/2}a_{j+1}^{-3\alpha/2}\phi(a_{j+1})\leqq a_j^{-3\alpha/2}\phi(a_j)\,, \tag{17}$$

and

$$\sum_{n=l_k}^{m_k} P(0\leqq S_n^{(k-1)}\leqq 1)\geqq a_k^{\alpha/2}/\phi(a_k) \qquad \text{for} \quad k\geqq 1\,, \tag{18}$$

where $l_k=[a_k^{\alpha}/2]$ and $m_k=[a_k^{\alpha}]$. Let

$$p_j=\min(a_j^{-3\alpha/2}\phi(a_j),\ 1/2)\,.$$

Then we have by (16)

$$\sum_{j=1}^{\infty} P(|X_1^{(j)}|\neq 0)=2\sum_{j=1}^{\infty} p_j<\infty\,.$$

Hence by Borel-Cantelli lemma $\sum_{j=1}^{\infty} X_n^{(j)}$ converges a.s. to a random variable X_n for each $n\geqq 1$. Thus we obtain a sequence $\{X_n\}$ of i.i.d. random variables satisfying (3). From (17) we have $2a_{j-1}<a_j$ and therefore $a_1+\cdots+a_{j-1}<a_j$ for $j\geqq 2$. Hence it follows from (17) that for $2a_{j-1}\leqq x\leqq 2a_j$

(19) $$P(|X_1|>x)\leqq P(X_1^{(k)}\neq 0 \text{ for some } k\geqq j)$$
$$\leqq 2\sum_{k=j}^{\infty} p_k \leqq 2\sum_{k=j}^{\infty} a_k^{-3\alpha/2}\phi(a_k)$$
$$\leqq 2\sum_{i=0}^{\infty}(2^{-3\alpha/2})^i a_j^{-3\alpha/2}\phi(a_j)$$
$$\leqq 2(1-2^{-3\alpha/2})^{-1}2^{3\alpha/2}x^{-3\alpha/2}\phi(x)\,.$$

The last inequality follows from the fact that $\phi(x)$ is increasing and $x^{-3\alpha/2}\phi(x)$ is decreasing. (19) shows that X_1 satisfies (8).

It remains to prove that $S_n=X_1+\cdots+X_n$ does not satisfy (6). Again we suppose that $h=1$ and $\varepsilon=1$. Let

$N_k=N(a_k^\alpha, [a_k, a_k+1])$
$\quad$ = no. of $n\leqq m_k$ such that $0\leqq S_n-a_k\leqq 1$,
N_k' = no. of $n\leqq m_k$ such that $0\leqq S_n^{(k)}-a_k\leqq 1$,
N_k'' = no. of n such that $l_k\leqq n\leqq m_k$ and $0\leqq S_n^{(k-1)}\leqq 1$,
$A_k=\{X_m^{(j)}=0$ for every $1\leqq m\leqq m_k$, $j\geqq k+1\}$,
$B_k=\{$there exists only one $m\leqq l_k$ satisfying $X_m^{(k)}=a_k$, and $X_l^{(k)}=0$ for every $l\neq m$, $1\leqq l\leqq m_k\}$.

Then we obtain

(20) $$EN_k\geqq EN_k\cdot 1_{A_k}=EN_k'\cdot 1_{A_k}=EN_k'\cdot P(A_k)\geqq EN_k'\cdot 1_{A_kB_k}\cdot P(A_k)$$
$$\geqq EN_k''\cdot 1_{A_kB_k}\cdot P(A_k)\geqq EN_k''\cdot P(A_k)^2P(B_k)\,.$$

It follows from (16) that

$$m_k\sum_{j=k+1}^{\infty}p_j\leqq a_k^\alpha\sum_{j=k+1}^{\infty}a_j^{-3\alpha/2}\phi(a_j)\leqq\sum_{j=k+1}^{\infty}a_j^{-\alpha/2}\phi(a_j)\to 0$$

as $k\to\infty$, and therefore

(21) $$P(A_k)=\prod_{j=k+1}^{\infty}(1-2p_j)^{m_k}\to 1 \qquad \text{as} \quad k\to\infty\,.$$

It is easy to see that

(22) $$P(B_k)\sim 2^{-1}m_kp_k(1-2p_k)^{m_k}$$
$$\sim 2^{-1}a_k^\alpha a_k^{-3\alpha/2}\phi(a_k)[1-2a_k^{-3\alpha/2}\phi(a_k)]^{m_k}$$
$$\sim 2^{-1}a_k^{-\alpha/2}\phi(a_k)\,.$$

At last we have from (18) that

$$(23)\qquad EN_k'' = \sum_{n=l_k}^{m_k} P(0 \leqq S_n^{(k-1)} \leqq 1)$$

$$= \sum_{n=l_k}^{m_k} 2^{-1} P(|S_n^{(k-1)}| \leqq 1)$$

$$\geqq 2^{-1} a_k^{\alpha/2} / \phi(a_k),$$

for $k \geqq 1$. From (20), (21), (22), and (23) we have

$$\limsup_{k \to \infty} EN_k \geqq 1/4 .$$

Therefore

$$\limsup_{x \to \infty} EN(x^\alpha, [x, x+1]) \geqq 1/4 .$$

This completes the proof.

Remark. We can modify the above construction to show the existence of $\{X_n\}$ satisfying (8) and

$$\limsup_{x \to \infty} EN(x^\alpha, [x, x+h]) = \infty .$$

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