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C^*-CROSSED PRODUCTS BY R

By

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ABSTRACT. We study crossed products of C^*-dynamical systems $(A, \mathbf{R}, \alpha)$ when α is asymptotically abelian. In particular we show that the crossed product is simple if and only if the system has neither ground states nor ceiling states, provided that A is α-simple and unital and α is faithful. In the appendix we study the case where α is uniformly continuous, and remark that it gives us proof of some of the well-known results on inner implementability of derivations.

1. Introduction

Given a C^*-dynamical system $(A, \mathbf{R}, \alpha)$ one forms another C^*-algebra, $A\times_{\alpha}\mathbf{R}$, called the crossed product of A by $\mathbf{R}$ (or by α). A characterization of the crossed product being prime (or simple) has been given in terms of spectral property of α (cf. [9], [4]). In this note we shall further study the ideal structure of the crossed product under some assumption (given below).

If A has a ground state or a ceiling state, the crossed product is not simple. In [6] Kusuda has elaborated on this point to show that the existence of a ground state (or a ceiling state) is equivalent to the existence of a proper ideal of the crossed product which is monotonely increasing up to the whole algebra under (the inverse of) the dual action. In our case we shall show that any primitive ideal of the crossed product is contained in an ideal of the kind just described. Hence in particular the crossed product is simple if and only if there are neither ground states nor ceiling states.

The situation we shall deal with is the following:

Assumption 1.1. *Let A be a C^*-algebra and α a strongly continuous one-parameter automorphism group of A. They are assumed to satisfy:*

(i) *A has an identity,*

(ii) *A is α-simple, i.e., A does not have any non-tirvial α-invariant ideals (an ideal is always assumed to be closed and two-sided),*

(iii) *α is faithful, i.e., α_t is the identity automorphism if and only if $t=0$.*
It is further assumed that

(iv) *There exists an automorphism γ of A with $\gamma\circ\alpha_t=\alpha_t\circ\gamma$, $t\in\mathbf{R}$, such that γ is asymptotically abelian, i.e., for any x, y in A,*

$$\lim_{|n|\to\infty}\|\gamma^n(x)y-y\gamma^n(x)\|=0\,.$$

We shall give main results in Sect. 2 and the proofs in Sect. 3 and 4. We shall give a more precise description of ideals of the crossed product, in Sect. 5, when α is assumed to be periodic instead of (iii).

In the appendix we shall deal with the opposite case: α is supposed to be uniformly continuous. It is known that $A \times_\alpha \mathbf{R}$ is isomorphic to $A \otimes C_0(\mathbf{R})$ under an isomorphism which preserves A (regarded canonically as multipliers) if and only if α is inner (in the multiplier algebra). By using this our method gives different proof of some of the well-known derivation theorems. In particular we improve Pedersen's result [8] slightly: Any *-derivation δ of a separable C^*-algebra A is implemented by a positive element h of $M^\infty(A)$ with $\|h\| = \|\delta\|$ in the way $\delta(x) = i(hx - xh)$, $x \in A$. (The improvement is just on the norm of h.)

2. Results

We borrow most of the material on the crossed products from Pedersen's book [9]. For example, the crossed product $A \times_\alpha \mathbf{R}$ of a C^*-algebra A by a strongly continuous one-parameter automorphism group α of A is the closed linear span of elements of the form $a\lambda(f)$, $a \in A$, $f \in L^1(\mathbf{R})$, where A is regarded as multipliers of $A \times_\alpha \mathbf{R}$ and λ is the canonical group of unitary multipliers of $A \times_\alpha \mathbf{R}$. The dual action $\hat{\alpha}$ on $A \times_\alpha \mathbf{R}$ is the action of $\mathbf{R}$ satisfying

$$\hat{\alpha}_p(a) = a\,, \qquad \hat{\alpha}_p(\lambda(t)) = e^{ipt}\lambda(t)\,, \qquad a \in A\,, \quad t \in \mathbf{R}\,,$$

where $\hat{\alpha}$ also denotes the unique extension of $\hat{\alpha}$ to an action on $M(A \times_\alpha \mathbf{R})$, the multiplier algebra of $A \times_\alpha \mathbf{R}$.

Let (π, u) be a covariant representation of (A, α), i.e., π is a representation of A on some Hilbert space H, u is a strongly continuous one-parameter unitary group on H, and they satisfy

$$u(t)\pi(x)u(t)^* = \pi \circ \alpha_t(x)\,, \qquad x \in A\,, \quad t \in \mathbf{R}\,.$$

Then there is associated a representation $\pi \times u$ of $A \times_\alpha \mathbf{R}$ such that

$$(\pi \times u)(a\lambda(f)) = \pi(a) \int f(t)u(t)dt\,, \qquad a \in A\,, \quad f \in L^1(\mathbf{R})\,.$$

Theorem 2.1. *Under Assumption* 1.1, *let* (π, u) *be an irreducible covariant representation of* (A, α), *i.e.,* $\pi(A)' \cap u(\mathbf{R})' = \mathbf{C}1$. *Then either* $Sp(u) = \mathbf{R}$ *or* $\mathrm{Sp}(u)$ *is semi-bounded. In the case* $Sp(u) = \mathbf{R}$, $\pi \times u$ *is a faithful representation of* $A \times_\alpha \mathbf{R}$.

To an invariant state ϕ of (A, α) there is associated a covariant representation (π_ϕ, u_ϕ): π_ϕ is the GNS representation of A associated with ϕ, on the Hilbert space H_ϕ, with the canonical cyclic vector Ω_ϕ, and u_ϕ is defined by

$$u_\phi(t)\pi_\phi(x)\Omega_\phi = \pi_\phi \circ \alpha_t(x)\Omega_\phi\,, \qquad x \in A\,, \quad t \in \mathbf{R}\,.$$

We remark that the above theorem is well-known for (π_ϕ, u_ϕ) with an extreme invariant state ϕ if α itself is asymptotically abelian (cf. [3]).

We call ϕ a ground state if $\mathrm{Sp}(u_\phi) \subset \mathbf{R}_+$, and a ceiling state if $\mathrm{Sp}(u_\phi) \subset \mathbf{R}_-$.

Corollary 2.2. *Under Assumption* 1.1 $A\times_\alpha\mathbf{R}$ *is simple if and only if* (A, α) *has neither ground states nor ceiling states.*

To prove the corollary it suffices to show that if there is a covariant representation (π, u) with $\mathrm{Sp}(u)$ semi-bounded, then there is a ground state or a ceiling state for (A, α). But this is well-known (cf. [6]).

We note that $A\times_\alpha\mathbf{R}$ is prime under Assumption 1.1. This follows from the general theory [9, 8.11] by showing that the Connes spectrum of α is $\mathbf{R}$.

3. Invariant states

We denote by $H^\alpha(A)$ the set of non-zero α-invariant hereditary C^*-subalgebras of A. We recall the reader the following (cf. [4]):

Theorem 3.1. *Let* (π, u) *be a covariant representation of* (A, α). *Then* $\pi\times u$ *is a faithful representation of* $A\times_\alpha\mathbf{R}$ *if and only if* π *is faithful and*

$$\mathrm{Sp}(u\,|\,[\pi(B)H])=\mathbf{R}$$

for all $B\in H^\alpha(A)$.

Now assuming π is faithful we give a preliminary consideration on an irreducible covariant representation (π, u) of (A, α). If π is not irreducible one has the non-trivial action $\beta_t=\mathrm{Ad}\,u_t$ of $\mathbf{R}$ on $\pi(A)'$. Let $B\in H^\alpha(A)$. Since for any $x'\in\pi(A)'$, $x'[\pi(B)H]$ is contained in $[\pi(B)H]$ and $x'[\pi(B)H]=(0)$ implies $x'=0$, we obtain

$$Sp(u\,|\,[\pi(B)H])+\mathrm{Sp}(\beta)\subset\mathrm{Sp}(u\,|\,[\pi(B)H])\,.$$

Since β is ergodic, $\mathrm{Sp}(\beta)$ forms a closed group. There are two possibilities.

(1) β is faithful. In this case $\pi\times u$ is faithful since $\mathrm{Sp}(\beta)=\mathbf{R}$ and so $\mathrm{Sp}(u\,|\,[\pi(B)H])=\mathbf{R}$ for any $B\in H^\alpha(A)$.

(2) β is periodic. If λ is defined by the equality $\mathrm{Sp}(\beta)=\lambda\mathbf{Z}$, then $\mathrm{Sp}(u\,|\,[\pi(B)H])$ is invariant under the addition of λ, i.e., the kernel of $\pi\times u$ is invariant under $\hat{\alpha}_\lambda$.

Lemma 3.2. *Under Assumption* 1.1 *let* ω *be an extreme* α- *and* γ-*invariant state of* A. *Then*

$$\mathrm{Sp}(u_\omega\,|\,[\pi_\omega(B)H_\omega])+\mathrm{Sp}(u_\omega)\subset\mathrm{Sp}(u_\omega\,|\,[\pi_\omega(B)H_\omega])$$

for all $B\in H^\alpha(A)$, *where* u_ω *is the unitary group corresponding to* α. *In particular* $\mathrm{Sp}(u_\omega)$ *forms a closed subsemigroup of* $\mathbf{R}$ *which generates* $\mathbf{R}$ *as a closed group.*

Proof. Fix $B\in H^\alpha(A)$. Notice that π_ω is faithful. Let $p\in\mathrm{Sp}(u_\omega\,|\,[\pi_\omega(B)H_\omega])$, $q\in\mathrm{Sp}(u_\omega)$, and $\varepsilon>0$. Let R_B be the right (closed) ideal of A generated by B. Then there are $x\in R_B$ and $y\in A$ such that

$$\mathrm{Sp}_\alpha(x)\subset p+(-\varepsilon, \varepsilon)\,,\qquad \pi_\omega(x)\Omega_\omega\neq 0\,,$$

$$\mathrm{Sp}_\alpha(y)\subset q+(-\varepsilon, \varepsilon)\,,\qquad \pi_\omega(y)\Omega_\omega\neq 0\,.$$

(See [1] or [9] for the spectral theory.) We assert that $\pi_\omega(x\gamma^n \circ \alpha_t(y))\Omega_\omega \neq 0$ for some n and t. Then the first result follows because

$$\pi_\omega(x\gamma^n \circ \alpha_t(y))\Omega_\omega \in [\pi_\omega(B)H_\omega]\,,$$

$$\mathrm{Sp}_{u_\omega}(\pi_\omega(x\gamma^n \circ \alpha_t(y))\Omega_\omega) \subset p+q+(-2\varepsilon, 2\varepsilon)\,.$$

To prove that $\pi_\omega(x\gamma^n \circ \alpha_t(y))\Omega_\omega \neq 0$ for some n and t, we use the asymptotic abelianess with respect to γ and the extremality of ω (cf. [1]). The average of $\|\pi_\omega(x\gamma^n \circ \alpha_t(y))\Omega_\omega\|^2$ over $n \in \mathbf{Z}$ and $t \in \mathbf{R}$ tends to $\|\pi_\omega(y)\Omega_\omega\|^2 \|\pi_\omega(x)\Omega_\omega\|^2$, which implies the desired result.

The second statement follows from taking A for B. Since α is faithful, the closed group generated by $\mathrm{Sp}(u_\omega)$ must be $\mathbf{R}$.

Lemma 3.3. *Let ω be an α- (and γ-) invariant state of A. Then there exists an extreme α- (and γ-) invariant state ω_0 such that*

$$(*) \qquad \mathrm{Sp}(u_{\omega_0} \mid [\pi_{\omega_0}(B)H_{\omega_0}]) \subset \mathrm{Sp}(u_\omega \mid [\pi_\omega(B)H_\omega])$$

for all $B \in H^\alpha(A)$.

Proof. Let S be the set of invariant states ω_0 satisfying $(*)$ for all $B \in H^\alpha(A)$. Then S is a face of the invariant states, since if an invariant state ω_1 satisfies $\omega_1 < \lambda\omega_2$ with $\omega_2 \in S$ and $\lambda > 0$, then $(\pi_{\omega_1}, u_{\omega_1})$ is just the restriction of $(\pi_{\omega_2}, u_{\omega_2})$ to a subspace of H_{ω_2}. To show that S is closed, let ω_1 be a weak* limit point of S. Then ω_1 is invariant. For $p \notin \mathrm{Sp}(u_\omega \mid [\pi_\omega(B)H_\omega])$, there is $\varepsilon > 0$ such that for $x \in R_B$ with $\mathrm{Sp}(x) \subset (p-\varepsilon, p-\varepsilon)$,

$$\omega(x^*x) = 0\,.$$

Hence $\omega_2(x^*x) = 0$ for such x, and for any ω_2 in S, and so for $\omega_2 = \omega_1$. This implies that $p \notin \mathrm{Sp}(u_{\omega_1} \mid [\pi_{\omega_1}(B)H_{\omega_1}])$. Thus S is closed. Take any extreme point of S, which satisfies $(*)$ and is extremely invariant.

Lemma 3.4. *Under Assumption* 1.1 *let ϕ be an extreme α-invariant state. Then there exists a closed subsemigroup G of $\mathbf{R}$ which generates $\mathbf{R}$ as a closed group such that*

$$\mathrm{Sp}(u_\phi \mid [\pi_\phi(B)H_\phi]) + G \subset \mathrm{Sp}(u_\phi \mid [\pi(B)H_\phi])$$

for all $B \in H^\alpha(A)$.

Proof. Choose a net N_ι such that

$$\frac{1}{2N_\iota + 1} \sum_{n=-N_\iota}^{N_\iota} \phi \circ \gamma^n$$

converges weakly to a state ω. Then ω is α- and γ-invariant. Fix $B \in H^\alpha(A)$. Let $p \in \mathrm{Sp}(u_\phi \mid [\pi_\phi(B)H_\phi])$, $q \in \mathrm{Sp}(u_\omega)$, and $\varepsilon > 0$. Then there are $x \in R_B$ and $y \in A$ such that

$$\mathrm{Sp}_\alpha(x) \subset p+(-\varepsilon, \varepsilon)\,, \qquad \pi_\phi(x)\Omega_\phi \neq 0\,,$$

$$\mathrm{Sp}_\alpha(y) \subset q+(-\varepsilon, \varepsilon)\,, \qquad \pi_\omega(y)\Omega_\omega \neq 0\,.$$

We assert that $\pi_\phi(x\gamma^n \circ \alpha_t(y))\Omega_\phi \neq 0$ for some n and t. Then it follows that

$$\mathrm{Sp}\,(u_\phi\,|\,[\pi_\phi(B)H_\phi]) + \mathrm{Sp}\,(u_\omega) \subset \mathrm{Sp}\,(u_\phi\,|\,[\pi_\phi(B)H_\phi]) \,.$$

To prove that $\pi_\phi(x\gamma^n \circ \alpha_t(y))\Omega_\phi \neq 0$ for some n and t, we may suppose that

$$\frac{1}{2N_\iota + 1} \sum_{n=-N_\iota}^{N_\iota} \pi_\varphi \circ \gamma^n(y^*y)$$

converges weakly to an element T in $\pi_\phi(A)'' \cap \pi_\phi(A)'$. Then if $\pi_\phi(x\gamma^n \circ \alpha_t(y))\Omega_\phi = 0$ for all n and t, we obtain

$$(\Omega_\phi,\ \pi_\phi(x^*x)u_\phi(t)Tu_\phi(t)^*\Omega_\phi) = 0 \,.$$

Since $\pi_\phi(A)' \cap u_\phi(\mathbf{R})' = \mathbf{C}1$, this implies, by taking the average over t,

$$(\Omega_\phi,\ \pi_\phi(x^*x)\Omega_\phi)(\Omega_\phi,\ T\Omega_\phi) = 0 \,.$$

Since $(\Omega_\phi, T\Omega_\phi) = \omega(y^*y)$, this is a contradiction. Now we can obtain the conclusion by Lemmas 3.2 and 3.3.

Lemma 3.5. *Let G be a closed subsemigroup of* $\mathbf{R}$ *which generates* $\mathbf{R}$ *as a closed group. Then either $G = \mathbf{R}$, $G \subset \mathbf{R}_+$, or $G \subset \mathbf{R}_-$. If $G \subset \mathbf{R}_+$, then for any $\varepsilon > 0$, there exists $N > 0$ such that*

$$G + (-\varepsilon, \varepsilon) \supset [N, \infty) \,.$$

Proof. The first part is well-known (cf. [3]).

Let $\tilde{G}$ be the subgroup generated by $G \subset \mathbf{R}_+$. Fix $\lambda \in G$ with $\lambda > 0$. For any positive integer n there exist $g_1, \cdots, g_n$ in $\tilde{G}$ such that

$$0 \equiv g_0 < g_1 < \cdots < g_n < g_{n+1} \equiv \lambda \,, \qquad \max_{0 \le k \le n} |g_{k+1} - g_k| < \lambda/n \,.$$

since g_k is of the form $h_1 - h_2$ with $h_i \in G$, there is h in G such that $h + g_k \in G$ for all k. Then for any $t \ge h$ there are k $(0 \le k \le n)$ and a non-negative integer j, such that

$$|\,t - (h + g_k + j\lambda)\,| < \lambda/2n \,.$$

Lemma 3.6. *Under Assumption* 1.1 *let (π, u) be a covariant representation of (A, α), and let $B \in H^\alpha(A)$. Then $Sp\,(u)$ is semi-bounded if and only if* $\mathrm{Sp}\,(u\,|\,[\pi(B)H])$ *is semi-bounded.*

Proof. Suppose that $\mathrm{Sp}\,(u)$ is not semi-bounded. We denote by $R_B{}^\alpha(-N, N)$ the α-spectral subspace of R_B corresponding to $(-N, N)$. Since A is α-simple and unital, the linear span of $R_B{}^*R_B$ contains the identity. Thus for sufficiently large N, there exist $x_1, \cdots, x_k$ in $R_B{}^\alpha(-N, N)$ such that

$$\sum_{i=1}^{k} x_i{}^*x_i \ge 1$$

This implies that for any non-zero vector Φ, $\pi(x_i)\Phi$ is a non-zero vector of $[\pi(B)H]$ for some i. Hence $\mathrm{Sp}\,(u\,|\,[\pi(B)H])$ is not semi-bounded.

The following is a special case of Theorem 2.1.

Proposition 3.7. *Under Assumption* 1.1 *let ϕ be an extreme α-invariant state of A. Then either* $\mathrm{Sp}(u_\phi)=\mathbf{R}$ *or* $\mathrm{Sp}(u_\phi)$ *is semi-bounded. In the case $Sp(u_\phi)=\mathbf{R}$, $\pi\times u$ is a faithful representation of $A\times_\alpha\mathbf{R}$.*

Proof. Suppose that $\mathrm{Sp}(u_\phi)$ is not semi-bounded. By Lemmas 3.3 and 3.4, supposing $G\subset\mathbf{R}_+$, for any $\varepsilon>0$ there exists N such that

$$\mathrm{Sp}(u_\phi\,|\,[\pi_\phi(B)H_\phi])+[N,\infty)\subset\mathrm{Sp}(u_\phi\,|\,[\pi_\phi(B)H_\phi])+(-\varepsilon,\varepsilon)\,.$$

Since $\mathrm{Sp}(u_\phi\,|\,[\pi_\phi(B)H_\phi])$ is not semi-bounded below, the left-hand side equals $\mathbf{R}$, which implies that $\mathrm{Sp}(u_\phi\,|\,[\pi_\phi(B)H_\phi])$ is dense in $\mathbf{R}$ and so equal to $\mathbf{R}$. Hence $\pi\times u$ is faithful by 3.1. The case $G\subset\mathbf{R}_-$ can be treated in a similar way.

4. Proof of Theorem 2.1

Lemma 4.1. *Under Assumption* 1.1 *let (π, u) be an irreducible covariant representation of (A,α). If π is not irreducible, $\pi\times u$ is faithful.*

Proof. Suppose that $\pi\times u$ is not faithful and that π is not irreducible. Then by the remark given after 3.1 there is $\lambda>0$ such that

$$\mathrm{Sp}(u\,|\,[\pi(B)H])+\lambda=\mathrm{Sp}(u\,|\,[\pi(B)H])$$

for all $B\in H^\alpha(A)$.

Let $p\in\mathrm{Sp}(u)$, $\varepsilon>0$ sufficiently small, and Φ a unit vector with

$$\mathrm{Sp}_u(\Phi)\subset(p-\varepsilon,p+\varepsilon)\,.$$

Define a state ϕ_T of A by

$$\phi_T(x)=\frac{1}{2T}\int_{-T}^{T}(\Phi,\pi\circ\alpha_t(x)\Phi)dt\,,\quad x\in A\,.$$

Let ϕ be a weak limit point of (ϕ_T) as $T\to\infty$. Then ϕ is α-invariant and satisfies

$$\mathrm{Sp}(u_\phi\,|\,[\pi_\phi(B)H_\phi])\subset\mathrm{Sp}(u\,|\,[\pi(B)H])+[-\varepsilon,\varepsilon]-p\,.$$

By Lemma 3.3 we may suppose that ϕ is an extreme α-invariant state. Then by 3.4 and 3.5, supposing $G\subset\mathbf{R}_+$, we have N such that

$$[N,\infty)\subset\mathrm{Sp}(u\,|\,[\pi(B)H])+(-2\varepsilon,2\varepsilon)-p\,.$$

Since $\mathrm{Sp}(u\,|\,[\pi(B)H])$ is invariant under the addition of λ we obtain

$$\mathrm{Sp}(u\,|\,[\pi(B)H])+(-2\varepsilon,2\varepsilon)=\mathbf{R}$$

which is a contradiction for some $B\in H^\alpha(A)$ and $\varepsilon>0$.

Proof of Theorem 2.1. Suppose that $\pi\times u$ is not faithful. Let Φ be a unit vector as before, i.e., Φ satisfies

$$\mathrm{Sp}_u(\Phi)\subset(p-\varepsilon,p+\varepsilon)\,.$$

Define a state ω_N of A by

$$\omega_N(x) = \frac{1}{2N+1} \sum_{n=-N}^{N} (\Phi, \pi \circ \gamma^n(x)\Phi) .$$

Let ω be a weak limit point of (ω_N). Note that any weak limit point of

$$\frac{1}{2N+1} \sum_{n=-N}^{N} \pi \circ \gamma^n(x)$$

is a multiple of the identity operator because π is irreducible. Thus one can conclude that ω is α- and γ-invariant.

Let $\lambda \in \mathrm{Sp}(u \mid [\pi(B)H])$, $\mu \in \mathrm{Sp}(u_\omega)$, and $\varepsilon > 0$. Then there exist $x \in R_B$ and $y \in A$ such that

$$\mathrm{Sp}_\alpha(x) \subset (\lambda - p - 2\varepsilon, \lambda - p + 2\varepsilon) , \qquad \pi(x)\Phi \neq 0 ,$$

$$\mathrm{Sp}_\alpha(y) \subset (\mu - \varepsilon, \mu + \varepsilon) , \qquad \pi(y)\Omega_\omega \neq 0 .$$

We shall assert that $\pi(x\gamma^n(y))\Phi \neq 0$ for some n. Then since

$$\mathrm{Sp}_u(\pi(x\gamma^n(y))\Phi) \subset \lambda + \mu + (-4\varepsilon, 4\varepsilon) ,$$

we obtain

$$\mathrm{Sp}(u \mid [\pi(B)H]) + (-4\varepsilon, 4\varepsilon) \supset \mathrm{Sp}(u \mid [\pi(B)H]) + \mathrm{Sp}(u_\omega) .$$

If $\mathrm{Sp}(u \mid [\pi(B)H])$ is not semi-bounded, Lemmas 3.3 and 3.5 imply that

$$\mathrm{Sp}(u \mid [\pi(B)H]) + (-5\varepsilon, 5\varepsilon) = \mathbf{R} ,$$

which is a contradiction for some $B \in H^\alpha(A)$ and $\varepsilon > 0$. Hence $\mathrm{Sp}(u \mid [\pi(B)H])$ is semi-bounded for some B and so is $\mathrm{Sp}(u)$ by 3.6.

To prove that $\pi(x\gamma^n(y))\Phi \neq 0$ for some n, we choose a net (N_ι) such that

$$\lim \frac{1}{2N_\iota + 1} \sum_{n=-N_\iota}^{N_\iota} \pi \circ \gamma^n(y^* y) = \omega(y^* y)\mathbf{1} .$$

Hence

$$\lim \frac{1}{2N_\iota + 1} \sum_{n=-N_\iota}^{N_\iota} \|\pi(x\gamma^n(y))\Phi\|^2 = \|\pi(x)\Phi\|^2 \omega(y^* y) \neq 0 .$$

5. Periodic systems

When α is periodic, say α_1 is the identity automorphism of the C^*-algebra A, one forms the crossed product $A \times_\alpha \mathbf{T}$, where α is considered as an action of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ on A. Then we can adapt the preceding proof to this case to conclude:

Theorem 5.1. *Under Assumption* 1.1 (i), (ii), *and* (iv), *suppose that* α_t *is the identity automorphism if and only if* t *is an integer. Then for any non-zero primitive* I *of* $A \times_\alpha \mathbf{T}$, *there exist a ground or ceiling state* ϕ *of* A *and* $n \in \mathbf{Z} = \hat{\mathbf{T}}$ *such that* I *is the kernel* *of* $\pi_\phi \times u_\phi \circ \hat{\alpha}_n$.

Proof. Let (π, u) be an irreducible covariant representation of (A, α). Then one may conclude that $\mathrm{Sp}(u)$ is either $\mathbf{Z}$ or semi-bounded, and that if $\mathrm{Sp}(u)=\mathbf{Z}$, $\pi\times u$ is a faithful representation of $A\times_\alpha\mathbf{T}$. Suppose that $\mathrm{Sp}(u)$ is bounded below. If $\inf\mathrm{Sp}(u)=\lambda$, then take any unit vector Φ with $u_t\Phi=e^{i\lambda t}\Phi$. The state ϕ of A defined by

$$\phi(x)=(\Phi, \pi(x)\Phi)$$

is a ground state and π is equivalent to π_ϕ, since $[\pi(A)\Phi]$ is $(\pi\times u)(A\times_\alpha\mathbf{T})$-invariant, and so is the whole space. Thus we obtain the conclusion as the case $\mathrm{Sp}(u)$ is bounded above can be treated similarly.

This can be considered as a generalization of the result on the ideal structure of the crossed product of the *CAR* algebra by the gauge action, due to Bratteli.

Probably we can not expect this kind of result for general (non-periodic) α. Because there are states of finite energy which are disjoint from any of the ground states, in general, which might induce an ideal which is not of the type given in the theorem. Here a state ϕ is called to be of finite energy if there exists $p\geq 0$ such that $\phi(x^*x)=0$ for any $x\in A$ with $\mathrm{Sp}_\alpha(x)\subset(-\infty, -p)$. (Note that ϕ is a ground state if $p=0$.) The states of finite energy are not invariant in general but covariant (cf. [1], [9]).

Appendix

Let A be a C^*-algebra and δ a $*$-derivation of A. Then δ is bounded and generates a uniformly continuous one-parameter automorphism group α of A. We study the crossed product $A\times_\alpha\mathbf{R}$ in order to know whether or not δ is inner.

In the proof of the following lemma we use the Dauns-Hoffmann theorem.

Lemma A.1. *Let B be a C^*-algebra and β a strongly continuous one-parameter automorphism group of B. Let I be a (closed two-sided) ideal of B. Suppose that $I+\beta_t(I)=B$ for non-zero t,*

$$\bigcap_{t\in\mathbf{R}}\beta_t(I)=(0),$$

and

$$\bigcup_{n=1}^{\infty}\bigcap_{|t|\geq n}\beta_t(I)$$

is dense in B. Let D be the quotient of B by I and let $C_0(\mathbf{R})$ be the continuous functions on $\mathbf{R}$ vanishing at infinity with the action τ of $\mathbf{R}$ by translations. Then (B, β) is isomorphic to $(D\otimes C_0(\mathbf{R}), \iota\otimes\tau)$, where ι is the trivial action of $\mathbf{R}$. More precisely the isomorphism F of B onto $D\otimes C_0(\mathbf{R})$ is given by

$$(Fx)(s)=\beta_{-s}(x)+I, \qquad s\in\mathbf{R}, \quad x\in B.$$

Proof. For $x\in B$, Fx defines a D-valued continuous function on $\mathbf{R}$. Since for s with $|s|\geq n$,

$$\|(Fx)(s)\| = \|x+\beta_s(I)\| \leq \|x + \bigcap_{|t|\geq n} \beta_t(I)\|,$$

$\|(Fx)(s)\|$ goes to zero as $|s|\to\infty$. Since $\bigcap \beta_t(I)=(0)$,

$$\|Fx\| \equiv \sup \|(Fx)(s)\| = \|x\|.$$

Since $F\beta_t(x)(x)=\beta_{t-s}(x)+I=Fx(s-t)$, it follows that $F\circ\beta_t=\iota\otimes\tau_t\circ F$. In this way we can conclude that F defines an isomorphism of B into $D\otimes C_0(\mathbf{R})$.

Now we have to show that F is surjective. Let J be a primitive ideal of B. Then for large n, $\bigcap\{\beta_t(I); |t|\geq n\}$ is not contained in J, which implies

$$\bigcap\{\beta_t(I); |t|\leq n\}\subset J.$$

For each k, there exists $t_k\in[-n, n]$ such that

$$\bigcap\{\beta_t(I); |t-t_k|<1/k\}\subset J.$$

Let t_0 be a limit point of (t_k). Then since

$$\bigcup_{k=1}^{\infty}\bigcap\{\beta_t(I); |t-t_0|<1/k\}$$

is dense in $\beta_{t_0}(I)$ (cf. [4]), it follows that $\beta_{t_0}(I)\subset J$. Since $\beta_{t_0}(I)+\beta_t(I)=B$ for $t\neq t_0$, t_0 is unique. By setting $\Phi(J)=t_0$ we define a map Φ from Prim B to $\mathbf{R}$. Then Φ is continuous, because for a closed subset S of $\mathbf{R}$,

$$\Phi^{-1}(S)=\{J\in\operatorname{Prim}B; J\supset\bigcap\{\beta_t(I), t\in S\}\}.$$

For a bounded continuous function f on $\mathbf{R}$, $f\circ\Phi$ defines a bounded continuous function on Prim B, and can be regarded as a central multiplier of B [9, 4.4.8]. Denoting by F again the unique extension of F to a map of $M(B)$, one calculates:

$$F(f\circ\Phi)(s)=\beta_{-s}(f\circ\Phi)+I$$

which is mapped into

$$f\circ\Phi+\beta_s(I)=f(s)1$$

under the isomorphism of $D=B/I$ with $B/\beta_s(I)$ defined by

$$\beta_{-s}(x)+I\longmapsto x+\beta_s(I), \quad x\in B.$$

This shows that $F(f\circ\Phi)=f$, i.e., $1\otimes C_b(\mathbf{R})$ are multipliers of the image of F. By using this fact we can easily conclude that F is surjective.

We now show how to use this lemma to prove Sakai's theorem [10] that every derivation of a simple C^*-algebra is inner. Let δ be a $*$-derivation of A and α the automorphism group generated by δ. In an irreducible representation π of A there exists a uniformly continuous unitary group u such that $u_t\pi(x)u_{-t}=\pi\circ\alpha_t(x)$, $x\in A$, $t\in\mathbf{R}$. Let $B=A\times_\alpha\mathbf{R}$ and I the kernel of $\pi\times u$. Let q be the quotient map of B onto $D\equiv B/I$. Then q has a unique extension to a map from $M(B)$ into $M(D)$, which is denoted by q again. Since u is uniformly continuous, we have that $q(B)\supset q(A)\cong A$, i.e., for each $a\in A$ there is $x\in B$ such that $(a-x)B\subset I$. Hence there exists a maximal ideal (including I) of B, say I again.

If $p\neq 0$, since $I\neq\hat{\alpha}_p(I)$ and $I+\hat{\alpha}_p(I)$ is closed, we must have that $B=I+\hat{\alpha}_p(I)$. Since $q(\lambda(t))$ is uniformly continuous, it is easy to conclude that the remaining assumptions on I in Lemma A.1 are satisfied. Hence

$$B\cong D\otimes C_0(\mathbf{R})=C_0(D;\mathbf{R}).$$

Suppose $f\in C_0(D;\mathbf{R})$ has compact support. Then $x=F^{-1}f$ belongs to $\bigcap\{\hat{\alpha}_p(I);|p|\geq n\}$ for some $n>0$. For $g\in L^1(\mathbf{R})\cap L^2(\mathbf{R})$, $\lambda(g)^*x\lambda(g)$ is $\hat{\alpha}$-integrable and

$$\int\hat{\alpha}_p(\lambda(g)^*x\lambda(g))dp\in A.$$

Thus

$$\int_{-n}^{n}F(\lambda(g)^*x\lambda(g))(p)dp\in q(A)$$

because the map q of $M(B)$ into $M(D)$ is continuous with respect to $\sigma(M(B), B^*)$- and $\sigma(M(D), D^*)$-topology (the kernel of q is the closure of I in the strict topology). This implies that

$$\int f(p)dp\in q(A)$$

for any $f\in C_0(D;\mathbf{R})$ with compact support. Hence we can conclude that $D=q(A)\cong A$. Since $\dot{\lambda}(t)=q(\lambda(t))$ is a multiplier of D, by identifying D with A under the map of A onto $q(A)=D$, one obtains that $\dot{\lambda}(t)$ is a multiplier of A. Thus the generator of $\dot{\lambda}$ is also a multiplier of A.

Apparently this method can be applied to the situation treated in [5], where α is not assumed to be uniformly continuous, but α^* is assumed to be strongly continuous, and under this assumption it is shown that there exists an ideal I of $A\times_\alpha\mathbf{R}$ such that $q(A\times_\alpha\mathbf{R})\supset q(A)$, that is what we need in the above argument.

As another application of Lemma A.1 we consider $M^\infty(A)$ with a derivation which leaves A invariant, when A is separable, to prove Pedersen's result [8] that such a derivation is inner.

Here $M^\infty(A)$ is defined as follows (cf. [2], [8]). The set of essential ideals of A forms a directed set when ordered under the inverse inclusion and the set of the multiplier algebras of those ideals forms an inductive system, whose inductive limit is the C^*-algebra $M^\infty(A)$.

Let δ be a $*$-derivation of A. Then δ has a unique extension $\bar{\delta}$ to a $*$-derivation of $M^\infty(A)$ with $\|\bar{\delta}\|=\|\delta\|$. Let α be the automorphism group generated by $\bar{\delta}$ and form the crossed product $M^\infty(A)\times_\alpha\mathbf{R}$. Note that for any essential ideal J of A, naturally,

$$J\times_\alpha\mathbf{R}\subset A\times_\alpha\mathbf{R}\subset M(J)\times_\alpha\mathbf{R}\subset M^\infty(A)\times_\alpha\mathbf{R}.$$

First we consider $A\times_\alpha \mathbf{R}$, and let I be a maximal ideal of $A\times_\alpha \mathbf{R}$ satisfying

$$(*)\qquad \bigcap \hat{\alpha}_p(I)=(0)\,.$$

Lemma A.2. *Let* $\varepsilon>0$ *and let*

$$I(\varepsilon)=\bigcap\{\hat{\alpha}_q(I);\ |q|\le\varepsilon/2\}\,.$$

Let $p>\varepsilon$ *or* $p<-\varepsilon$. *Then the ideal* J *of* A *defined by*

$$(**)\qquad \bigcap \hat{\alpha}_q(I(\varepsilon)+\hat{\alpha}_p(I(\varepsilon)))=J\times_\alpha \mathbf{R}$$

is essential in A.

Proof. It is known (cf. [9]) that any $\hat{\alpha}$-invariant ideal of $A\times_\alpha \mathbf{R}$ is given as in the right hand side of (**). For any primitive ideal P of $A\times_\alpha \mathbf{R}$ containing $J\times_\alpha \mathbf{R}$ there must be r such that

$$P\supset\hat{\alpha}_r(I(\varepsilon)+\hat{\alpha}_p(I(\varepsilon)))\,.$$

Then since $P\supset\hat{\alpha}_r(I(\varepsilon))$ and $P\supset\hat{\alpha}_{r+p}(I(\varepsilon))$, there exist δ_1 and δ_2 in $[-\varepsilon/2,\varepsilon/2]$ such that

$$P\supset\hat{\alpha}_{r+\delta_1}(I)\,,\qquad P\supset\hat{\alpha}_{r+p+\delta_2}(I)\,.$$

Hence we can conclude that

$$P\supset\bigcap_q \hat{\alpha}_q(I+\hat{\alpha}_p(I(2\varepsilon)))\equiv K\times\mathbf{R}\,.$$

Since this is true for any primitive ideal P with $P\supset J\times_\alpha \mathbf{R}$, we obtain that $J\times_\alpha \mathbf{R}\supset K\times_\alpha \mathbf{R}$, i.e., $J\supset K$. Thus it suffices to show that K is essential in A. Since $I\not\supset\hat{\alpha}_p(I(2\varepsilon))$, if $K^\perp\equiv\{x\in A;\ xK=(0)\}$ were non-zero, $I+\hat{\alpha}_p(I(2\varepsilon))\cap K^\perp\times_\alpha \mathbf{R}$ would be strictly greater than I and satisfy (*) instead of I, which contradicts the maximality of I.

Lemma A.3. *I is closed in $A\times_\alpha \mathbf{R}$ with the strict topology as a multiplier algebra of $J\times_\alpha \mathbf{R}$, for any essential ideal J of A.*

Proof. The closure of I equals

$$\bar{I}=\{x\in A\times_\alpha \mathbf{R};\ ax\in I,\ a\in J\}\,.$$

Since $\bar{I}\cap J\times_\alpha \mathbf{R}=I\cap J\times_\alpha \mathbf{R}$,

$$\left(\bigcap\hat{\alpha}_q(\bar{I})\right)\cap J\times_\alpha \mathbf{R}=\bigcap(\hat{\alpha}_q(\bar{I})\cap J\times_\alpha \mathbf{R})=(0)\,.$$

Hence

$$\bigcap\hat{\alpha}_q(\bar{I})=(0)$$

which implies that $\bar{I}=I$ since $\bar{I}\supset I$.

Let J be an essential ideal of A and let $I(J)$ be the closure of I in $M(J)\times_\alpha \mathbf{R}$ with the strict topology as a multiplier algebra of $J\times_\alpha \mathbf{R}$, or equivalently, with $\sigma(M(J)\times_\alpha \mathbf{R},\ (J\times_\alpha \mathbf{R})^*)$-topology, i.e.,

$$I(J)=\{x\in M(J)\times_\alpha \mathbf{R};\ ax\in I,\ a\in J\}\ .$$

Clearly $I(J)$ is an ideal of $M(J)\times_\alpha \mathbf{R}$ and by Lemma A.3 $I(J)\cap A\times_\alpha \mathbf{R}=I$. If J_1 and J_2 are essential ideals of A with $J_1\subset J_2$, then $I(J_1)\cap M(J_2)\times_\alpha \mathbf{R}\supset I(J_2)$. Let I^∞ be the norm closure of the union of $I(J)$ with all essential ideals J of A. Then of course I^∞ is an ideal of $M^\infty(A)\times_\alpha \mathbf{R}$ and $I^\infty\cap A\times_\alpha \mathbf{R}$ equals I. Hence the ideal K of $M^\infty(A)$ defined by

$$\bigcap \hat{\alpha}_q(I^\infty)=K\times_\alpha \mathbf{R}$$

satisfies $K\cap A=(0)$. As $K\cap J=(0)$ implies $K\cap M(J)=(0)$ for an essential ideal J of A, K must be (0).

It is known that the sum of two norm-closed ideals is closed. For the strict topology we have the following

Lemma A.4. *Let B be a separable C^*-algebra and let I and J be ideals of $M(B)$ which are closed under the strict topology. Then $I+J$ is also closed under the strict topology.*

Proof. Let q be the quotient map of $M(B)$ onto $M(B)/I$ and let π be the unique extension of the quotient map of B onto $B/B\cap I$ to a normal map of $M(B)$ into $M(B/B\cap I)$, where $M(B)$ is regarded as in B^{**}. Then the kernel of π is

$$\{x\in M(B);\ ax,\ xa\in I\cap B,\ a\in B\}$$

which is I since I is closed. Thus $\pi=q$, by regarding $M(B)/I$ as a subalgebra of $M(B/B\cap I)$. Since B is separable, in fact $M(B)/I$ equals $M(B/B\cap I)$ by [9, 3.12.10].

Let x be a positive element of the closure of $I+J$ in $M(B)$ and let $b=q(x)$. Then b is in the closure of $q(B\cap J)=q(B\cap(I+J))$ in $M(q(B))$. Then there are sequences (x_n) in $q(B\cap J)_+$ and (y_n) in $(q(B\cap J)+\mathbf{C}1)_+$ such that $x_n\uparrow b$ and $y_n\downarrow b$ (cf. [9, 3.12.9]). Let (u_n) be a countable approximate unit for $B\cap I\cap J$.

Choose self-adjoint elements v_1 in $B\cap J$ and w_1 in $B\cap J+\mathbf{C}1$ such that $q(v_1)=x_1$, $q(w_1)=y_1$, and $0\le v_1\le w_1$. Put

$$v_1'=v_1+(w_1-v_1)^{1/2}u_1(w_1-v_1)^{1/2}\in B\cap J\ .$$

Then $q(v_1')=x_1$ and $v_1\le v_1'\le w_1$. In this way we choose (v_k), (v_k') in $B\cap J$, and (w_k) in $J\cap B+\mathbf{C}1$ such that

$$v_{k-1}\le v_k\le w_k\le w_{k-1}$$

$$q(v_k)=q(v_k')=x_k\,,\qquad q(w_k)=y_k$$

$$v_k'=v_k+(w_k-v_k)^{1/2}u_k(w_k-v_k)^{1/2}\ .$$

The proof goes exactly as the proof of [9, 3.12.10], with a simple check on [9, 1.5.10] due to the additional assumption $(v_k)\subset B\cap J$. Thus the limit a of (v_k) is a multiplier of B, and belongs to J since J is closed in $M(B)$. Since $q(a)=b$, $a-x$ belongs to I, i.e., $x\in I+J$.

Lemma A.5. *The set of elements x of $M(J\times_\alpha \mathbf{R})$ satisfying that*

$$p\to\hat{\alpha}_p(x)\,,\qquad t\to\lambda(t)x\,,\qquad \text{and}\qquad t\to x\lambda(t)$$

are norm-continuous, is equal to $M(J)\times_\alpha\mathbf{R}$.

Proof. The set of those elements clearly includes $M(J)\times_\alpha\mathbf{R}$, and forms an **R**-product [9, 7.8.2]. To prove the converse inclusion, it suffices to show that any element x of $M(J\times_\alpha\mathbf{R})$ satisfying $\hat{\alpha}_p(x)=x$, $p\in\mathbf{R}$, and $t\to\lambda(t)x\lambda(t)^*$ is continuous, belongs to $M(J)$. For such an element x, there is a net (x_i) is $J\times_\alpha\mathbf{R}$ which converges to x in the strict topology. For $f\in L^1(\mathbf{R})\cap L^2(\mathbf{R})$ and $a\in J$,

$$\int\hat{\alpha}_p(\lambda(f)^*x_ia\lambda(f))dp\in J$$

which converges in norm to

$$\int\hat{\alpha}_p(\lambda(f)^*xa\lambda(f))dp=\int\lambda(t)^*xa\lambda(t)\,|f(t)|^2dt\,.$$

Since f is arbitrary, this implies that $xa\in J$, and similarly $ax\in J$. Thus $x\in M(J)$. Cf. [9, 7.10.4].

Now we resume the proof. For $\varepsilon>0$, let

$$I^\infty(\varepsilon)=\bigcap\{\hat{\alpha}_q(I^\infty);\ |q|\leq\varepsilon/2\}\,.$$

Fix p with $|p|>\varepsilon$. Then by Lemma A.2 there exists an essential ideal J of A such that

$$I^\infty(\varepsilon)+\hat{\alpha}_p(I^\infty(\varepsilon))\supset J\times_\alpha\mathbf{R}\,.$$

Now by Lemma A.4 for any element x in $M(J)\times_\alpha\mathbf{R}$, there exist a and b in the closure of $I^\infty(\varepsilon)\cap J\times_\alpha\mathbf{R}$, and $\hat{\alpha}_p(I^\infty(\varepsilon))\cap J\times_\alpha\mathbf{R}$, respectively, in $M(J\times_\alpha\mathbf{R})$ such that $x=a+b$. Then for $f\in L^1(\mathbf{R})$ and $g\in L^1(\mathbf{R})$ with supp $(g)\subset[-\varepsilon/2,\ \varepsilon/2]$, one has

$$a_1=\lambda(f)^*\int\hat{\alpha}_q(a)g(q)dq\lambda(f)\in M(J)\times_\alpha\mathbf{R}\,,$$

$$b_1=\lambda(f)^*\int\hat{\alpha}_q(b)g(q)dq\lambda(f)\in M(J)\times_\alpha\mathbf{R}\,.$$

Then since $I^\infty\cap M(J)\times_\alpha\mathbf{R}$ is closed in $M(J)\times_\alpha\mathbf{R}$ (as a multiplier algebra of $J\times_\alpha\mathbf{R}$), we obtain

$$a_1\in I^\infty\,,\qquad b_1\in\hat{\alpha}_p(I^\infty)\,.$$

Hence it follows for $x=1$ that $\lambda(f)^*\lambda(f)\in I^\infty+\hat{\alpha}_p(I^\infty)$, which implies that

$$I^\infty+\hat{\alpha}_p(I^\infty)\supset M(J)\times_\alpha\mathbf{R}\,.$$

We can now slightly improve Pedersen's theorem:

Theorem A.6. *Let A be a separable C^*-algebra and δ a $*$-derivation. Let $\bar{\delta}$ be the* *unique extension of* δ *to a* $*$*-derivation of* $M^{\infty}(A)$. *Then there exists a positive* h *in* $M^{\infty}(A)$ *with* $\|h\| = \|\delta\|$ *such that* $\bar{\delta}(x) = i(hx - xh)$, $x \in M^{\infty}(A)$.

Proof. In the atomic representation π of A there exists a uniformly continuous unitary group u in $\pi(A)''$ such that the generator H of u satisfies that $0 \leq H \leq \|\delta\|$, and

$$u_t \pi(x) u_t^* = \pi \circ \alpha_t(x), \qquad x \in A$$

where α is the automorphism group generated by δ. We take for I in the above argument a maximal ideal I of $A \times_{\alpha} \mathbf{R}$ including $\ker \pi \times u$ such that $\bigcap \hat{\alpha}_p(I) = (0)$. (The closure I of the union of an increasing family of ideals J of $\pi \times u(A \times_{\alpha} \mathbf{R})$ with $J \cap \pi(A) = (0)$ satisfies that $I \cap \pi(A) = (0)$.) From the above proof it follows that $M^{\infty}(A) \times_{\alpha} \mathbf{R}/I^{\infty} \cong M^{\infty}(A)$. Then the generator h of the image of $\lambda(\cdot)$ under the quotient map clearly satisfies the above properties.

Apparently the same method can be applied to a single automorphism α of a separable C^*-algebra A if the Borchers spectrum of α is trivial. Because in this case there exists a maximal ideal I of $A \times_{\alpha} \mathbf{Z}$ such that $\bigcap \hat{\alpha}_p(I) = (0)$ and for this I, $I \neq \hat{\alpha}_p(I)$ for non-zero $p \in \mathbf{T}$. Then it would follow that $I^{\infty} + \hat{\alpha}_p(I^{\infty}) = M^{\infty}(A) \times_{\alpha} \mathbf{Z}$ and that α is inner in $M^{\infty}(A)$ (see [7, 6.6]).

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