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UNIVERSALLY WEAKLY INNER ONE-PARAMETER AUTOMORPHISM GROUPS OF C^*-ALGEBRAS

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ABSTRACT. If a one-parameter automorphism group of a C^*-algebra has a strongly continuous action on its dual, it is shown to be universally weakly inner. If a single automorphism of a separable C^*-algebra is extendible in each irreducible representation, it is shown to be universally weakly inner.

1. Introduction

Let A be a C^*-algebra and α a one-parameter automorphism group of A. One may consider the following four conditions on α:

(i) α is uniformly continuous, i.e., $\|\alpha_t - \iota\| \to 0$ as $t \to 0$ (where ι denotes the identity automorphism).

(ii) α is universally weakly inner, i.e., there exists a weakly* continuous one-parameter unitary group u in the second dual A^{**} of A such that $\alpha_t(x) = u_t x u_t^*$, $x \in A$.

(iii) α^* is strongly continuous, i.e., $\|\phi \circ \alpha_t - \phi\| \to 0$ as $t \to 0$ for any $\phi \in A^*$ (or, equivalently, for any pure state ϕ of A).

(iv) α is strongly continuous, i.e., $\|\alpha_t(x) - x\| \to 0$ as $t \to 0$ for any $x \in A$.

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv); (i)$\Rightarrow$(ii) follows from Sakai's theorem on derivations ([9], 4.1) and the other implications are standard (see, e.g., [1], 3.1.8). In general (iv)$\not\Rightarrow$(iii) and (ii)$\not\Rightarrow$(i). In this note we shall show that (iii) implies (ii).

For the proof we introduce the following notion: α is said to be *almost uniformly continuous* if, for any α-invariant (closed two-sided) proper ideal I of A, the action on the quotient A/I induced by α has a non-zero invariant hereditary C^*-subalgebra on which the induced action is uniformly continuous.

Now we state our main result:

Theorem 1.1. *Let A be a C^*-algebra and α a one-parameter automorphism group of A. Then the following conditions are equivalent*:

(1) *α^* is strongly continuous,*

(2) *α is almost uniformly continuous,*

(3) *α is universally weakly inner.*

For another characterization of those automorphism groups, see Elliott [3] (and also [2]).

If α is uniformly continuous on an α-invariant hereditary C^*-subalgebra B of A, and if F is a subset of A whose α-spectrum is compact, then α is uniformly continuous on the closed linear span of $\alpha_t(x)B\alpha_s(y^*)$, $x, y \in F$, $t, s \in \mathbf{R}$ (e.g., as shown by using [1], 3.2.42). Thus if the C^*-algebra is simple and unital, 'almost uniform continuity' implies 'uniform continuity'. In this way the above theorem implies 7 in [4] and, similarly 2.4 in [5]. In fact the proof of the theorem is obtained by combining the techniques used in those papers.

We show that (1)⇒(2) in Sect. 2, and that (2)⇒(3) in Sect. 3. We give a similar result on single automorphisms of separable C^*-algebras in Sect. 4. Finally we show that the Connes spectrum of an automorphism of a separable C^*-algebra A depends only on its action on the spectrum $\hat{A}$ of A.

2. Proof (1) ⇒ (2)

In this section α denotes a one-parameter automorphism group of a C^*-algebra A and is assumed to have a strongly continuous action on the dual A^*.

Since α is extendible in every irreducible representation, it leaves any (primitive) ideal invariant. If I is a (closed two-sided) ideal of A, the induced action $\dot{\alpha}$ on the quotient A/I is defined by

$$\dot{\alpha}_t \circ q(x) = q \circ \alpha_t(x)\,, \qquad x \in A\,,$$

where q is the quotient map from A onto A/I. If $\phi \in (A/I)^*$, then

$$\|\phi \circ \dot{\alpha}_t - \phi\| = \|(\phi \circ \dot{\alpha}_t - \phi) \circ q\| = \|\phi \circ q \circ \alpha_t - \phi \circ q\|\,.$$

This implies that $\dot{\alpha}^*$ is strongly continuous on $(A/I)^*$.

By using this fact, to prove that α is almost uniformly continuous, it suffices to show that there exists a non-zero α-invariant hereditary C^*-algebra B of A such that $\alpha \,|\, B$ is uniformly continuous.

Lemma 2.1. *Let B_0 be a non-zero hereditary C^*-algebra of A and $r < 1$. Then there exist $\delta > 0$ and a non-zero hereditary C^*-subalgebra B of B_0 such that*

$$\inf\{\|x\alpha_t(x)\|;\ x \in B,\ 0 \le x,\ \|x\| = 1,\ 0 < t < \delta\} > r\,.$$

Proof. Suppose the contrary. Then there are B_0 and $r < 1$ such that for any $\delta > 0$ and B the infimum is not larger than r. Then as in the proof of 2.1 in [5], we may find e_1, $a_1 \in B_0$, and $t_1 \in (0, 1)$ such that

$$e_1, a_1 \ge 0\,, \quad \|e_1\| = \|a_1\| = 1\,, \quad e_1 a_1 = a_1\,, \quad \|e_1 \alpha_{t_1}(e_1)\| \le r\,.$$

Similarly we find e_2, $a_2 \in \overline{a_1 B_0 a_1}$, and $t_2 \in (0, 1/2)$ satisfying the same conditions as for (e_1, a_1, t_1). In this way we construct e_n, $a_n \in \overline{a_{n-1} B_0 a_{n-1}}$, and $t_n \in (0, 1/2^n)$. Since $e_n e_{n-1} = e_n$ for all n, there is a pure state ϕ of A satisfyig that $\phi(e_n) = 1$ for all n. Then

$$\|\phi - \phi \circ \alpha_{t_n}\| \ge \phi(e_n) - \phi \circ \alpha_{t_n}(e_n) \ge 1 - \|e_n \alpha_{t_n}(e_n)\| \ge 1 - r\,.$$

Thus $t \longmapsto \phi \circ \alpha_t$ is not norm-continuous, which is a contradiction.

Lemma 2.2. *Let B_0 be a non-zero hereditary C^*-subalgebra of A and $\varepsilon>0$. Then there exist $\delta>0$ and $e\in B_0$ with $e\geq 0$, $\|e\|=1$, such that*

$$\sup\{\|e(\alpha_t(x)-x)e\|;\ x\in A,\ \|x\|=1,\ 0<t<\delta\}<\varepsilon .$$

Proof. Suppose the contrary. There are B_0 and $\varepsilon>0$ such that for any $\delta>0$ and e the supremum is not smaller than ε. Then we can adopt the proof of Lemma 6 in [4] with obvious modifications and would obtain a pure state ϕ with the property that $t\longmapsto \phi\circ\alpha_t$ is not norm-continuous, which is a contradiction.

Lemma 2.3. *In the atomic representation π of A, there exists a unitary group V in $\pi(A)''$ such that*

$$V_t\pi(x)V_t^*=\pi\circ\alpha_t(x)\,, \qquad x\in A\,,$$

and

$$I\equiv\{x\in A;\ t\longmapsto V_t\pi(x) \text{ is norm-continuous}\}$$

is a non-zero ideal of A.

Proof. By Lemma 2.2 there exist $\delta>0$ and $e\in A$ with $e\geq 0$, $\|e\|=1$, such that

$$(*) \qquad \sup\{\|e(\alpha_t(x)-x)e\|;\ x\in A,\ \|x\|=1,\ 0<t<\delta\}<1/3\,.$$

We may suppose that there is $a\in A$ with $a\geq 0$, $\|a\|=1$, $ea=a$. Let $r<1$ be a number given for $\varepsilon=1/3$ in the following lemma. By Lemma 2.1, there exist $\delta_1\in(0,\delta)$ and a non-zero hereditary C^*-subalgebra B_1 of $\overline{aAa}$ such that

$$\inf\{\|x\alpha_t(x)\|;\ x\in B_1,\ 0\leq x,\ \|x\|=1,\ 0<t<\delta_1\}>r\,.$$

We may further suppose that there is $e_1\in\overline{aAa}$ with $e_1\geq 0$, such that

$$B_1=\{x\in A;\ e_1x=xe_1=x\}\,.$$

Note that $(*)$ holds for e_1, δ_1 instead of e, δ respectively.

Let ρ be an irreducible representation of A with $\rho(B_1)\neq(0)$. Then there exists a weakly continuous unitary group u on H_ρ such that

$$u_t\rho(x)u_t^*=\rho\circ\alpha_t(x)\,, \qquad x\in A\,.$$

Let Φ be a unit vector in $[\rho(B_1)H_\rho]$. Then $\rho(e_1)\Phi=\Phi$, and Φ satisfies

$$|(\Phi, u_t\Phi)|\geq r$$

for $t\in(0,\delta_1)$ (see ([7], 5.3) or proof of ([5], 1.1)). Then, since r is chosen for $\varepsilon=1/3$ in the following lemma, we have for some $\lambda\in\mathbf{R}$,

$$\|\Phi-e^{i\lambda t}u_t\Phi\|<1/3\,, \qquad t\in[0,\delta_1]\,.$$

Now we denote $e^{i\lambda t}u_t$ by u_t. As in the proof of Lemma 5 in [4],

$$\|(\rho(e_1)u_t-\rho(e_1))\rho(x)\Phi\| \le \|\rho(e_1)u_t\rho(x)u_t^*\rho(e_1)\Phi-\rho(e_1)\rho(x)\rho(e_1)\Phi\|+\|\rho(x)\|\cdot\|u_t^*\Phi-\Phi\|$$
$$\le \|e_1(\alpha_t(x)-x)e_1\|+\|x\|\cdot\|\Phi-u_t\Phi\|$$

Hence, since ρ is irreducible,

$$\|\rho(e_1)u_t-\rho(e_1)\|<2/3\,, \qquad 0<t<\delta_1\,.$$

For each irreducible representation ρ with $\rho(B_1)\neq(0)$, we fix u in the above way, and for ρ with $\rho(B_1)=(0)$, we choose u arbitrarily (such that $u_t\rho(x)u_t^*=\rho\alpha_t(x)$, $x\in A$). By taking the direct sum V of u so obtained in the atomic representation π of A, we have that

$$\|\pi(x)V_t-\pi(x)\|\le 2/3\,, \qquad 0<t<\delta_1$$

for $x\in B_1$ with $\|x\|=1$. This implies that $B_1\subset I$ (Lemma 4 in [4]).

Remark. In the above proof we have used that

$$\sup\{\|e(\alpha_t(x)-x)e\|;\ x\in A,\ \|x\|=1,\ 0<t<\delta\}$$

can be small for some e and δ. Instead of this we could use, with straightforward modifications, that

$$\sup\{\|\alpha_t(x)-x\|;\ x\in B,\ \|x\|=1,\ 0<t<\delta\}$$

can be small for some non-zero hereditary C^*-subalgebra B of A and some $\delta>0$, which is shown by Lemma 2.1 and ([7], 5.1).

Lemma 2.4. *Let $\varepsilon>0$, Φ a unit vector, and u a weakly continuous unitary group. Then there exist $r<1$, which depends only on ε, and $\lambda\in\mathbf{R}$ such that if for some $\delta>0$*

$$|(\Phi, u_t\Phi)|\ge r\,, \qquad t\in(0,\delta)\,,$$

then

$$\|\Phi-e^{i\lambda t}u_t\Phi\|<\varepsilon\,, \qquad t\in(0,\delta)\,.$$

Proof. Suppose that r is close to 1. Define f as a continuous function on $[0,\delta]$ by

$$\exp if(t)=(\Phi, u_t\Phi)/|(\Phi, u_t\Phi)|\,, \qquad f(0)=0\,.$$

By changing u_t by $\exp(-if(\delta)t/\delta)u_t$, we may suppose that $f(\delta)=0$. Now for $t\in[0,\delta]$,

$$|(\Phi, u_{t+s}\Phi)-(\Phi, u_t\Phi)(\Phi, u_s\Phi)|=|(\Phi-e^{-if(t)}u_t\Phi, u_{t+s}\Phi)-(\Phi, u_t\Phi-e^{if(t)}\Phi)(\Phi, u_s\Phi)|$$
$$\le 2\|\Phi-e^{-if(t)}u_t\Phi\|<2\sqrt{2}\sqrt{1-r}\,.$$

Hence for $t, s\in[0,\delta]$ with $t+s\in[0,\delta]$

$$|\exp if(t+s)-\exp if(t)+if(s)|\le|(\Phi, u_{t+s}\Phi)-(\Phi, u_t\Phi)(\Phi, u_s\Phi)|+1-r+1-r^2$$
$$\le 2\sqrt{2}\sqrt{1-r}+2-r-r^2\equiv 2\sin\theta/2\,,$$

where θ is chosen from $(0, \pi/6)$ for r sufficiently close to 1. Thus, using the continuity of f, we have

$$|f(t+s)-f(t)-f(s)| \leq \theta .$$

Let t_0 be a point of $[0, \delta]$ such that

$$|f(t_0)| = \max\{|f(t)|;\ 0 \leq t \leq \delta\} .$$

Suppose that $f(t_0) > 3\theta$. Then $t_0 > \delta/2$. Because, otherwise $|f(2t_0) - 2f(t_0)| \leq \theta$ implies that $f(2t_0) \geq 2f(t_0) - \theta > f(t_0) + 2\theta$, a contradiction. Since $|f(t_0) + f(\delta - t_0)| \leq \theta$, we obtain

$$f(\delta - t_0) \leq -f(t_0) + \theta$$

and so

$$f(2\delta - 2t_0) \leq 2f(\delta - t_0) + \theta \leq -2f(t_0) + 3\theta < -f(t_0)$$

which is a contradiction. Hence $f(t_0) \leq 3\theta$. Similarly $f(t_0) \geq -3\theta$ i.e., $|f(t)| \leq 3\theta$ for $t \in [0, \delta]$.

Lemma 2.5. *The Borchers spectrum* $\mathbf{T}_B(\alpha_t)$ *of* α_t *is* $\{1\}$ *for each* $t \in \mathbf{R}$ *(see* ([8], 8.8) *for Borchers spectrum).*

Proof. Using Lemma 2.1 we take $\delta > 0$ and a non-zero hereditary C^*-subalgebra B of A such that

$$\inf\{\|x\alpha_t(t)\|;\ x \in B,\ 0 \leq x,\ \|x\| = 1,\ 0 < t < \delta\} > 1/2 .$$

If I denotes the ideal generated by B, then $T_B(\alpha_t | I) = \{1\}$ for all t. Otherwise there exist $t \in (0, \delta)$ and an ideal J of I such that α_t is freely acting on J. Then, since $B \cap J \neq (0)$, we must have

$$\inf\{\|x\alpha_t(x)\|;\ x \in B \cap J,\ x \geq 0,\ \|x\| = 1\} = 0$$

which is a contradiction (cf. [6]).

Let (I_i) be a maximal family of such ideals such that $I_i I_j = (0)$ for $i \neq j$. Then, by Lemma 2.1, the ideal generated by (I_i) is essential in A. Therefore it follows that $\mathbf{T}_B(\alpha_t) = \{1\}$.

Now we come to the proof of α being almost uniformly continuous.

Let V be the unitary group on the atomic representation space given in Lemma 2.3. Let H be the infinitesimal generator of V and E its spectral resolution:

$$H = \int \lambda dE(\lambda) .$$

Let, for $n > 0$,

$$D_n = \{x \in A;\ E[-n, n]\pi(x)E[-n, n] = \pi(x)\} .$$

Then D_n is an α-invariant hereditary C^*-subalgebra of A and the restriction of α to D_n is uniformly continuous. Hence we have only to show that D_n is non-zero for large n.

Since $T_B(\alpha_1)=\{1\}$, by ([8], 8.8.7) there is a non-zero α_1-invariant hereditary C^*-subalgebra B of A such that $B\subset I$ and

$$\mathrm{Sp}(\alpha_1 \mid B)\subset\{e^{is};\ |s|<\pi/2\}.$$

Then, for an irreducible representation ρ with $\rho(B)\neq(0)$, there is $\lambda\in\mathbf{R}$ such that

$$\mathrm{Sp}(u_1 \mid [\rho(B)H_\rho])\subset\{e^{i(\lambda+s)};\ |s|<\pi/4\},$$

where u is the unitary group which implements α (chosen in the proof of Lemma 2.3). Since $u_t u_1 u_t^*=u_1$ and u_t^* maps the subspace $[\rho\circ\alpha_t(B)H_\rho]$ onto $[\rho(B)H_\rho]$,

$$\begin{aligned}\mathrm{Sp}(u_1 \mid [\rho\circ\alpha_t(B)H_\rho])&=\mathrm{Sp}(u_t u_1 u_t^* \mid [\rho\circ\alpha_t(B)H_\rho])\\&=\mathrm{Sp}(u_1 \mid [\rho(B)H_\rho]).\end{aligned}$$

Let B_1 be the α-invariant hereditary C^*-subalgebra generated by B, i.e., the closed linear span of $\alpha_t(B)I\alpha_s(B)$, $0\le t, s\le 1$. Then, since $[\rho(B_1)H_\rho]$ is the closed linear span of $[\rho\circ\alpha_t(B)H_\rho]$, $0\le t\le 1$, we obtain

$$\mathrm{Sp}(u_1 \mid [\rho(B_1)H_\rho])=\mathrm{Sp}(u_1 \mid [\rho(B)H_\rho]),$$

which implies

$$\mathrm{Sp}(\mathrm{Ad}\,u_1 \mid \rho(B_1))\subset\{e^{is};\ |s|<\pi/2\}.$$

Since this is true for any irreducible representation ρ with $\rho(B_1)\neq(0)$, which is equivalent to $\rho(B)\neq(0)$, we have

$$\mathrm{Sp}(\alpha_1 \mid B_1)\subset\{e^{is};\ |s|\le\pi/2\}.$$

Then one can define $\delta=\mathrm{Log}\,\alpha_1 \mid B_1$ as an operator on B_1, which is a *-derivation commuting with $\alpha_s \mid B_1$, $s\in\mathbf{R}$. Define an automorphism group β of B_1 by

$$\beta_t=\alpha_t\circ e^{-t\delta}=e^{-t\delta}\circ\alpha_t.$$

Let h be a self-adjoint element of $\pi(B_1)''$ satisfying

$$\pi\circ\delta(x)=[ih, \pi(x)], \qquad x\in B_1.$$

Since $V_t\in\pi(A)''$, and $(V_t h V_t^*-h) \mid [\pi(B_1)H_\pi]\in\pi(B_1)'$, it follows that

$$V_t h V_t^*=h.$$

Since β is periodic, there exists $e\in B_1{}^\beta$ such that $e\ge 0$, $\|e\|=1$ and the hereditary C^*-subalgebra

$$D=\{x\in B_1;\ ex=xe=x\}$$

is non-zero. Since $t\to e^{-ith}V_t\pi(e)$ is norm-continuous, the restriction of $H-h$ to the range P of $\pi(D)$ must be bounded. Since the spectral projection of $H-h$ corresponding to the interval $[-p, p]$ is majorized by $E[-p-\|h\|, p+\|h\|]$, there exists n such that $P\le E[-n, n]$. This implies that $D\subset D_n$, which concludes the proof.

Remark. If α is almost uniformly continuous on A, there exists an α-invariant hereditary C^*-subalgebra B of A such that B generates an essential ideal of A and $\alpha \mid B$ is uniformly continuous. In particular the Borchers spectrum of α is trivial.

To show this one notes that if $\alpha \mid B$ is uniformly continuous, then, for any $\varepsilon > 0$, there exists a non-zero α-invariant hereditary C^*-subalgebra B_1 of B such that the generator of $\alpha \mid B_1$ has norm less than ε (e.g., pick up a non-zero x of B whose α-spectrum is contained in $(\|\delta\| - \varepsilon/2, \|\delta\|]$, where δ is the generator of $\alpha \mid B$, and let B_1 be the closed linear span of $\alpha_t(x) B \alpha_s(x^*)$, $t, s \in \mathbf{R}$). One can form a maximal family (B_i) of α-invariant hereditary C^*-subalgebras such that $B_i A B_j = (0)$ for $i \neq j$ and the generator of $\alpha \mid B_i$ has norm less than one for each i. Then the closed linear span of (B_i) has the desired property.

3. Proof (2) ⇒ (3)

In this section we show that if α is almost uniformly continuous, then it is universally weakly inner. The proof is straightforward if we admit the following:

Lemma 3.1. *Let B be an α-invariant hereditary C^*-subalgebra of A such that $\alpha \mid B$ is uniformly continuous. Let I be the ideal generated by B. Then I is α-invariant and $\alpha \mid I$ is universally weakly inner.*

Proof. This follows from the fact that $\alpha \mid B$ is universally weakly inner ([9], 4.1.7). See, e.g. ([8], 8.9.1).

Lemma 3.2. *Let I and J be α-invariant ideals of A with $I \subset J$. Suppose that $\alpha \mid I$ and $\dot{\alpha} \mid J/I$ are universally weakly inner, where $\dot{\alpha}$ is the action on A/I induced by α. Then $\alpha \mid J$ is universally weakly inner.*

Proof. The universal representation of I extends uniquely to a representation of J through the canonical map from J into the multiplier algebra of I. The universal representation of J/I can be regarded as a representation of J. Those operations are consistent with the actions α, $\dot{\alpha}$, and the direct sum of those representations is quasi-equivalent to the universal representation of J.

Lemma 3.3. *Let (I_i) be a family of α-invariant ideals of A. Suppose that $\alpha \mid I_i$ is universally weakly inner for each i. Then the restriction of α to the ideal generated by (I_i) is universally weakly inner.*

Proof. Assuming the index set to be a well-ordered set, define for each ordinal k, J_k to be the ideal generated by I_i, $i < k$.

Let k be an ordinal and suppose that $\alpha \mid J_i$ is universally weakly inner for each $i < k$. If k is isolated, $J_k/J_{k-1} = I_{k-1}/J_{k-1}$, and so $\alpha \mid J_k$ is universally weakly inner. If k is a limit ordinal, J_k is the closure of the union of J_i, $i < k$. By considering the universal representation of J_i as a subrepresentation of the one of J_k we can easily conclude that $\alpha \mid J_k$ is universally weakly inner.

Now we can prove that α is universally weakly inner under the assumption that α

is almost uniformly continuous. Let I be the maximal α-invariant ideal of A such that $\alpha\,|\,I$ is universally weakly inner. If $I \neq A$, we find a non-zero $\dot{\alpha}$-invariant hereditary C^*-subalgebra B of A/I such that $\dot{\alpha}\,|\,B$ is uniformly continuous. By Lemmas 3.1 and 3.2 this contradicts the maximality of I.

4. Single automorphisms

We give a result on single automorphisms similar to the one-parameter case.

We call an automorphism α of a C^*-algebra A to be *almost derivable* if, for any α-invariant ideal I of A, the automorphism $\dot{\alpha}$ of A/I induced by α has a non-zero $\dot{\alpha}$-invariant hereditary C^*-subalgebra B of A/I on which $\dot{\alpha}$ is derivable, i.e., $\dot{\alpha}\,|\,B = \exp\delta$ with some $*$-derivation δ on B.

Theorem 4.1. *Let α be an automorphism of a separable C^*-algebra A. Then the following conditions are equivalent:*

(1) *α is extendible in every irreducible representation,*

(2) *α is almost derivable,*

(3) *α is universally weakly inner.*

Proof. Suppose that α satisfies (1). Then α does not have an ideal on which α is freely acting, i.e., $\mathbf{T}_B(\alpha) = \{1\}$ ([6], [7]). Then there exists a non-zero α-invariant hereditary C^*-subalgebra B of A such that $\alpha\,|\,B = \exp\delta$ with some $*$-derivation δ of B. Since the above procedure applies to quotients of A, we have the implication (1)$\Rightarrow$(2). Since automorphisms of the type $\exp\delta$ are universally weakly inner, we obtain the implication (2)$\Rightarrow$(3) by the same reasoning as given in Sect. 3. (3)$\Rightarrow$(1) is obvious.

The Connes spectrum $\mathbf{T}(\alpha)$ of an automorphism α of a separable C^*-algebra A depends only on the behaviour of α on the spectrum $\hat{A}$ of A. Namely,

Proposition 4.2. *Let α, β be automorphisms of a separable C^*-algebra A. Suppose that $\pi \circ \alpha = \pi \circ \beta$, $\pi \in \hat{A}$. Then the set of α-invariant ideals of A equals the set of β-invariant idelas, and $\mathbf{T}(\dot{\alpha}\,|\,J/I) = \mathbf{T}(\dot{\beta}\,|\,J/I)$. for any pair I, J of invariant ideals with $I \subset J$.*

Proof. The statement on invariant ideals is obvious.

Suppose that there is a pair I, J such that $\mathbf{T}(\dot{\alpha}\,|\,J/I) \neq \mathbf{T}(\dot{\beta}\,|\,J/I)$. Since $\pi \circ \dot{\alpha} = \pi \circ \dot{\beta}$ for $\pi \in (J/I)^{\wedge}$, we may assume that $\mathbf{T}(\alpha) \neq \mathbf{T}(\beta)$. Further by ([6], 3.1) we may as well assume that $\mathbf{T}_B(\alpha) = \mathbf{T}(\alpha)$ and $\mathbf{T}_B(\beta) = \mathbf{T}(\beta)$. Suppose that the order of $\mathbf{T}(\alpha)$ is finite, say n. Then there is a non-zero α-invariant hereditary C^*-subalgebra B of A such that

$$\mathrm{Sp}\,(\alpha^n\,|\,B) \subset \{e^{is};\ |s| < 2\pi/3\}\,.$$

Let I be the ideal generated by B. Then there is a unitary u of I^{**} such that $\alpha^n(x) = uxu^*$, $x \in I$, and $\bar{\alpha}(u) = u$, where $\bar{\alpha}$ denotes the unique extension of $\alpha\,|\,I$ to I^{**}. Since $\beta \circ \alpha^{-1}$ is extendible in every irreducible representation, there is a unitary w of I^{**} such that $\beta(x) = w\alpha(x)w^*$, $x \in I$. Then $\beta^n\,|\,I$ is implemented by $v = w\bar{\alpha}(w) \cdots \bar{\alpha}^{n-1}(w)u$. Since $\bar{\beta}(v) = v$ by calculation and $\mathbf{T}_B(\beta^n\,|\,I) = \{1\}$, there exists, for any $\varepsilon > 0$, a non-zero

β-invariant hereditary C^*-subalgebra B_1 of I such that

$$\mathrm{Sp}(\beta^n \mid B_1) \subset \{e^{is};\ |s| < \varepsilon\}$$

(cf. [6], 3.3). This implies that $\mathbf{T}(\beta)$ is contained in the subgroup of order n of $\mathbf{T}$, i.e., $\mathbf{T}(\beta) \subset \mathbf{T}(\alpha)$.

By changing the roles of α, β, we would eventually have $\mathbf{T}(\alpha) = \mathbf{T}(\beta)$, which is a contradiction.

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