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ASYMPTOTIC BEHAVIOR OF STATISTICAL ESTIMATORS CONSTRUCTED ON ABSOLUTELY REGULAR SEQUENCES

By

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1. Introduction.

Recently, in a series of papers, Ibragimov and Khas'minskii studied the asymptotic behavior of a normalized likelihood function of the form

$$Z_n(\theta)=\frac{\prod_1^n f(X_j, \theta_0+\theta/\varphi(n))}{\prod_1^n f(X_j, \theta_0)}$$

treating it as a random function of θ where $(X_1, \cdots, X_n)$ is a repeated sample of size n from a population P_θ depending on an unknown parameter $\theta\in\Theta$. Specially, in [3], they proved that under some conditions Z_n converges weakly to some Gaussian process Z. Their results give many powerful tools in the investigation of asymptotic problems for statistics.

In 1977, Bakirov [1] reported that he proved the analogous weak convergence theorem when $\{X_k\}$ is a strictly stationary sequence of random variables satisfying some absolute regularity condition. But, the conditions he considered are difficult to check in a certain sense.

The object of this paper is to prove the weak convergence theorem for observations satisfying some absolute regularity conditions under different restrictions from Bakirov's (Theorem 4.4).

2. Assumptions and notations.

Let Θ be an open set in the real line R^1. Suppose that a family of probability space $(\Omega, F, P_\theta, \theta\in\Theta)$ is given. Let $\{\xi_n, -\infty<n<\infty\}$ be a strictly stationary sequence of random variables with values in the measurable space $(\mathscr{X}, \mathscr{A})$. Let $\xi_m{}^n=(\xi_m, \cdots, \xi_n)$ denote a random element in the direct product

$$\mathscr{X}^{n-m+1}=\mathop{\times}_{i=m}^{n} \mathscr{X}_i \quad (\mathscr{X}_i=\mathscr{X}, i=m, \cdots, n),$$

let $\mathscr{F}_m{}^n$ denote the smallest σ-algebra generated by $\xi_m{}^n$, and let $P_\theta{}^n$ be the projection of the measure P_θ on $\mathscr{F}_1{}^n$.

We assume that the sequence $\{\xi_n\}$ satisfies an absolute regularity (a.r.) condition

(2.1) $$\beta_n=\beta(n)=\sup_{\theta\in\Theta}\beta(n,\theta)\downarrow 0$$

as $n\to\infty$ where for each $\theta\in\Theta$

(2.2) $$\beta(n,\theta)=E_\theta\left\{\sup_{A\in\mathscr{F}_n^\infty}|P_\theta(A\,|\,F_{-\infty}^0)-P_\theta(A)|\right\}\cdot$$

and $E_\theta(\cdot)$ denotes the expectation with respect to $P_\theta(\cdot)$.

Let $(\mathscr{X},\mathscr{A},\nu)$ be a measurable space with σ-finite measure ν. Put

$$\mathscr{A}^n=\bigtimes_{i=1}^n\mathscr{A}_i\quad(\mathscr{A}_i=\mathscr{A},i=1,\cdots,n)\quad(n\geqq 1).$$

For each $\theta\in\Theta$ let $\hat{P}_\theta$ and $\hat{P}_\theta{}^n$ $(n\geqq m\geqq 1)$ be the probability measures defined respectively by

$$\hat{P}_\theta(A)=P_\theta(\xi_{-\infty}^\infty\in A)(A\in\mathscr{A}^\infty)$$

and

$$\hat{P}_\theta{}^n(B)=P_\theta(\xi_1{}^n\in B)(B\in\mathscr{A}^n)\,.$$

We assume that the measure $\hat{P}_\theta{}^n$ is absolutely continuous with respect to the product measure $\nu^n=\nu\times\cdots\times\nu$ and defines the probability density

(2.3) $$f(x_1{}^n,\theta)=\frac{d\hat{P}_\theta{}^n}{d\nu^n}(x_1{}^n)$$

which is $\mathscr{A}^n\times\mathscr{B}$-measurable. Here, $x_q{}^p\in\mathscr{X}^{p-q+1}$ and $\mathscr{B}$ is the smallest σ-algebra of all Borel subsets of R^1. For $n\geqq p\geqq 1$ and $\theta\in\Theta$ let $f(x_p{}^n\,|\,x_1{}^{p-1},\theta)$ be the conditional probability density function or the probability density function, i.e.,

$$P_\theta(\xi_1{}^{p-1}\in B,\xi_p{}^n\in A)=\int_B f(x_1{}^{p-1},\theta)\nu^{p-1}(dx_1{}^{p-1})\int_A f(x_p{}^n\,|\,x_1{}^{p-1},\theta)\nu^{n-p+1}(dx_p{}^n)$$
$$(B\in\mathscr{A}^{p-1}\text{ and }A\in\mathscr{A}^{n-p+1})$$

where $f(x_1{}^n\,|\,x_1{}^0,\theta)=f(x_1{}^n,\theta)$ $(m=1)$ denotes the probability density function.

Let us now formulate the restrictions to be imposed on the family $\hat{P}_\theta$, $\theta\in\Theta$ which will be used below (cf. [1] and [3]).

Conditions of group I. I_1. The parameter set Θ is an open interval (bounded or unbounded) of R^1.

I_2. For any n the functions $f(x_n\,|\,x_1{}^{n-1},\theta)$ are defined for all $x_1{}^n\in\mathscr{X}^n$ and $\theta\in\Theta$ and $\mathscr{A}^n\times\mathscr{B}$-measurable.

I_3. If $\theta\neq\theta'$, then $\hat{P}_\theta=\hat{P}_{\theta'}$. More precisely, if $\theta\neq\theta'$, then for all n and p $(n\geqq p)$ and for ν^{p-1}-almost all $x_1{}^{p-1}$

$$\int_{X^{n-p+1}} |f(x_p{}^n \mid x_1{}^{p-1}, \theta) - f(x_p{}^n \mid x_1{}^{p-1}, \theta')| \, v^{n-p+1}(dx_p{}^n) > 0 \,. \tag{2.5}$$

Whenever the integrations with respect to $x_p{}^n$ are over all of X^{n-p+1} we shall agree to omit the region of integration and to write $v(dx_p{}^n)$ instead of $v^{n-p+1}(dx_p{}^n)$.

Conditions of group II. II_1. For any n ($\geqq 1$) and for any fixed $x_1{}^n$, the function $f(x_n \mid x_1{}^{n-1}, \theta)$ is defined and continuously twice differentiable in the closure Θ^c of Θ.

We put

$$f'(x_p{}^n \mid x_1{}^{p-1}, \theta) = \frac{\partial}{\partial \theta} f(x_p{}^n \mid x_1{}^{p-1}, \theta) \,. \tag{2.6}$$

II_2. For all n ($\geqq 1$) and for v^{n-1}-almost all $x_1{}^{n-1}$

$$\int f'(x_n \mid x_1{}^{n-1}, \theta) v(dx_n) = \frac{\partial}{\partial \theta} \int f(x_n \mid x_1{}^{n-1}, \theta) v(dx_n) = 0 \,. \tag{2.7}$$

For any n and k ($n \geqq k$), let

$$U_{n,k}(\theta) = \begin{cases} \dfrac{f'(\xi_n \mid \xi_k{}^{n-1}, \theta)}{f(\xi_n \mid \xi_k{}^{n-1}, \theta)} \,, & \text{if } \ f(\xi_n \mid \xi_k{}^{n-1}, \theta) > 0 \\ 0 \,, & \text{otherwise.} \end{cases} \tag{2.8}$$

II_3. There exists a number δ (>0) such that

$$\sup_{\theta \in \Theta} \sup_{n \geqq 1} \ E_\theta |U_{n,1}(\theta)|^{4+\delta} < \infty \tag{2.9}$$

and

$$\sup_{\theta \in \Theta} \sup_{n \geqq 1} E_\theta \left| \frac{\partial U_{n,1}(\theta)}{\partial \theta} \right|^{2+\delta} < \infty \,. \tag{2.10}$$

For each n ($\geqq 1$) and for each $\theta \in \Theta$, put

$$I(\xi_n \mid \xi_1{}^{n-1}, \theta) = E_\theta |U_{n,1}(\theta)|^2 \tag{2.11}$$

and for each $\theta \in \Theta$

$$I(\theta) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} g_i(\theta) \tag{2.12}$$

where

$$g_i(\theta) = I(\xi_i \mid \xi_1{}^{i-1}, \theta) + 2 \sum_{j=i+1}^{\infty} E_\theta\{U_{i,1}(\theta) U_{j,1}(\theta)\} \qquad (i = 1, 2, \cdots) \tag{2.13}$$

(The existence of limits in the right-hand sides of (2.12) and (2.13), respectively, are verified in Propositions 3.2 and 3.3 (below).)

II_4. $I(\theta)$ is a positive and continuous function of $\theta \in \Theta^c$.

II_5. There exists a number d ($\geqq 0$) such that

$$\sup_{\theta \in \Theta}(1+|\theta|)^{-d} I(\theta) < \infty . \tag{2.14}$$

Conditions of group III. III_1. The sequence $\{\beta(n)\}$ defined by (2.1) satisfy the condition

$$\sum_n k\beta_n^* < \infty \tag{2.15}$$

where $\beta_n^* = \beta_n^{\delta/2(2+\delta)}$ and δ is the same one in II_3.

III_2. For any $c>0$ and k ($\geqq 1$), let

$$\gamma(k, \theta, c) = \sup_{|\theta'-\theta| \leqq c} \sup_{n \geqq k} \{E_\theta | U_{n,1}(\theta') - U_{n,n-k}(\theta')|^4\}^{1/4} \tag{2.16}$$

There exists a positive number c_0 such that

$$\gamma_k = \gamma(k) = \sup_{\theta \in \Theta} \gamma(k, \theta, c_0) = O(k^{-2}) \tag{2.17}$$

as $k \to \infty$.

Conditions of group IV. IV_1. There exists a positive number d_1 such that for all θ and θ' in Θ

$$\sup_{n \geqq 1} \operatorname{ess.sup}_{x_1^{n-1}} |\theta - \theta'|^{d_1} \int \sqrt{f(x_n | x_1{}^{n-1}, \theta) f(x_n | x_1{}^{n-1}, \theta')}\, \nu(dx_n) < \infty . \tag{2.18}$$

IV_2. Instead of III_1 and III_2, $\beta(n) = O(e^{-\lambda_1 n})$ and $\gamma(n) = O(e^{-\lambda_2 n})$ hold as $n \to \infty$, where λ_1 and λ_2 are some positive constants.

Remark 2.1. If Conditions I–III are satisfied, then it is obvious that for any fixed p, q and $\theta \in \Theta \{f'(\xi_{n+1}^{n+p} | \xi_{n-q}^n, \theta) / f(\xi_{n+1}^{n+p} | \xi_{n-q}^n, \theta)\}$, $n = 0, \pm 1, \pm 2, \cdots$ is a strictly stationary sequence satisfying the a.r. condition with the same coefficient β_n as that of $\{\xi_n\}$.

Remark 2.2. It is obvious that the condition $\gamma(n) = O(e^{-\lambda_2 n})$ is satisfied if $\{\xi_n\}$ possesses the r-th order Markov property, where $r \geqq 1$, i.e., for any $\theta \in \Theta$

$$P_\theta(\xi_{n+1} \in A | \xi_0, \cdots, \xi_n) = P_\theta(\xi_{n+1} \in A | \xi_{n-r+1}, \cdots, \xi_n) \qquad (n \geqq r-1) .$$

In what follows, K (with or without subscript) will stand for a quantity not depending on the parameters occurring in the discussion and the same letter K will be used to denote different constants even within the same formula. For any $s \in R^1$ $[s]$ denotes the largest integer p such that $p \leqq s$. Instead of $E_\theta(\cdot)$, $P_\theta(\cdot)$ and $U_{n,p}(\theta)$ where θ is the true value of the parameter, let us agree to write $E(\cdot)$, $P(\cdot)$ and $U_{n,p}$. For any integrable random variable $|\eta|^r$ we write

$$\|\eta\|_r = \{E|\eta|^r\}^{1/r} \qquad (r>0) .$$

3. Preliminary results.

In this section, we always assume that Conditions I, III and II_1–II_4 are satisfied. The next lemma is a special case of Deo's result ([2], Lemma 1).

Lemma 3.1. *Let r_1, r_2, r_3 be positive numbers such that $r_1^{-1}+r_2^{-1}+r_3^{-1}=1$. Suppose that X and Y are random variables measurable with respect to $\mathscr{F}^0_{-\infty}$, $\mathscr{F}_n$ respectively and assume further that $\|X\|_{r_1}<\infty$ and $\|Y\|_{r_2}<\infty$. Then for all $\theta\in\Theta$*

$$|E_\theta(XY)-E_\theta(X)E_\theta(Y)|\leqq 10\beta_n^{1/r_3}\|X\|_{r_1}\|Y\|_{r_2}\,. \tag{3.1}$$

For $q<p\leqq n$, let

$$V_{n,q,p}(\theta)=U_{n,q}(\theta)-U_{n,p}(\theta) \tag{3.2}$$

and for brevity, put $U_{n,1}(\theta)=U_n(\theta)$ and $V_{n,1,p}(\theta)=V_{n,p}(\theta)$ $(1\leqq p<n)$. Since $U_{n,p}\in F_p^n$ and by II_2 $EU_{n,p}=0$ for all n and p $(p\leqq n)$, so by Lemma 3.1 and II_3

$$\begin{aligned} |EU_iU_{j,s}| &\leqq |EU_i\cdot EU_{j,s}|+10\|U_i\|_{2+\delta}\|U_{j,s}\|_{2+\delta}\beta_{s-i}^{\delta/(2+\delta)} \\ &\leqq K\beta^*_{s-i} \qquad (i<s\leqq j)\,. \end{aligned} \tag{3.3}$$

Further by Hölder's inequality and (2.17)

$$\begin{aligned} |EU_{i,q}\cdot V_{j,s,p}| &\leqq \|U_{i,q}\|_{2+\delta}\|V_{j,s,p}\|_{(2+\delta)/(1+\delta)} \\ &\leqq K\gamma_{j-p} \qquad (s\leqq p<j) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} |EV_{j,q,p}|^2 &\leqq \|V_{j,q,p}\|_{2+\delta}\|V_{j,q,p}\|_{(2+\delta)/(1+\delta)} \\ &\leqq K\gamma_{j-p} \qquad (q\leqq p<j)\,. \end{aligned} \tag{3.5}$$

Hence, for any s $(i<s<j)$

$$\begin{aligned} |EU_iU_j| &\leqq |EU_iU_{j,s}|+|EU_iV_{j,s}| \\ &\leqq K\{\beta^*_{s-i}+\gamma_{j-s}\}\,. \end{aligned} \tag{3.6}$$

Further, if $q<p<i<s<j$, then $V_{i,q,p}\in F^i_{-\infty}$ and so by II_2 and Lemma 3.1

$$\begin{aligned} |EV_{i,q,p}U_{j,p}| &\leqq |EV_{i,q,p}U_{j,s}|+|EV_{i,q,p}V_{j,p,s}| \\ &\leqq K\gamma_{i-p}\{\beta^*_{s-i}+\gamma_{j-s}\}\,. \end{aligned} \tag{3.7}$$

Proposition 3.1. *For each $\theta\in\Theta$ $\lim_{n\to\infty} I(\xi_n\,|\,\xi_1^{n-1},\theta)$ exists and is finite.*

Proof. Let $\theta\in\Theta$ be fixed. Since for any n and q $(q<n)$

$$|U_n|^2=|U_{n,n-q+1}|^2+2U_{n,n-q+1}V_{n,n-q+1}+|V_{n,n-q+1}|^2\,,$$

so using Remark 2.1, (3.4) and (3.5)

$$E|U_q|^2-K_1\gamma_q \leqq I(\xi_n|\xi_1^{n-1},\theta)\leqq E|U_q|^2+K_1\gamma_q+\gamma_q^2\,.$$

As $\gamma_q \downarrow 0$ ($q\to\infty$), so for any $\varepsilon>0$ we can choose p such that

$$2K_1\gamma_p+\gamma_p^2<\varepsilon\,.$$

Let $n_0\geqq p$. Then, for all m, n ($>n_0$)

$$|I(\xi_m|\xi_1^{m-1},\theta)-I(\xi_n|\xi_1^{n-1},\theta)|<\varepsilon\,,$$

which implies that

$$\lim_{n\to\infty} I(\xi_n|\xi_1^{n-1},\theta)$$

exists. The finiteness of the limit is now obvious, and so the proof is completed.

Proposition 3.2. *The series in* (2.12) *are absolutely convergent and the limit in* (2.11) *exists and is finite.*

Proof. Let $\theta\in\Theta$ be fixed. To prove the first-part it is enough to show that for each i ($\geqq 1$) the series

$$h_i=h_i(\theta)=\sum_{j=i+1}^{\infty} EU_iU_j$$

is absolutely convergent. In (3.6) putting $s=[(i+j)/2]+1$ for each j and using III we obtain

$$\sum_{j=i+1}^{\infty}|EU_iU_j|\leqq \|U_i\|_2\|U_{i+1}\|_2+\|U_i\|_2\|U_{i+2}\|_2$$
$$+K\sum_{j=i+2}^{\infty}\left\{\beta^*\left(\left[\frac{j-i}{2}\right]\right)+\gamma\left(\left[\frac{i+j}{2}\right]\right)\right\}\leqq K\,,$$

which is the desired result.

To prove the latter-half, it suffices to show that for each $\theta\in\Theta$ $\lim_{i\to\infty} h_i(\theta)$ exists and is finite, since by Proposition 3.1

$$\lim_{n\to\infty} I(\xi_n|\xi_1^{n-1},\theta)$$

exists and is finite. Let i be sufficiently large. For any j ($>i$) and q ($1<q<i$)

$$EU_iU_j=EU_{i,i-q}U_{j,i-q}+EU_{i,i-q}V_{j,i-q}+EV_{i,i-q}U_{j,i-q}+EV_{i,i-q}V_{j,i-q}\,.$$

So, by (3.4)–(3.7)

$$\sum_{j=i+1}^{\infty}\{|EU_{i,i-q}V_{j,i-q}|+|EV_{i,i-q}U_{j,i-q}|+|EV_{i,i-q}V_{j,i-q}|\}$$
$$\leqq K\sum_{j=i+1}^{\infty}\{\gamma_{j-i+q}+\gamma_q(\beta^*_{s_j-i}+\gamma_{j-s_j})+\gamma_q\gamma_{j-i+q}\}$$
$$=M(q)\quad(\text{say})\,,$$

where $s_j=[(i+j)/2]+1$ for each j ($\geqq i+1$). $M(q)$ is obviously independent on i ($>q$) and $\theta\in\Theta$, and from III $M(q)\to 0$ as $q\to\infty$. On the other hand, by stationarity $EU_{i,i-q}U_{j,i-q}=EU_{q+1}U_{j-i+q+1}$. So for all q sufficiently large and for all i ($>q$)

$$h_{q+1}-M(q)\leqq h_i\leqq h_{q+1}+M(q)\,.$$

Hence, by the method used in the proof of Proposition 3.1 we can show that $\{h_i\}$ is a Cauchy sequence, i.e., $\lim_{i\to\infty} h_i$ exists. The finiteness of the limit is easily obtained. So, the proof is completed.

Proposition 3.3. *For any $\theta\in\Theta$*

$$I(\theta)=\lim_{n\to\infty}\frac{1}{n}E\left|\frac{f'(\xi_1{}^n,\theta)}{f(\xi_1{}^n,\theta)}\right|^2 \tag{3.8}$$

Proof. Put $W_n=|f'(\xi_1{}^n,\theta)/f(\xi_1{}^n,\theta)|^2$ and let h_i be as before. Then

$$EW_n^2=E\left|\sum_{j=1}^n U_j\right|^2=\sum_{j=1}^n EU_j{}^2+2\sum_{1\leqq i<j\leqq n}EU_iU_j\,.$$

We note that from III$_1$ $\beta_k=o(k^{-2})$ since $\{\beta_k\}$ is a nonincreasing sequence. Hence, from (3.6) and III$_2$ we obtain that for an arbitrarily fixed integer q ($\geqq 1$)

$$\begin{aligned}\sum_{i=1}^{n-1}\left|h_i-\sum_{j=i+1}^n EU_iU_j\right| &\leqq \sum_{i=1}^{n-1}\sum_{j=n+1}^{\infty}|EU_iU_j|\\ &\leqq K\sum_{i=1}^{n-1}\sum_{j=n+1}^{\infty}\{\beta^*_{s_j-i}+\gamma_{j-s_j}\}\\ &\leqq K\left[q+\sum_{i=1}^{n-1}\sum_{j=n+1}^{\infty}\{\beta^*_{s_j-i}+\gamma_{j-s_j}\}\right]\\ &\leqq K\left[q+n\sum_{k=q+1}^{\infty}\left\{\beta^*\left(\left[\frac{k+q}{2}\right]\right)+\gamma\left(\left[\frac{k+q}{2}\right]\right)\right\}\right]\\ &\leqq K\{q+nq^{-1}\}\end{aligned}$$

where $s_j=[(i+j)/2]+1$ for each j ($\geqq i+1$). Now, putting $q=[n^{1-\alpha}]$ ($0<\alpha<1$), we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\sum_{i=1}^{n-1}\left(h_i-\sum_{j=i+1}^n EU_iU_j\right)\right|=0\,.$$

So, from Proposition 3.2 we have the desired conclusion.

Proposition 3.4. *Let $q=q(n)$ be an integer-valued function such that $q=o(n)$ as $n\to\infty$. Then for any $\theta\in\Theta$*

$$\lim_{n\to\infty}\frac{1}{n}E\left|\frac{f'(\xi_1{}^n\,|\,\xi^0_{-q+1},\theta)}{f(\xi_1{}^n\,|\,\xi^0_{-q+1},\theta)}\right|^2=I(\theta) \tag{3.9}$$

Proof. Let W_n be as before. Since $\{\xi_n\}$ is strictly stationary to prove (3.9) it is enough to show

$$\lim_{n\to\infty}\frac{1}{n}E\left|\frac{f'(\xi_{q+1}^{n+q}\mid\xi_1{}^q,\theta)}{f(\xi_{q+1}^{n+q}\mid\xi_1{}^q,\theta)}\right|^2=I(\theta)\,.$$

As

$$E\left|\frac{f'(\xi_{q+1}^{n+q}\mid\xi_1{}^q,\theta)}{f(\xi_{q+1}^{n+q}\mid\xi_1{}^q,\theta)}\right|^2=EW_{n+q}^2+EW_q{}^2-2EW_{n+q}W_q$$

and by Proposition 3.3 and Schwarz's inequality

$$\lim_{n\to\infty}\frac{1}{n}EW_{n+q}^2=\lim_{n\to\infty}\frac{1}{n+q}EW_{n+q}^2=I(\theta)\,,$$

$$EW_q{}^2=O(q)=o(n)$$

$$|EW_{n+q}W_q|\leqq\|W_{n+q}\|_2\|W_q\|_2=o(n)\,,$$

so we have (3.9) and the proof is completed.

4. Limiting behavior of the stochastic process $Z_n(\theta)$.

The following lemma plays a fundamental role in proving main results.

Lemma 4.1. (See [4], Theorem 1.) *Let $\{t_j{}^{(i)};\, j=1, 2, \cdots;\, i=1, 2\}$ be any set of integers such that $t_1{}^{(1)}\leqq t_1{}^{(2)}<t_2{}^{(1)}\leqq t_2{}^{(2)}<t_3{}^{(1)}\leqq t_3{}^{(2)}<\cdots$. For each j, let η_j be $F_{t_j^{(1)}}^{t_j^{(2)}}$-measurable. If*

$$\min_{1\leqq j\leqq k-1}(t_{j+1}^{(1)}-t_j{}^{(2)})=q\,,$$

then

$$P\left(\sum_{j=1}^k\eta_j{}^*<x\right)-2k\beta_q\leqq P\left(\sum_{j=1}^k\eta_j<x\right)\leqq P\left(\sum_{j=1}^k\eta_j{}^*<x\right)+2k\beta_q$$

where $\eta_1{}^*, \cdots, \eta_k{}^*$ are independent random variables and for each j, $\eta_j{}^*$ has the same distribution function (df) as that of η_j.

Now, for each $\theta\in\Theta$ put

$$Z_n(\theta)=\frac{f(\xi_1{}^n,\theta_0+\theta n^{-1/2})}{f(\xi_1{}^n,\theta_0)}=\exp Y_n(\theta)\,,$$

(4.1)

$$Y_n(\theta)=\log\frac{f(\xi_1{}^n,\theta_0+\theta n^{-1/2})}{f(\xi_1{}^n,\theta_0)}\,.$$

First, corresponding to Theorem 2.1 in [3], we prove the following theorem.

Theorem 4.1. *If Conditions* I–III *are satisfied, then as $n\to\infty$ the finite-dimensional distributions of the stochastic process $Z_n(\theta)$ converge to the finite-dimensional distributions of the process*

$$Z(\theta)=\exp\left\{\theta\sqrt{I_0}\zeta-\frac{\theta^2}{2}I_0\right\} \tag{4.2}$$

where ζ is the standardized normal random variables, and $I_0=I(\theta_0)$.

To prove the theorem we need a number of lemmas corresponding to lemmas in [3]. In the followings we always assume that Conditions I–III hold. We introduce some notations. Let $p=p(n)=[n^\alpha]$ for some $\alpha(1/2<\alpha<1)$ and $q=q(n)=[n^{1/3}]$. Let $l=p+q$ and $k=[n/l]$. Further, let $\varepsilon=\varepsilon_n=bn^{-1/2}$ where b is some positive constant. Put

$$\begin{aligned}
y_j&=y_j(\theta)=\log\frac{f(\xi_{jl+1}^{jl+p},\theta_0+\theta n^{-1/2})}{f(\xi_{jl+1}^{jl+p},\theta_0)},\\
\tilde{y}_j&=\log\frac{f(\xi_{jl+1}^{jl+p}\mid\xi_1{}^{jl},\theta_0+\theta n^{-1/2})}{f(\xi_{jl+1}^{jl+p}\mid\xi_1{}^{jl},\theta_0)}-y_j \quad (0\leqq j\leqq k-1),\\
z_j&=z_j(\theta)=\log\frac{f(\xi_{jl+p+1}^{(j+1)l},\theta_0+\theta n^{-1/2})}{f(\xi_{jl+p+1}^{(j+1)l},\theta_0)},\\
\tilde{z}_j&=\log\frac{f(\xi_{jl+p+1}^{(j+1)l}\mid\xi_1{}^{jl+p},\theta_0+\theta n^{-1/2})}{f(\xi_{jl+p+1}^{(j+1)l}\mid\xi_1{}^{jl+p},\theta_0)}-z_j \quad (0\leqq j\leqq k-1),\\
z_k&=Y_n(\theta)-\sum_{j=0}^{k-1}(y_j+\tilde{y}_j)-\sum_{j=0}^{k-1}(z_j+\tilde{z}_j).
\end{aligned} \tag{4.3}$$

We note that for fixed $\theta\in\Theta$ $\{y_1,\cdots,y_k\}$ are identically distributed and satisfy the a.r. condition with coefficient $\beta(n)$.
For any $\tau>0$, let

$$A_{n\tau}=\left\{x_1{}^p\colon\left|\log\frac{f(x_1{}^p,\theta+\varepsilon_n)}{f(x_1{}^p,\theta)}\right|>\tau\right\} \tag{4.4}$$

(here we agree to set $0/0=1$). Further, let

$$a_m(s)=\frac{f'(x_1{}^m,s)}{\sqrt{f(x_1{}^m,s)}}. \tag{4.5}$$

Lemma 4.2.

$$\lim_{n\to\infty}\frac{1}{p\varepsilon}\int_\theta^{\theta+\varepsilon}\int_{A_{n\tau}}a_p{}^2(s)\nu(dx_1{}^p)ds=0. \tag{4.6}$$

Proof. Let

$$B_{\varepsilon\alpha}=\{x_1{}^p\colon|\sqrt{f(x_1{}^p,\theta+\varepsilon)}-\sqrt{f(x_1{}^p,\theta)}|>\alpha\sqrt{f(x_1{}^p,\theta)}\}$$

From Jensen's inequality, Proposition 3.3 and II_4 it follows that

$$\begin{aligned}\int_{B_{\varepsilon\alpha}} f(x_1{}^p, \theta)\nu(dx_1{}^p) &\leqq \frac{1}{\alpha^2}\int_{B_{\varepsilon\alpha}} (\sqrt{f(x_1{}^p, \theta+\varepsilon)}-\sqrt{f(x_1{}^p, \theta)})^2\nu(dx_1{}^p) \\ &\leqq \frac{\varepsilon}{4\alpha^2}\int_{B_{\varepsilon\alpha}}\int_\theta^{\theta+\varepsilon} a_p{}^2(s)ds\nu(dx_1{}^p) \\ &\leqq \frac{\varepsilon^2}{4\alpha^2}\left\{\frac{1}{\varepsilon}\int_\theta^{\theta+\varepsilon}\int a_p{}^2(s)\nu(dx_1{}^p)ds\right\} \\ &\leqq \frac{\varepsilon^2}{4\alpha^2} pI(\theta)(1+o(1)) \\ &= O(k^{-1}).\end{aligned}$$

Hence, by the method used in the proof of Lemma 2.1 in [3] we have that

$$\max_{|\theta-\theta_0|<\varepsilon}\int_{A_{n\tau}} f(x_1{}^p, \theta_0)\nu(dx_1{}^p)=O(k^{-1}). \tag{4.7}$$

Now, (4.6) follows from (4.7) and A.1 (below), (see the proof of Theorem 2.6 in [3]) and the proof is completed.

Lemma 4.3. *For any* τ (>0)

$$\sup_{|\theta-\theta_0|<\varepsilon}\int_{A_{n\tau}} f(x_1{}^p, \theta_0)\nu(dx_1{}^p)=o(k^{-1}) \quad (n\to\infty) \tag{4.8}$$

The proof of this lemma is easily proved by Lemma 4.3 and the method used in the proof of Lemma 2.1 in [3] and so is omitted.

Remark 4.1. As in ([3], Remark 1), we can prove that for each j

$$P(|y_j|>\tau)=\int_{A_{n\tau}} f(x_1{}^p, \theta_0)\nu(dx_1{}^p)=o(k^{-1}).$$

Lemma 4.4. *For any* $\theta\in\Theta$

$$\lim_{n\to\infty}\frac{1}{p\varepsilon^2}L(\theta, p, \varepsilon)=\frac{1}{4}I(\theta) \tag{4.9}$$

where

$$L(\theta, p, \varepsilon)=\int(\sqrt{f(x_1{}^p, \theta+\varepsilon)}-\sqrt{f(x_1{}^p, \theta)})^2\nu(dx_1{}^p).$$

Proof. We note that

$$\frac{\partial a_p(s)}{\partial s}=\frac{1}{2}\sqrt{f(x_1{}^p, s)}\sum_{i=1}^p\sum_{j=1}^p\left\{\frac{f'(x_i|x_1{}^{i-1}, s)}{f(x_i|x_1{}^{i-1}, s)}\frac{f'(x_j|x_1{}^{j-1}, s)}{f(x_j|x_1{}^{j-1}, s)}+\frac{2}{p}\frac{\partial}{\partial s}\frac{f'(x_j|x_1{}^{j-1}, s)}{f(x_j|x_1{}^{j-1}, s)}\right\}.$$

By Jensen's inequality and A2 (below)

$$
\begin{aligned}
J_p &= \int |a_p(s) - a_p(\theta)|^2 \nu(dx_1{}^p) \\
&\leqq t^2 \frac{1}{t} \int_\theta^{\theta+t} \left| \frac{\partial a_p(s)}{\partial s} \right|^2 \nu(dx_1{}^p) ds \\
&\leqq \frac{t^2}{4} \sup_{\theta \leqq s \leqq \theta+t} E_s \left[\sum_{i=1}^p \sum_{j=1}^p \left\{ U_i(s) U_j(s) + \frac{2}{p} \frac{\partial U_j(s)}{\partial s} \right\} \right]^2 \leqq K t^2 p^2
\end{aligned}
$$

and so by Proposition 3.3

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{p\varepsilon} \left| \int_\theta^{\theta+\varepsilon} \int a_p(\theta)(a_p(t) - a_p(\theta)) \nu(dx_1{}^p) dt \right| &\leqq \lim_{n\to\infty} \frac{1}{p\varepsilon} \int_\theta^{\theta+\varepsilon} \left\{ \int a_p{}^2(\theta) \nu(dx_1{}^p) \right\}^{1/2} J_p{}^{1/2} dt \\
&\leqq K \lim_{n\to\infty} \varepsilon p^{1/2} = 0 \,.
\end{aligned}
$$

Hence, we obtain that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{p\varepsilon^2} L(\theta, p, \varepsilon) &= \lim_{n\to\infty} \frac{1}{4p\varepsilon^2} \int \left\{ \int_\theta^{\theta+\varepsilon} a_p(t) dt \right\}^2 (dx_1{}^p) \\
&\geqq \frac{1}{4} \lim_{n\to\infty} \frac{1}{p\varepsilon^2} \left\{ \varepsilon^2 \int a_p{}^2(\theta) \nu(dx_1{}^p) \right. \\
&\qquad \left. + 2\varepsilon \int a_p(\theta) \int_\theta^{\theta+\varepsilon} (a_p(t) - a_p(\theta)) dt \nu(dx_1{}^p) \right\} \\
&\geqq \frac{1}{4} \lim_{n\to\infty} \frac{1}{p} \int a_p{}^2(\theta) \nu(dx_1{}^p) = \frac{1}{4} I(\theta) \,.
\end{aligned}
$$

On the other hand, by Jensen's inequality

$$
\begin{aligned}
(4.10) \qquad L(\theta, p, \varepsilon) &\leqq \frac{1}{4} \varepsilon^2 \left\{ \frac{1}{\varepsilon} \int_\theta^{\theta+\varepsilon} \int a_p{}^2(t) \nu(dx_1{}^p) dt \right\} \\
&\leqq \frac{1}{4} \varepsilon^2 \max_{\theta \leqq t \leqq \theta+\varepsilon} \int a_p{}^2(t) \nu(dx_1{}^p) \,.
\end{aligned}
$$

Hence, by II_4 and Proposition 3.3.

$$
\limsup_{n\to\infty} \frac{1}{p\varepsilon^2} L(\theta, p, \varepsilon) \leqq \frac{1}{4} \limsup_{n\to\infty} \max_{\theta \leqq t \leqq \theta+\varepsilon} \frac{1}{p} \int a_p{}^2(t) \nu(dx_1{}^p) = \frac{1}{4} I(\theta) \,.
$$

Thus, we have the lemma.

Corollary. *For any* $\theta \in \Theta$

$$
\begin{aligned}
(4.11) \qquad \lim_{n\to\infty} \frac{1}{p\varepsilon_n{}^2} \int &\left(\sqrt{f(x_1{}^p \mid x^0_{-q+1}, \theta+\varepsilon_n)} - \sqrt{f(x_1{}^p \mid x^0_{-q+1}, \theta)} \right)^2 \\
&\times f(x^0_{-q}, \theta) \nu(dx^p_{-q}) = \frac{1}{4} I(\theta) \,.
\end{aligned}
$$

Proof. The proof is easily obtained from the above method and Proposition 3.4.

The following two lemmas are proved by completely analogous methods to the proofs of Lemmas 2.3 and 2.4 in [3] and so are omitted.

Lemma 4.5. *As* $n\to\infty$

$$\int\sqrt{f(x_1{}^p,\theta+\varepsilon_n)f(x_1{}^p,\theta)}\,\nu(dx_1{}^p)=1-\frac{1}{8}I(\theta)k^{-1}+o(k^{-1}) \tag{4.12}$$

and

$$\int\sqrt{f(x_1{}^p\,|\,x_{-q}^0,\theta+\varepsilon_n)f(x_1{}^p\,|\,x_{-q}^0,\theta)}\,f(x_{-q}^0,\theta)\nu(dx_{-q}^p)=1-\frac{1}{8}I(\theta)k^{-1}+o(k^{-1})\,. \tag{4.13}$$

Lemma 4.6. *For any positive* τ

$$\int_{\bar{A}_{n\tau}}\log\frac{f(x_1{}^p,\theta_0+\varepsilon_n)}{f(x_1{}^p,\theta_0)}f(x_1{}^p,\theta_0)\nu(dx_1{}^p)=-\frac{1}{2}k^{-1}I_0+o(k^{-1}) \tag{4.14}$$

and

$$\int_{\bar{A}_{n\tau}}\log^2\frac{f(x_1{}^p,\theta_0+\varepsilon_n)}{f(x_1{}^p,\theta_0)}f(x_1{}^p,\theta_0)\nu(dx_1{}^p)=k^{-1}I_0+o(k^{-1})\,. \tag{4.15}$$

Here, $\bar{A}$ *denotes the complementary set of* A.

Lemma 4.7. *For any* $\tau>0$

$$P\left(\left|\sum_{j=0}^{k-1}\tilde{y}_j\right|>\tau\right)\to 0\quad(n\to\infty) \tag{4.16}$$

and

$$P\left(\left|\sum_{j=0}^{k-1}\tilde{z}_j\right|>\tau\right)\to 0\quad(n\to\infty)\,. \tag{4.17}$$

Proof. Let $V_{m,r}(t)$ be the one defined in Section 3. Let $\rho_n=\theta n^{-1/2}$. For $m>r>1$, let

$$\begin{aligned} v_{m,r}&=\log\frac{f(\xi_m\,|\,\xi_1{}^{m-1},\theta_0+\rho_n)}{f(\xi_m\,|\,\xi_1{}^{m-1},\theta_0)}-\log\frac{f(\xi_m\,|\,\xi_{m-r}^{m-1},\theta_0+\rho_n)}{f(\xi_m\,|\,\xi_{m-r}^{m-1},\theta_0)}\\ &=\int_{\theta_0}^{\theta_0+\rho_n}V_{m,r}(t)dt\,. \end{aligned}$$

Then, by Jensen's inequality and III_2

$$E\,|v_{m,r}|^2\leqq\rho_n{}^2E\left[\frac{1}{\rho_n}\int_{\theta_0}^{\theta_0+\rho_n}|V_{m,r}(t)|^2dt\right]\leqq\rho_n{}^2\gamma_{m-r}^2$$

and so by Schwarz's inequality

$$|Ev_{m,r}v_{m',r'}|\leqq \rho_n{}^2\gamma_{m-r}\gamma_{m'-r'}\,.$$

Hence, it follows from III that for each j $(0\leqq j\leqq k-1)$

$$E\tilde{y}_j{}^2=E\left|\sum_{i=1}^{p} v_{jl+i,jl}\right|^2\leqq K\rho_n{}^2\,.$$

So, using the fact that $k=O(n^{1-\alpha})=o(n^{1/2})$

$$E\left|\sum_{j=0}^{k-1}\tilde{y}_j\right|^2\leqq Kk^2\rho_n{}^2=o(1)\,,$$

which implies (4.16). Similarly, we have (4.17) and the proof is completed.

Lemma 4.8. *For any* $\tau>0$

$$P\left(\left|\sum_{j=0}^{k} z_j\right|>\tau\right)\to 0 \quad (n\to\infty)\,. \tag{4.18}$$

Proof. Since for each j z_j is $F_{jl+p+1}^{(j+1)l}$-measurable, so by Lemma 4.1 we have

$$P\left(\left|\sum_{j=0}^{k} z_j\right|>2\tau\right)\leqq P\left(\left|\sum_{j=0}^{k-1} z_j{}^*\right|>\tau\right)+2k\beta_q+P(|z_k|>\tau)$$

where $z_0{}^*, z_1{}^*, \cdots, z_{k-1}^*$ are independent and for each j $z_j{}^*$ has the same *df* as that of z_j.

We note that from Remark 4.1 and the method used in the proof of Lemma 4.7 we obtain

$$P(|z_k|>\tau)\to 0 \qquad (n\to\infty)\,,$$

and that $k\beta_q\to 0$ as $n\to\infty$. So, to prove (4.18), it suffices to show

$$P\left(\left|\sum_{j=0}^{k-1} z_j{}^*\right|>\tau\right)\to 0 \quad (n\to\infty)\,. \tag{4.19}$$

Let σ be a positive number such that $\sigma<\tau$. For each j $(0\leqq j\leqq k-1)$, let

$$\bar{z}_j{}^*=\begin{cases} z_j{}^* & (|z_j{}^*|\leqq\sigma)\,,\\ 0 & (|z_j{}^*|>\sigma)\,.\end{cases}$$

Then, by Remark 4.1 and Lemma 4.6

$$P(|\bar{z}_j{}^*|\geqq\sigma)=o(qn^{-1})=o(n^{-2/3})$$

$$Ez_j{}^*=-\frac{q\theta^2}{2n}I_0+o(n^{-2/3})\quad (0\leqq j\leqq k-1)$$

$$\mathrm{Var}\, z_j{}^*=\frac{q\theta^2}{n}I_0+o(n^{-2/3})$$

and so for all n sufficiently large

$$P\left(\left|\sum_{j=0}^{k-1} z_j^*\right|>\tau\right)\leqq P\left(\left|\sum_{j=0}^{k-1} z_j^*\right|>\frac{\tau}{2}\right)+P\left(\max_{0\leqq j\leqq k-1}|z_j^*|>\sigma\right)$$

$$\leqq P\left(\left|\sum_{j=0}^{k-1}(z_j^*-Ez_j^*)\right|>\frac{\tau}{4}\right)+k\max_{0\leqq j\leqq k-1}P(|z_j^*|>\sigma)=o(1),$$

which implies (4.19), and the proof is completed.

Proof of Theorem 4.1. By Lemmas 4.7 and 4.8 it suffices to prove the following two statements:

A. The distributions of the random variables $\sum_{j=0}^{k-1} y_j(\theta)$ converge to the distribution of the random variable $\theta\sqrt{I_0}\,\zeta-(1/2)\theta^2 I_0$.

B. As $n\to\infty$, the difference

$$\left(\frac{\sum_{j=0}^{k-1} y_j(\theta_2)}{\theta_2}+\frac{I_0}{2}\theta_2\right)-\left(\frac{\sum_{j=0}^{k-1} y_j(\theta_1)}{\theta_1}+\frac{I_0}{2}\theta_1\right)$$

converges in probability to zero for all θ_1 and θ_2.

Since for each $\theta\in\Theta$ $y_0(\theta), y_1(\theta), \cdots, y_{k-1}(\theta)$ satisfy to a.r. condition with β_n and $y_j(\theta)$ is F_{jl+1}^{jl+p}-measurable ($j=0, 1, \cdots, k-1$), so by Lemma 4.1

$$P\left(\sum_{j=0}^{k-1} y_j^*(\theta)<u\right)-2k\beta_q\leqq P\left(\sum_{j=0}^{k-1} y_j(\theta)<u\right)\leqq P\left(\sum_{j=0}^{k-1} y_j^*(\theta)<u\right)+2k\beta_q,$$

where the $y_j^*(\theta)$ are i.i.d. random variables with the same *df* as that of $y_1(\theta)$. Hence, to prove Theorem 4.1 it is enough to show that statements *A* and *B* are fulfilled by $\{y_j^*(\theta)\}$ instead of $\{y_j(\theta)\}$ since by III_1 $k\beta_q\to 0$ as $n\to\infty$. But, using Lemmas 4.3–4.6 we can verify the statements by the identical method to the one in the proof of Theorem 2.1 in [3] and so the proof is omitted.

Theorem 4.2. *If Conditions* I–III *hold, then there exist two positive numbers* b_0 *and* K_0 *such that for any pair* $(\theta_1, \theta_2)(|\theta_2-\theta_1|\leqq b_0)$

$$E|Z_n^{1/2}(\theta_2)-Z_n^{1/2}(\theta_1)|\leqq K_0|\theta_2-\theta_1|^2. \tag{4.20}$$

Proof. Putting $m=n$ in (4.10) and using Proposition 3.3 we have the relation

$$\int(\sqrt{f(x_1^n, \theta+bn^{-1/2})}-\sqrt{f(x_1^n, \theta)})^2\nu(dx_1^n)\leqq K\frac{b^2}{4}$$

for all n sufficiently large. From this (4.20) is easily obtained.

Theorem 4.3. *Suppose that Conditions* I, II *and* IV *hold. Then, for any positive* N, *there exist an* n_0 *and a constant* c_N *depending only on* N *such that, for* $n>n_0$

$$P\left(\sup_{|\theta|>A} Z_n(\theta)>\frac{1}{A^N}\right)\leqq\frac{c_N}{A^N} \tag{4.21}$$

$$P\left(\sup_{r\leqq|\theta|\leqq r+1} Z_n(\theta)\geqq\frac{1}{r^N}\right)\leqq\frac{c_N}{r^N}\quad (r\geqq 1) \tag{4.22}$$

To prove Theorem 4.3 we need some lemmas.

Lemma 4.9. *If Conditions* I, II *and* IV *hold, then for any* K_1 *and* N *there is a positive* c_N *such that, in the region* $|\theta|<K_1n^{1/2}$

$$P\left(Z_n(\theta)>\frac{1}{|\theta|^N}\right)\leqq\frac{c_N}{|\theta|^N} \tag{4.23}$$

holds for all n sufficiently large.

Proof. Let $p=[n^\alpha]$, $q=[n^{1-\alpha}]$ $(1/2<\alpha<1)$ and $k=[n/2p]$. For each j $(0\leqq j\leqq k-1)$ put

$$w_j=\log\frac{f(\xi_{2jp+1}^{(2j+1)p}\,|\,\xi_{2jp-q}^{2jp},\,\theta_0+\theta n^{-1/2})}{f(\xi_{2jp+1}^{(2j+1)p}\,|\,\xi_{2jp-q}^{2jp},\,\theta_0)},$$

and

$$\tilde{w}_j=\log\frac{f(\xi_{2jp+1}^{(2j+1)p}\,|\,\xi_1{}^{2jp},\,\theta_0+\theta n^{-1/2})}{f(\xi_{2jp+1}^{(2j+1)p}\,|\,\xi_1{}^{2jp},\,\theta_0)}-w_j\,.$$

To prove (4.23) it suffices to show that

$$P\left(\sum_{j=1}^{k-1}(w_j+\tilde{w}_j)>-N\log|\theta|\right)<K|\theta|^{-N} \tag{4.24}$$

since the rest is analogously estimated.

As for each j $(1\leqq j\leqq k-1)$ w_j is $F_{2jp-q}^{(2j+1)p}$-measurable, so from Lemma 4.1 we obtain

$$P\left(\sum_{j=1}^{k-1}w_j>-2N\log|\theta|\right)\leqq P\left(\sum_{j=1}^{k-1}w_j{}^*>-2N\log|\theta|\right)+2k\beta_p$$

where $w_1{}^*,\cdots,w_{k-1}^*$ are i.i.d. random variables each having the same *df* as that of w_1. Since $|\theta|<K_1n^{-1/2}$ and $k\beta_p=o(n^{1/2}e^{-\lambda_1 n})$, so $2k\beta_p<|\theta|^{-N}$ for any N if n is sufficiently large. On the other hand, using Lemma 4.5, the method of the proof of Lemma 2.6 in [3] and the fact that $\{w_j{}^*\}$ are i.i.d. we have the relation

$$P\left(\exp\left(\sum_{j=1}^{k-1}w_j{}^*\right)>e^{-C_1\theta^2}\right)<e^{-C_1\theta^2}$$

for some $C_1>0$. Hence, for any N (>0)

$$P\left(\sum_{j=1}^{k-1}w_j>-2N\log|\theta|\right)<K|\theta|^{-N} \tag{4.25}$$

Now, we prove that

$$P\left(\left|\sum_{j=1}^{k-1} \tilde{w}_j\right| > N \log|\theta|\right) < K|\theta|^{-N} \tag{4.26}$$

holds for any $N\,(>0)$. As in the proof of Lemma 4.7 we have that for all n sufficiently large and for each j $(1 \leqq j \leqq k-1)$.

$$\begin{aligned} E\tilde{w}_j^{\,2} &= E\left|\sum_{i=1}^{p} v_{2jp+i,2jp-q}\right|^2 \\ &\leqq p^2 \max_{1 \leqq i \leqq p} \|v_{2jp+i,2jp-q}\|_2^2 \\ &\leqq Kp^2\gamma_q^{\,2} = o(p^2 e^{-2\lambda_2 q}) . \end{aligned}$$

Hence, we have

$$\begin{aligned} P\left(\left|\sum_{j=1}^{k-1} \tilde{w}_j\right| > N \log|\theta|\right) &\leqq \frac{1}{(N\log|\theta|)^2} E\left|\sum_{j=1}^{k-1} \tilde{w}_j\right|^2 \\ &\leqq \frac{1}{(N\log|\theta|)^2} k^2 \max_{1 \leqq j \leqq k-1} E\tilde{w}_j^{\,2} \\ &= \frac{1}{(N\log|\theta|)^2} o(ne^{-\lambda_2 q}) \end{aligned}$$

and so (4.26) is obtained if $|\theta| < K_1 n^{1/2}$. Combining (4.25) and (4.26) we have (4.24). Thus, the proof is completed.

Lemma 4.10. *Suppose that Conditions* I, II *and* IV *are satisfied. Then, to any positive N, there are numbers n_0 and K_2 such that*

$$P\left(Z_n(\theta) > \frac{1}{|\theta|^N}\right) < \frac{c_N}{|\theta|^N} \tag{4.27}$$

for all $n > n_0$ and all $|\theta| > K_2 n^{1/2}$.

Proof. By IV

$$\begin{aligned} &P(Z_n(\theta) > |\theta|^{-N}) \leqq |\theta|^{N/2} EZ_n^{1/2}(\theta) \\ &\quad = |\theta|^{N/2} \int \prod_{j=1}^{n} \sqrt{f(x_j \mid x_1^{j-1}, \theta_0) f(x_j \mid x_1^{j-1}, \theta_0 + \theta n^{-1/2})}\, \nu(dx_1^n) \\ &\quad \leqq |\theta|^{N/2} (|\theta|^{-1} n^{1/2})^{nd_1} . \end{aligned}$$

Hence, if K_2 and n_0 are sufficiently large, then

$$(|\theta|^{-1} n^{1/2})^{nd_1} < |\theta|^{-3/2N}$$

and so the proof is completed.

Proof of Theorem 4.3. The proof is identical to the proof of Theorem 2.3 in [3] (using Lemmas 4.9 and 4.10 instead of Lemmas 2.7 and 2.8 in [3]) and so is omitted.

Let $C_0(-\infty, \infty)=C_0$ be the space of functions which are continuous on $(-\infty, \infty)$ and for which $\lim_{|x|\to\infty} f(x)=0$ endowed with the usual uniform metric. If $\Theta=R^1$, put $\hat{Z}_n(\theta)=Z_n(\theta)$ and if $\Theta=(a, b)\neq R^1$, define a process $\hat{Z}_n(\theta)$ as follows;

$$\hat{Z}_n(\theta)=\begin{cases} Z_n(\theta) & \text{if } \sqrt{n}(a-\theta_0)<\theta<\sqrt{n}(b-\theta_0)\,, \\ 0 & \text{if } \theta\leqq\sqrt{n}(a-\theta_0)-1 \text{ or } \theta\geqq\sqrt{n}(b-\theta_0)+1\,, \\ \text{linear and continuous in all other intervals.} \end{cases}$$

Then it is clear that $\hat{Z}_n(\theta)$ belongs to C_0 with probability one.

As in [3], we have the following result.

Theorem 4.4. *Suppose that Conditions* I, II *and* IV *are satisfied. Then the distributions in C_0 generated by the process $\hat{Z}_n$ converge as $n\to\infty$ to the distribution generated by Z. In particular, if h is a continuous functional on C_0, then for all x*

$$\lim_{n\to\infty} P(h(\hat{Z}_n)<x)=P(h(Z)<x)\,.$$

As in [3], from Theorem 4.4 we have the following result: Suppose that Conditions I, II and IV hold. Define the maximum likelihood estimator $\hat{\theta}_n$ to be one of the solutions of the equation

$$f(\xi_1{}^n, \hat{\theta}_n)=\max_{\theta\in\Theta^c} f(\xi_1{}^n, \theta)\,.$$

Then the distribution of the difference $\sqrt{n}(\hat{\theta}_n-\theta_0)$ is asymptotically normal with parameters $(0, I_0{}^{-1})$.

Appendix

First, we prove the following statement.

A.1. Suppose that Conditions I–III are satisfied. Then,

$$\sup_{\theta\in\Theta} E\left|\frac{f'(\xi_1{}^n, \theta)}{f(\xi_1{}^n, \theta)}\right|^4 \leqq Kn^2\,. \tag{A.1}$$

Proof. Let U_i be the one defined in Section 3. First, we note that for any fixed n

$$\begin{aligned} E\left|\frac{f'(\xi_1{}^n, \theta)}{f(\xi_1{}^n, \theta)}\right|^4 &= E\left|\sum_{i=1}^{n} U_i\right|^4 \\ &\leqq K\sum_{i=1}^{n-3}\{\textstyle\sum_i^{(1)}+\sum_i^{(2)}+\sum_i^{(3)}\}C_{i,j,k,l} \end{aligned} \tag{A.2}$$

where $C_{i,j,k,l}=EU_iU_jU_kU_l$, and $\sum_i^{(1)}$, $\sum_i^{(2)}$ and $\sum_i^{(3)}$ denote, respectively, the components of $\sum_{j,k,l}$ for which $j-i\geqq\max\{k-j,\ l-k\}$, $k-j\geqq\max\{j-i,\ l-k\}$ and $l-k\geqq\max\{j-i,\ k-l\}$.

(a) Consider the case in which $j-i\geqq\max\{k-j,\ l-k\}$. Let $s_j=[(i+j)/2]$ and put

$$w_t^{(0)}=U_{t,s_j}\,,\qquad w_t^{(1)}=U_t-U_{t,s_j}\quad (t>s_j)\,.$$

Then, it is obvious that if $t>s_j$

$$\|w_t^{(1)}\|_{(4+\delta)/(1+\delta)}\leqq \gamma^*_{t-s_j}\,. \tag{A.3}$$

Hence, by II_2, (5.2) and Lemma 3.1

$$\begin{aligned}C_{i,j,k,l}&\leqq |EU_iw_j^{(0)}w_k^{(0)}w_l^{(0)}|+\sum_{a+b+c>0}|EU_iw_j^{(a)}w_k^{(b)}w_l^{(c)}|\\&\leqq K\{\|U_i\|_{k+\delta}\|w_j^{(0)}w_k^{(0)}w_l^{(0)}\|^3_{4+\delta}\beta^*_{j-s_j}\\&\qquad+\|w_j^{(1)}\|_{(4+\delta)/(1+\delta)}+\|w_k^{(1)}\|_{(4+\delta)/(1+\delta)}+\|w_l^{(1)}\|_{(4+\delta)/(1+\delta)}\}\\&\leqq K\{\beta^*_{j-s_j}+\gamma^*_{j-s_j}+\gamma^*_{k-s_j}+\gamma^*_{l-s_j}\}\,.\end{aligned}$$

Since $\max\left\{\sum_{k=1}^{n}k^2\beta_k^*,\ \sum_{k=1}^{n}k^2\gamma_k^*\right\}\leqq Kn$, so

$$\begin{aligned}\sum_{i=1}^{n-3}\sum\nolimits_i^{(1)}C_{i,j,k,l}&\leqq K\left[n^2+\sum_{i=1}^{n-3}\sum_{i+1<j\leqq n}(j-i+1)^2\left\{\beta^*\left(\left[\frac{j-i}{2}\right]\right)+\gamma^*\left(\left[\frac{j-i}{2}\right]\right)\right\}\right]\\&\leqq Kn^2\,.\end{aligned} \tag{A.4}$$

(b) Let $k-j\geqq\max\{j-i,\,l-k\}$ and put $s_k=[(k+j)/2]$. Then, as in Remark 3.1, from Lemma 3.1 we obtain

$$\begin{aligned}C_{i,j,k,l}&\leqq|EU_iU_jU_{k,s_k}U_{l,s_k}|+K\gamma^*_{k-s_k}\\&\leqq|EU_iU_j|\cdot|EU_{k-s_k}U_{l-s_k}|+K\gamma^*_{k-s_k}\\&\leqq K\left[\left\{\beta^*\left(\left[\frac{k+j}{2}\right]\right)+\gamma^*\left(\left[\frac{j+i}{2}\right]\right)\right\}\left\{\beta^*\left(\left[\frac{l-k}{2}\right]\right)\right.\right.\\&\qquad\left.\left.+\gamma^*\left(\left[\frac{l+k}{2}\right]\right)\right\}+\gamma^*\left(\left[\frac{k+j}{2}\right]\right)\right].\end{aligned}$$

So

$$\sum_{i=1}^{n-3}\sum\nolimits_i^{(2)}C_{i,j,k,l}\leqq Kn^2\,. \tag{A.5}$$

(c) If $l-k\geqq\max\{j-i,\,k-j\}$, we put $s_l=[(l+k)/2]$. Then

$$C_{i,j,k,l}\leqq|EU_iU_jU_kU_{l,s_l}|+K\gamma^*_{l-s_l}\leqq K\{\beta^*_{s_l-k}+\gamma^*_{l-s_l}\}$$

and so

$$\sum_{i=1}^{n-3}\sum\nolimits_i^{(3)}C_{i,j,k,l}\leqq Kn^2\,. \tag{A.6}$$

Hence, from (A.2)–(A.6), (A.1) follows and the proof is completed.

Finally, we have the following statement by the above method and so the proof is omitted.

A2. Under the conditions of A1,

$$E\left[\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{U_i(\theta)U_j(\theta)+\frac{2}{m}\frac{\partial U_j(\theta)}{\partial\theta}\right\}\right]^2 \leqq Km^2 . \tag{A.7}$$

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