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RATES OF CONVERGENCE IN THE INVARIANCE PRINCIPLE FOR WEAKLY DEPENDENT RANDOM VARIABLES

By

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1. Introduction and results

Let $\{X_i, i\in \mathbf{Z}\}$ be a sequence of strictly stationary random variables. Let $\mathscr{F}_m{}^n$ denote σ-field generated by random variables $\{X_i, i=m, m+1, \cdots, n\}$. Suppose that the sequence $\{X_i\}$ satisfies the strong mixing condition, that is,

$$\alpha(n)\equiv \sup_{A\in\mathscr{F}_{-\infty}^{0},\, B\in\mathscr{F}_{n}^{\infty}} |P(AB)-P(A)P(B)| \downarrow 0$$

as $n\to\infty$.

Suppose $EX_1=0$ and $E|X_i|^r<\infty$ for some $r>2$. Under these assumptions, if

$$\sum_{i=1}^{\infty} (\alpha(i))^{(r-2)/r}<\infty ,$$

then

$$\sigma^2\equiv EX_1{}^2+2\sum_{i=1}^{\infty} E(X_1X_{i+1})<\infty . \tag{1.1}$$

(See e.g. [5].) Here suppose that $\sigma>0$.

Define a continuous polygonal line $\{X_n(t), 0\leqq t\leqq 1\}$ by

$$X_n(t)=\begin{cases}(nt)X_1/(\sigma n^{1/2}), & \text{for} \quad t\in[0, 1/n] , \\ \sum_{i=1}^{k} X_i/(\sigma n^{1/2})+(nt-k)X_{k+1}/(\sigma n^{1/2}) , & \\ & \text{for} \quad t\in(k/n), (k+1)/n] , \quad k=1, \cdots, n-1 .\end{cases}$$

Let $C=C[0, 1]$ be the space of all continuous functions on $[0, 1]$ with the uniform metric defined by

$$d(x, y)=\sup_{0\leqq t\leqq 1} |x(t)-y(t)| , \qquad \text{for} \quad x, y\in C ,$$

and let $\mathscr{C}$ be the smallest σ-field containing all open sets in C. Let P_n be distribution of $\{X_n(t)\}$ and W the Wiener measure on $(C, \mathscr{C})$. The Prokhorov-Lévy metric $\rho(\cdot, \cdot)$ on the space of probability measures on $(C, \mathscr{C})$ is defined by

$$\rho(R, Q)=\inf\{\varepsilon>0;\ R(B)\leqq\varepsilon+Q\{y;\ d(x, y)<\varepsilon,\ x\in B\}\ ,$$
$$Q(B)\leqq\varepsilon+R\{y;\ d(x, y)<\varepsilon,\ x\in B\}\ \text{for all}\ B\in\mathscr{C}\}\ ,$$

where R and Q are probability measures on $(C, \mathscr{C})$.

In this paper we shall show the following results concerning the rate of convergence of $\rho(P_n, W)$ to zero.

Theorem 1. *Let $\{X_i, i\in\mathbf{Z}\}$ be a sequence of strictly stationary random variables with $EX_1=0$ and $E|X_1|^r<\infty$ for some $r>2$. Suppose that the sequence $\{X_i\}$ satisfies the strong mixing condition with coefficient $\alpha(n)$ and that there exists s with $2<s<r$ such that*

$$\sum_{i=1}^{\infty}(\alpha(i))^{(r-s)/rs}<\infty\ . \tag{1.2}$$

Then (1.1) *holds. Suppose that $\sigma>0$, (and furthermore suppose $\sigma=1$ without loss of generarity).*

If $s\leqq 4$, then we have for any $\delta<s(s-2)/4(s-1)(s+1)$,

$$\rho(P_n, W)=o(n^{-\delta})$$

as $n\to\infty$.

If $r>4$ and (1.2) *holds for some s with $4<s<r$, then we have for any $\kappa<(s-4)/30(s+1)$,*

$$\rho(P_n, W)=o(n^{-2/15-\kappa})$$

as $n\to\infty$.

Corollary. *In Theorem* 1, *suppose that*

$$\alpha(n)=O(e^{-\gamma n})$$

for some $\gamma>0$ in place of (1.2). *If $r\leqq 4$, then we have for any $\delta<r(r-2)/4(r-1)(r+1)$*

$$\rho(P_n, W)=o(n^{-\delta})$$

as $n\to\infty$.

If $r>4$, then we have for any $\kappa<(r-4)/30(r+1)$

$$\rho(P_n, W)=o(n^{-2/15-\kappa})$$

as $n\to\infty$.

It should be mentioned that Yoshihara [10] gave the rate of convergence for the stationary sequence satisfying the absolutely regular condition which is,

$$\beta(n)\equiv E\left\{\sup_{A\in\mathscr{F}_n^{\infty}}|P(A\,|\,\mathscr{F}_{-\infty}^{0})-P(A)|\right\}\downarrow 0$$

as $n \to \infty$.

Theorem A (Yoshihara [10]). *Let $\{X_i\}$ be a strictly stationary and absolutely regular sequence of random variables with $EX_1=0$. If, for some $\varepsilon>0$, $E|X_1|^{4+\varepsilon}<\infty$ and*

$$\sum_{i=1}^{\infty} i(\beta(i))^{\varepsilon/(4+\varepsilon)}<\infty , \tag{1.3}$$

then

$$\rho(P_n, W)=O(n^{-1/8}(\log n)^{1/2})$$

as $n \to \infty$.

The absolutely regular condition is stronger than the strong mixing condition. Actually, for each n, $\alpha(n) \leqq \beta(n)$. However (1.3) does not imply (1.2) in general, thus our theorem is not a complete generalization of Theorem A. Although it is true for some small $\varepsilon>0$. But we emphasize that in our case we assume only the strong mixing condition. As far as the author knows, there are few results on the rate of convergence of the invariance principle under the strong mixing condition. Our technique of the proof is different from that of Theorem A.

2. Preliminaries

In this section, we state three lemmas. The first lemma is due to Davydov [4].

Lemma 1. *Let p, q and u be positive numbers with $p^{-1}+q^{-1}+u^{-1}=1$. Suppose that X is $\mathscr{F}_1{}^a$-measurable and Y is $\mathscr{F}_{a+n}^{\infty}$-measurable. Moreover suppose that $E|X|^p<\infty$ and $E|Y|^q<\infty$. Then*

$$|E(XY)-E(X)E(Y)| \leqq 10(\alpha(n))^{1/u}(E|X|^p)^{1/p}(E|Y|^q)^{1/q} .$$

Recently Yokoyama [9] gave an estimate of the p-th absolute moment of sums of strong mixing random variables as follows.

Lemma 2. *Let $\{X_i\}$ be a strictly stationary strong mixing sequence with $EX_1=0$ and $E|X_1|^r<\infty$ for some $r>2$. If there exists p with $2<p<r$ such that*

$$\sum_{i=1}^{\infty} i^{p/2-1}(\alpha(i))^{(r-p)/r}<\infty , \tag{2.1}$$

then there exists a positive constant K such that

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqq Kn^{p/2} , \qquad n \geqq 1 .$$

In what follows, as an absolute positive constant, we shall use a K which may be different in the different equations.

For some $0<v<1$, let $M=[n/[n^v]]+1$, where $[\cdot]$ denotes the integer part of $\cdot$. For $j=1, \cdots, M-1$, define $I_j=\{(j-1)[n^v]+1, (j-1)[n^v]+2, \cdots, j[n^v]\}$ and $I_M=$

$\{(M-1)[n^\nu]+1, (M-1)[n^\nu]+2, \cdots, n\}$. Let

$$y_j = \sum_{i \in I_j} n^{-1/2} X_i \qquad \text{for} \quad j=1, \cdots, M.$$

Using Lemmas 1 and 2 we can easily prove the following lemma.

Lemma 3. *Under the conditions of Theorem* 1, *for* $j=1, \cdots, M-1$,

$$|Ey_j^2 - n^{\nu-1}| \leqq Kn^{-1/2}, \tag{2.2}$$

and for any p *with* $2<p<2rs/(r+s)$,

$$E|y_j|^p \leqq Kn^{(\nu-1)p/2}. \tag{2.3}$$

Also we have

$$|Ey_M^2 - (n-(M-1)[n^\nu])n^{-1}| \leqq Kn^{-1} \tag{2.4}$$

and

$$E|y_M|^p \leqq Kn^{(\nu-1)p/2}. \tag{2.5}$$

Proof. From the stationarity, it suffices to show (2.2) only for $j=1$. By the ordinary argument (see e.g. Philipp-Stout [6], p. 28),

$$\begin{aligned} Ey_1^2 &= n^{-1}\left\{\sum_{i=1}^{[n^\nu]} EX_i^2 + 2\sum_{i=1}^{[n^\nu]-1}([n^\nu]-i)E(X_1X_{i+1})\right\} \\ &= n^{-1}\left\{[n^\nu]\left(EX_1^2 + 2\sum_{i=1}^{\infty} E(X_1X_{i+1})\right)\right. \\ &\qquad \left. -2[n^\nu]\sum_{i=[n^\nu]}^{\infty} E(X_1X_{i+1}) - 2\sum_{i=1}^{[n^\nu]-1} iE(X_1X_{i+1})\right\}. \end{aligned}$$

Applying Lemma 1 we have

$$|E(X_1X_{i+1})| \leqq 10(\alpha(i))^{(r-2)/r}(E|X_1|^r)^{1/r}(E|X_{i+1}|^r)^{1/r}.$$

Thus from condition (1.2) and the monotonicity of $\{\alpha(i)\}$, it follows that

$$[n^\nu]\sum_{i=[n^\nu]}^{\infty} E(X_1X_{i+1}) \leqq K[n^\nu]\sum_{i=[n^\nu]}^{\infty}(\alpha(i))^{(r-2)/r} \leqq K,$$

and

$$\sum_{i=1}^{[n^\nu]-1} iE(X_1X_{i+1}) \leqq K\sum_{i=1}^{[n^\nu]-1} i(\alpha(i))^{(r-2)/r} \leqq K.$$

Hence we have (2.2) for $j=1$.

Furthermore, if $2<p\leqq 2rs/(r+s)$, then condition (1.2) implies

$$\sum_{i=1}^{\infty} i^{-1+p/2}(\alpha(i))^{(r-s)/r} < \infty,$$

from which (2.3) follows because of Lemma 2. (2.4) and (2.5) are similarly shown.

3. Martingale representations

For $i=1, \cdots, M$, let $\mathscr{L}_i$ be the σ-field generated by the random variables $\{y_1, \cdots, y_i\}$ and $\mathscr{L}_0=\{\Omega, \phi\}$. For simplicity let $\mathscr{F}_j=\mathscr{F}_1{}^j$ for $j=1, 2, \cdots$ and $\mathscr{F}_0=\{\Omega, \phi\}$.

The idea of the proofs of the following lemmas in this section is the same as in the proof of Lemma 7.4.1 in Philipp-Stout [6].

Lemma 4. *Under the assumptions of Theorem* 1, *we can represent X_i in the form,*

$$X_i=\eta_i-d_{i+1}+d_i\,, \qquad i=1, \cdots, n\,, \tag{3.1}$$

where $\{\eta_i\}$ is a martingale difference sequence with respect to $\{\mathscr{F}_i\}$ and where

$$\|d_i\|_s\leqq K \qquad \text{and} \qquad \|\eta_i\|_s\leqq K\,, \quad i=1, \cdots, n\,, \tag{3.2}$$

where $\|\cdot\|_s=(E|\cdot|^s)^{1/s}$.

Proof. Define

$$d_i=\sum_{k=0}^{\infty} E(X_{i+k}|\mathscr{F}_{i-1})$$

and

$$\eta_i=X_i-d_i+d_{i+1}$$

for $1\leqq i\leqq n$. Then the representation (3.1) follows.

On the other hand, by the Minkovsky inequality, we have

$$\|d_i\|_s\leqq \sum_{k=0}^{\infty} \|E(X_{i+k}|\mathscr{F}_{i-1})\|_s\,.$$

Using Lemma 1 with $p=r$, $q=s/(s-1)$ and $u=sr/(r-s)$, and noting $EX_{i+k}=0$, we have

$$\begin{aligned}E|E(X_{i+k}|\mathscr{F}_{i-1})|^s&=E(E(X_{i+k}|\mathscr{F}_{i-1})E(X_{i+k}|\mathscr{F}_{i-1})|E(X_{i+k}|\mathscr{F}_{i-1})|^{s-2})\\&=E(X_{i+k}E(X_{i+k}|\mathscr{F}_{i-1})|E(X_{i+k}|\mathscr{F}_{i-1})|^{s-2})\\&\leqq 10(\alpha(k+1))^{(r-s)/rs}\|X_{i+k}\|_r\||E(X_{i+k}|\mathscr{F}_{i-1})|^{s-1}\|_{s/(s-1)}\,.\end{aligned}$$

Thus we have by assumption (1.2),

$$\|d_i\|_s\leqq 10\|X_1\|_r\sum_{k=0}^{\infty}(\alpha(k+1))^{(r-s)/rs}\leqq K\,,$$

and also by the Minkovsky inequality,

$$\|\eta_i\|_s\leqq \|X_i\|_s+\|d_i\|_s+\|d_{i+1}\|_s\leqq K\,.$$

The lemma is concluded.

Lemma 5. *We can represent y_i in the form*

$$y_i=\xi_i+v_i\,,\qquad i=1,\cdots,M\,, \tag{3.3}$$

where $\{\xi_i\}$ is a martingale difference sequence with respect to $\{\mathscr{L}_i\}$ and where

$$\|v_i\|_s\leqq Kn^{-1/2} \quad \text{and} \quad \|\xi_i\|_s\leqq Kn^{(\nu-1)/2}\,,\quad i=1,\cdots,M\,. \tag{3.4}$$

Proof. Let $\xi_i=y_i-E(y_i\,|\,\mathscr{L}_{i-1})$, and $v_i=E(y_i\,|\,\mathscr{L}_{i-1})$, then we have the representation (3.3). For the proof of (3.4), we note that

$$\begin{aligned}
E\,|\,E(y_i\,|\,\mathscr{L}_{i-1})\,|^s&=\sum_{k=1}^{[n^\nu]}E\{n^{-1/2}X_{(i-1)[n^\nu]+k}E(y_i\,|\,\mathscr{L}_{i-1})\,|\,E(y_i\,|\,\mathscr{L}_{i-1})\,|^{s-2}\}\\
&\leqq\sum_{k=1}^{[n^\nu]}10n^{-1/2}(\alpha(k))^{(r-s)/rs}\|X_{(i-1)[n^\nu]+k}\|_r\|E(y_i\,|\,\mathscr{L}_{i-1})\|_s^{s-1}\\
&\leqq Kn^{-1/2}\|E(y_i\,|\,\mathscr{L}_{i-1})\|_s^{s-1}\,,
\end{aligned}$$

by Lemma 1 and assumption (1.2). Then the proof of the lemma is concluded by the same reasoning as in the proof of Lemma 4.

Lemma 6. *We can represent v_i in the form*

$$v_i=\theta_i-g_{i+1}+g_i\,,\quad i=1,\cdots,M\,, \tag{3.5}$$

where $\{\theta_i\}$ is a martingale difference sequence with respect to $\{\mathscr{L}_{i-1}\}$ and where

$$\|g_i\|_s\leqq Kn^{-1/2} \quad \text{and} \quad \|\theta_i\|_s\leqq Kn^{-1/2}\,,\quad i=1,\cdots,M\,. \tag{3.6}$$

Proof. Define

$$g_i=\sum_{k=0}^{M-i}E(y_{i+k}\,|\,\mathscr{L}_{i-2}) \quad\text{and}\quad \theta_i=v_i-g_{i+1}+g_i \qquad \text{for}\quad 1\leqq i\leqq M\,,$$

where $\mathscr{L}_{-1}=\mathscr{L}_0=\{\phi,\Omega\}$. Then the representation (3.5) is obtained.

Using Lemma 1 we have

$$\begin{aligned}
E\,|\,E(y_{i+k}\,|\,\mathscr{L}_{i-2})\,|^s&=\sum_{m=1}^{[n^\nu]}E\{n^{-1/2}X_{(i+k-1)[n^\nu]+m}E(y_{i+k}\,|\,\mathscr{L}_{i-2})\,|\,E(y_{i+k}\,|\,\mathscr{L}_{i-2})\,|^{s-2}\}\\
&\leqq\sum_{m=1}^{[n^\nu]}10n^{-1/2}(\alpha((k+1)[n^\nu]+m))^{(r-s)/rs}\\
&\qquad\times\|X_{(i+k-1)[n^\nu]+m}\|_r\|E(y_{i+k}\,|\,\mathscr{L}_{i-2})\|_s^{s-1}\,.
\end{aligned}$$

Thus from the Minkovsky inequality and assumption (1.2) we have

$$\begin{aligned}
\|g_i\|_s&\leqq\sum_{k=0}^{M-i}\|E(y_{i+k}\,|\,\mathscr{L}_{i-2})\|_s\\
&\leqq Kn^{-1/2}\sum_{k=0}^{M-i}\sum_{m=1}^{[n^\nu]}(\alpha((k+1)[n^\nu]+m))^{(r-s)/rs}
\end{aligned}$$

$$=Kn^{-1/2}\sum_{k=[n^\nu]+1}^{(M-i+2)[n^\nu]}(\alpha(k))^{(r-s)/rs}$$

$$\leqq Kn^{-1/2}\,, \qquad i=1,\cdots,M\,.$$

The rest of the proof is similar to that of Lemma 4.

Lemma 7. *We can represent $y_i v_i$ in the form*

$$y_i v_i=\mu_i+u_i\,, \qquad i=1,\cdots,M\,, \tag{3.7}$$

where $\{\mu_i\}$ is a martingale difference sequence with respect to $\{\mathscr{L}_i\}$ and where

$$\|u_i\|_{s/2}\leqq Kn^{-1/2} \quad and \quad \|\mu_i\|_{s/2}\leqq Kn^{(\nu-2)/2}\,, \quad i=1,\cdots,M\,. \tag{3.8}$$

Proof. Let $u_i=E(y_i v_i\,|\,\mathscr{L}_{i-1})$ and $\mu_i=y_i v_i-u_i$ for $1\leqq i\leqq M$. Then the representation (3.7) follows. Since

$$u_i=E(y_i E(y_i\,|\,\mathscr{L}_{i-1})\,|\,\mathscr{L}_{i-1})=(E(y_i\,|\,\mathscr{L}_{i-1}))^2=v_i^2\,,$$

it follows from (3.4) in Lemma 5 that

$$\|u_i\|_{s/2}\leqq \|v_i\|_s^2\leqq Kn^{-1}\,, \qquad i=1,\cdots,M\,.$$

On the other hand by the Schwartz inequality and Lemmas 3 and 5, we have

$$\|y_i v_i\|_{s/2}\leqq \|y_i\|_s\|v_i\|_s\leqq Kn^{(\nu-2)/2}\,.$$

Thus by the Minkovsky inequality we have

$$\|\mu_i\|_{s/2}\leqq \|y_i v_i\|_{s/2}+\|u_i\|_{s/2}\leqq Kn^{(\nu-2)/2}\,, \quad i=1,\cdots,M\,,$$

and the statement of the lemma is concluded.

Lemma 8. *Let $h_i=y_i^2-Ey_i^2$. We can represent h_i in the form*

$$h_i=\zeta_i-w_{i+1}+w_i\,, \qquad i=1,\cdots,M\,, \tag{3.9}$$

where $\{\zeta_i\}$ is a martingale difference sequence with respect to $\{\mathscr{L}_i\}$ and where

$$\|w_i\|_{s/2}\leqq Kn^{\nu-1} \quad and \quad \|\zeta_i\|_{s/2}\leqq n^{\nu-1}\,, \quad i=1,\cdots,M\,. \tag{3.10}$$

Proof. Define

$$w_i=\sum_{k=0}^{M-i}E(h_{i+k}\,|\,\mathscr{L}_{i-1}) \quad \text{and} \quad \zeta_i=h_i+w_{i+1}-w_i \qquad \text{for} \quad 1\leqq i\leqq M\,,$$

where $w_{M+1}\equiv 0$. Then the representation (3.9) follows.

Let $\mathscr{L}_a^b$ be σ-field generated by $\{h_a, h_{a+1}, \cdots, h_b\}$. Since $\{h_i\}$ satisfies the strong mixing condition such that for any $1\leqq a\leqq M-1$ and $k\geqq 1$,

$$\sup_{A\in\mathscr{L}_1^a,\,B\in\mathscr{L}_{a+k}^M}|P(AB)-P(A)P(B)|\leqq\alpha((k-1)[n^\nu]+1)\,,$$

we have from Lemma 1,

$$E|E(h_{i+k}|\mathscr{L}_{i-1})|^{s/2}=E(h_{i+k}E(h_{i+k}|\mathscr{L}_{i-1})|E(h_{i+k}|\mathscr{L}_{i-1})|^{-2+s/2})$$
$$\leqq 10(\alpha(k[n^{\nu}]+1))^{(r-s)/rs}\|h_{i+k}\|_{rs/(r+s)}\||E(h_{i+k}|\mathscr{L}_{i-1})|^{(s-2)/2}\|_{s/(s-2)}$$
$$\leqq K(\alpha(k[n^{\nu}]+1))^{(r-s)/rs}\|y_{i+k}\|^2_{2rs/(r+s)}\|E(h_{i+k}|\mathscr{L}_{i-1})\|_{s/2}^{(s-2)/2}.$$

Thus from the Minkovsky inequality, Lemma 3 and assumption (1.2) it follows that

$$\|w_i\|_{s/2}\leqq \sum_{k=0}^{M-i}\|E(h_{i+k}|\mathscr{L}_{i-1})\|_{s/2}$$
$$\leqq Kn^{\nu-1}\sum_{k=0}^{M-i}(\alpha(k[n^{\nu}]+1))^{(r-s)/rs}$$
$$\leqq Kn^{\nu-1}$$

and

$$\|\zeta_i\|_{s/2}\leqq \|y_i\|_s^2+Ey_i^2+\|w_{i+1}\|_{s/2}+\|w_i\|_s\leqq Kn^{\nu-1}, \quad i=1,\cdots,M.$$

Hence the lemma is proved.

4. Proof of Theorem 1

We first prove the theorem in case $2<s\leqq 4$. Let $\nu=(s-2)/2(s-1)$, $\varepsilon_n=n^{-\delta}$ and $\lambda_n=n^{-2\delta}(\log n)^{-2}$. If we could construct a Brownian motion $\{B(t), 0\leqq t\leqq 1\}$ on a probability space $(\Omega, \mathscr{F}, P)$ such that

$$P\{d(X_n(t), B(t))\geqq \varepsilon_n\}=O(\varepsilon_n)=O(n^{-\delta}) \tag{4.1}$$

as $n\to\infty$, then we have by the same reasoning as in the proof of Theorem 1 in [2] that

$$\rho(P_n, W)=O(n^{-\delta}) \qquad \text{for any} \quad \delta<s(s-2)/4(s-1)(s+1), \tag{4.2}$$

which concludes the theorem. Hence we first have to define a Brownian motion with the property (4.1). To do that, we use the following lemma due to Strassen [8].

Lemma 9. *Let $\{z_i\}$ be random variables such that for all $i\geqq 1$, $E(z_i^2|z_{i-1},\cdots,z_1)$ exists and $E(z_i|z_{i-1},\cdots,z_1)=0$ a.s. Then on a probability space $(\Omega, \mathscr{F}, P)$, there is a standard Brownian motion $\{B(t)\}$ together with a sequence of non-negative random variables $\{T_i\}$ such that the joint distributions of $\{z_i\}$ are the same as those of*

$$\left\{B\left(\sum_{k=1}^{i}T_k\right)-B\left(\sum_{k=1}^{i-1}T_k\right)\right\}$$

Moreover, let $\mathscr{B}_i$ be the σ-field generated by

$$\left\{\{z_1,\cdots,z_i\};\quad B(t),\quad 0\leqq t\leqq \sum_{k=1}^{i}T_k\right\}.$$

Then we see that for all $i\geqq 1$, T_i is $\mathscr{B}_i$-measurable, $E(T_i|\mathscr{B}_{i-1})$ exists and

$$E(T_i \mid \mathscr{B}_{i-1}) = E(z_i^2 \mid \mathscr{B}_{i-1}) = E(z_i^2 \mid z_{i-1}, \cdots, z_1) \quad \text{a.s.} \tag{4.3}$$

Furthermore, for all $i \geqq 1$, if $E(|z_i|^r \mid z_{i-1}, \cdots, z_1)$ exists for some $r > 2$, then

$$ET_i^{r/2} \leqq KE|z_i|^r . \tag{4.4}$$

From Lemma 9 we can construct a Brownian motion $\{B^*(t)\}$ and a sequence of positive random variables $\{T_i^*\}$ on a probability space $(\Omega^*, \mathscr{F}^*, P^*)$ such that the joint distributions of $\{\xi_1, \xi_2, \cdots, \xi_M\}$ are the same as those of

$$\left\{B^*(T_1^*), B^*(T_1^* + T_2^*) - B^*(T_1^*), \cdots, B^*\left(\sum_{i=1}^{M} T_i^*\right) - B^*\left(\sum_{i=1}^{M-1} T_i^*\right)\right\}.$$

Moreover using Lemma A1 in [1] we can redefine $(\{\hat{y}_i\}, \{\hat{\xi}_i\}, \{\hat{T}_i\}, \{\hat{B}(t)\})$ on a common probability space $(\Omega, \mathscr{F}, P)$ such that the distributions of $(\{\hat{y}_i\}, \{\hat{\xi}_i\})$ and $(\{\hat{\xi}_i\}, \{\hat{T}_i\}, \{\hat{B}(t)\})$ are the same as those of $(\{y_i\}, \{\xi_i\})$ and

$$\left(\left\{B^*\left(\sum_{k=1}^{i} T_k^*\right) - B^*\left(\sum_{k=1}^{i-1} T_k^*\right)\right\}, \quad \{T_i^*\}, \quad \{B^*(t)\}\right),$$

respectively.

In what follows, for simplicity, we shall write $\{y_i\}$, $\{\xi_i\}$, $\{T_i\}$, and $\{B(t)\}$ for $\{\hat{y}_i\}$, $\{\hat{\xi}_i\}$, $\{\hat{T}_i\}$ and $\{\hat{B}(t)\}$.

Define M points on $[0, 1]$ such that $a_k = k[n^\nu]/n$, $k = 0, 1, \cdots, M-1$ and $a_M = 1$. Let $\tilde{X}_n(t)$ and $\tilde{B}_n(t)$ be continuous polygonal lines defined by

$$\tilde{X}_n(t) = X_n(a_k) + (X_n(a_{k+1}) - X_n(a_k))(t - a_k)/(a_{k+1} - a_k),$$
$$\text{for} \quad t \in [a_k, a_{k+1}], \quad k = 0, 1, \cdots, M-1,$$

and

$$\tilde{B}_n(t) = B(a_k) + (B(a_{k+1}) - B(a_k))(t - a_k)/(a_{k+1} - a_k),$$
$$\text{for} \quad t \in [a_k, a_{k+1}], \quad k = 0, 1, \cdots, M-1,$$

respectively.

Lemma 10. *As $n \to \infty$*

$$P\{d(X_n(t), \tilde{X}_n(t)) \geqq \varepsilon_n\} = o(\varepsilon_n).$$

Proof. From the definition of $\{X_n(t)\}$ and $\{\tilde{X}_n(t)\}$ we have

$$\begin{aligned} P\{d(X_n(t), \tilde{X}_n(t)) \geqq \varepsilon_n\} &\leqq M\, P\left\{\max_{1 \leqq k \leqq [n^\nu]} \left|\sum_{i=1}^{k} n^{-1/2} X_i - (k/[n^\nu]) \sum_{i=1}^{k} n^{-1/2} X_i\right| \geqq \varepsilon_n\right\} \\ &\leqq M\, P\left\{\max_{1 \leqq k \leqq [n^\nu]} \left|\sum_{i=1}^{k} n^{-1/2} X_i\right| \geqq \varepsilon_n/2\right\} \\ &\equiv A_1, \qquad \text{say}. \end{aligned} \tag{4.5}$$

Recall that $M=[n/[n^{\nu}]]+1=O(n^{1-\nu})$. Using Lemma 4 we have

$$(4.6)\qquad A_1 \leqq M\,P\left\{\max_{1\leqq k\leqq[n^{\nu}]}\left|\sum_{i=1}^{k} n^{-1/2}\eta_i\right|\geqq \varepsilon_n/4\right\}$$
$$+M\,P\left\{\max_{1\leqq k\leqq[n^{\nu}]}\left|\sum_{i=1}^{k} n^{-1/2}(d_{i+1}-d_i)\right|\geqq \varepsilon_n/4\right\}.$$

Applying the Kolmogorov maximal inequality, the Burkholder inequality for martingale difference sequences (see e.g. [3]) and Lemma 4 to the first term on the right hand side of (4.6), and noting that $E|\eta_i|^s<\infty$, we have

$$(4.7)\qquad M\,P\left\{\max_{1\leqq k\leqq[n^{\nu}]}\left|\sum_{i=1}^{k} n^{-1/2}\eta_i\right|\geqq \varepsilon_n/4\right\}\leqq KM\varepsilon_n{}^{-s}E\left|\sum_{i=1}^{[n^{\nu}]} n^{-1/2}\eta_i\right|^s$$
$$\leqq KM\varepsilon_n{}^{-s}n^{\nu(s-2)/2}\sum_{i=1}^{[n^{\nu}]}E|n^{-1/2}\eta_i|^s$$
$$\leqq K\varepsilon_n{}^{-s}n^{-s(s-2)/4(s-1)}$$
$$=o(\varepsilon_n)\,.$$

Moreover we have from the Chebyshev inequality and Lemma 4,

$$(4.8)\quad M\,P\left\{\max_{1\leqq k\leqq[n^{\nu}]}\left|\sum_{i=1}^{k} n^{-1/2}(d_{i+1}-d_i)\right|\geqq \varepsilon_n/4\right\}=M\,P\left\{\max_{1\leqq k\leqq[n^{\nu}]}|d_{k+1}-d_1|\geqq n^{1/2}\varepsilon_n/4\right\}$$
$$\leqq M\sum_{k=1}^{[n^{\nu}]}P\{|d_{k+1}-d_1|\geqq n^{1/2}\varepsilon_n/4\}$$
$$\leqq M\sum_{k=1}^{[n^{\nu}]}(n^{1/2}\varepsilon_n/4)^{-s}E|d_{k+1}-d_1|^s$$
$$\leqq Kn^{1-s/2}\varepsilon_n{}^{-s}$$
$$=o(\varepsilon_n)\,.$$

Thus we have $A_1=o(\varepsilon_n)$ from (4.6), (4.7) and (4.8) and conclude this lemma.
The following lemma is due to Borovkov (Lemma 2 in [2]).

Lemma 11. *As* $n\to\infty$,

$$P\{d(B(t),\tilde{B}_n(t))\geqq\varepsilon_n\}=o(\exp(-n^{\delta}))=o(\varepsilon_n)\,,$$

for some $\delta>0$.

Finally we shall prove the following

Lemma 12. *As* $n\to\infty$,

$$P\{d(\tilde{X}_n(t),\tilde{B}_n(t))\geqq\varepsilon_n\}=o(\varepsilon_n)\,.$$

Proof. By the definition of $\{\tilde{X}_n(t)\}$ and $\{\tilde{B}_n(t)\}$ and by Lemma 4 we have

(4.9)

$$\begin{aligned} P\{d(\tilde{X}_n(t), \tilde{B}_n(t)) \geqq \varepsilon_n\} &= P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} y_i - B(a_k)\right| \geqq \varepsilon_n\right\} \\ &\leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} \xi_i - B(a_k)\right| \geqq \varepsilon_n/2\right\} + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} v_i\right| \geqq \varepsilon_n/2\right\} \\ &\equiv D_1 + D_2, \qquad \text{say}. \end{aligned}$$

We first estimate D_1. Denote

$$Z_k = \sum_{i=1}^{k} (T_i - ET_i).$$

By Lemma 5 we have

$$\begin{aligned} D_1 &\leqq P\left\{\max_{1\leqq k\leqq M}\left|B\left(Z_k + \sum_{i=1}^{k} ET_i\right) - B(a_k)\right| \geqq \varepsilon_n,\ \max_{1\leqq k\leqq M}|Z_k| < 3\lambda_n\right\} \\ &\quad + P\left\{\max_{1\leqq k\leqq M}|Z_k| \geqq 3\lambda_n\right\} \\ &\equiv E_1 + E_2, \qquad \text{say}. \end{aligned} \tag{4.10}$$

According to the proof of Philipp-Stout [6], we use (4.3) in Lemma 9 to estimate E_2. Then we have

$$\begin{aligned} E_2 &\leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} (T_i - E(T_i \mid \mathscr{B}_{i-1}))\right| \geqq \lambda_n\right\} \\ &\quad + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} (E(\xi_i^2 \mid \xi_{i-1}, \cdots, \xi_1) - \xi_i^2)\right| \geqq \lambda_n\right\} \\ &\quad + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} (\xi_i^2 - E\xi_i^2)\right| \geqq \lambda_n\right\} \\ &\equiv L_1 + L_2 + L_3, \qquad \text{say}. \end{aligned} \tag{4.11}$$

Since $\{T_i - E(T_i \mid \mathscr{B}_{i-1})\}$ and $\{\xi_i^2 - E(\xi_i^2 \mid \xi_{i-1}, \cdots, \xi_1)\}$ are martingale difference sequences, respectively, we have, using the Kolmogorov maximal inequality and the Burkholder inequality for martingale difference sequences,

$$L_1 \leqq K\lambda_n^{-s/2} \sum_{i=1}^{M} E\,|T_i - E(T_i \mid \mathscr{B}_{i-1})|^{s/2}$$

and

$$L_2 \leqq K\lambda_n^{-s/2} \sum_{i=1}^{M} E\,|\xi_i^2 - E(\xi_i^2 \mid \xi_{i-1}, \cdots, \xi_1)|^{s/2}.$$

Using the Hölder inequality for the conditional expectation and Lemmas 9 and 5, we have

$$E|E(T_i|\mathscr{B}_{i-1})|^{s/2} \leqq ET_i^{s/2} \leqq KE|\xi_i|^s \leqq Kn^{(\nu-1)s/2},$$

and

$$E|E(\xi_i^2|\xi_{i-1}, \cdots, \xi_1)|^{s/2} \leqq E|\xi_i|^s \leqq Kn^{(\nu-1)s/2}.$$

Hence

$$L_1 \leqq Kn^{(\nu-1)(s-2)/2}\lambda_n^{-s/2} = o(\varepsilon_n) \tag{4.12}$$

and

$$L_2 = o(\varepsilon_n). \tag{4.13}$$

We next estimate L_3. By the definition of ξ_i we have

$$\begin{aligned} L_3 \leqq & P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k (y_i^2 - Ey_i^2)\right| \geqq \lambda_n/3\right\} + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k (y_iv_i - E(y_iv_i))\right| \geqq \lambda_n/6\right\} \\ & + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k (v_i^2 - Ev_i^2)\right| \geqq \lambda_n/3\right\} \\ \equiv & L_{31} + L_{32} + L_{33}, \qquad \text{say}. \end{aligned} \tag{4.14}$$

As to L_{31}, using Lemma 8, the Kolmogorov maximal inequality and the Burkholder inequality for martingale difference sequences, we have

$$\begin{aligned} L_{31} & \leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k \zeta_i\right| \geqq \lambda_n/6\right\} + P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k (w_{i+1} - w_1)\right| \geqq \lambda_n/6\right\} \\ & \leqq K\lambda_n^{-s/2}E\left|\sum_{i=1}^M \zeta_i\right|^{s/2} + \sum_{i=1}^M P\{|w_{i+1} - w_1| \geqq \lambda_n/6\} \\ & \leqq K\lambda_n^{-s/2}\sum_{i=1}^M E|\zeta_i|^{s/2} + K\lambda_n^{-s/2}\sum_{i=1}^M E|w_{i+1} - w_1|^s \\ & \leqq Kn^{(\nu-1)(s-2)/2}\lambda_n^{-s/2} \\ & = o(\varepsilon_n). \end{aligned} \tag{4.15}$$

We next handle L_{32}. Note that from Lemmas 5 and 3,

$$\sum_{i=1}^M |E(y_iv_i)| = \sum_{i=1}^M |E(y_i(Y_i|\mathscr{L}_{i-1}))| = \sum_{i=1}^M Ey_i^2 = O(n^{-\nu}) = o(\lambda_n).$$

Thus from Lemma 7 we have

$$\begin{aligned} L_{32} & \leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k y_iv_i\right| \geqq \lambda_n/6 - \sum_{i=1}^M |E(y_iv_i)|\right\} \\ & \leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^k y_iv_i\right| \geqq \lambda_n/8\right\} \end{aligned} \tag{4.16}$$

$$\leqq P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k}\mu_i\right|\geqq\lambda_n/16\right\}+P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k}u_i\right|\geqq\lambda_n/16\right\}:$$

From the Chebyshev inequality, the Minkovsky inequality and Lemma 7, it follows that

$$P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k}u_i\right|\geqq\lambda_n/16\right\}\leqq K\lambda_n^{-s/2}E\left(\sum_{i=1}^{M}|u_i|\right)^{s/2} \tag{4.17}$$

$$\leqq K\lambda_n^{-s/2}\left(\sum_{i=1}^{M}\|u_i\|_{s/2}\right)^{s/2}$$

$$\leqq K\lambda_n^{-s/2}n^{(1-2\nu)s/4}$$

$$=o(\varepsilon_n)\,.$$

Since $\{\mu_i\}$ is a martingale difference sequence, we have

$$P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}\mu_i\right|\geqq\lambda_n/16\right\}\leqq K\lambda_n^{-s/2}E\left|\sum_{i=1}^{M}\mu_i\right|^{s/2}$$

$$\leqq K\lambda_n^{-s/2}\sum_{i=1}^{M}E|\mu_i|^{s/2}$$

$$\leqq K\lambda_n^{-s/2}\sum_{i=1}^{M}(E|y_iv_i|^{s/2}+E|u_i|^{s/2})\,.$$

From Lemma 7 we have

$$E|y_iv_i|^{s/2}\leqq Kn^{(\nu-2)s/4}\qquad\text{and}\qquad E|u_i|^{s/2}\leqq Kn^{-s/2}\,,$$

thus it follows that

$$P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k}\mu_i\right|\geqq\lambda_n/16\right\}\leqq K\lambda_n^{-s/2}n^{1-s/2+(s-4)\nu/4}+K\lambda_n^{-s/2}n^{1-\nu-s/2} \tag{4.18}$$

$$=o(\varepsilon_n)\,.$$

Combining (4.16), (4.17) and (4.18) we have

$$L_{32}=o(\varepsilon_n)\,. \tag{4.19}$$

As to L_{33}, we have from Lemma 5

$$L_{33}\leqq P\left\{\max_{1\leqq k\leqq M}\sum_{i=1}^{k}v_i^2\geqq\lambda_n/3-\sum_{i=1}^{M}Ev_i^2\right\} \tag{4.20}$$

$$\leqq P\left\{\sum_{i=1}^{M}v_i^2\geqq\lambda_n/3-o(\lambda_n)\right\}$$

$$\leqq P\left\{\sum_{i=1}^{M}v_i^2\geqq\lambda_n/4\right\}$$

$$\leqq K\lambda_n^{-s/2} E\left(\sum_{i=1}^{M} v_i^2\right)^{s/2}$$
$$\leqq K\lambda_n^{-s/2}\left(\sum_{i=1}^{M} \|v_i^2\|_{s/2}\right)^{s/2}$$
$$\leqq Kn^{-\nu s/2}\lambda_n^{-s/2}$$
$$=o(\varepsilon_n)\,,$$

From (4.14), (4.15), (4.19) and (4.20), it follows that as $n\to\infty$,

$$L_3=o(\varepsilon_n)\,. \tag{4.21}$$

From (4.11), (4.12), (4.13) and (4.21) we obtain

$$E_2=o(\varepsilon_n)\,. \tag{4.22}$$

We next estimate E_1 in (4.10) following Rosenkrantz [7]. Since $E(y_iv_i)=Ev_i^2$ for all $1\leqq i\leqq M$, we have from Lemmas 3, 5 and 9

$$\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} ET_i-a_k\right|\leqq \max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k} Ey_i^2-a_k\right|+\sum_{i=1}^{M} Ev_i^2 \tag{4.23}$$
$$=o(\lambda_n)\,.$$

Recalling

$$Z_k=\sum_{i=1}^{k}(T_i-ET_i),$$

we have from (4.23)

$$E_1=P\left\{\max_{1\leqq k\leqq M}\left|B\left(Z_k+\sum_{i=1}^{k} ET_i\right)-B(a_k)\right|\geqq\varepsilon_n/2,\ \max_{1\leqq j\leqq M}|Z_j|<3\lambda_n\right\} \tag{4.24}$$
$$\leqq\sum_{k=1}^{M} P\left\{\sup_{|t|\leqq \max_{1\leqq j\leqq M}|\sum_{i=1}^{j} ET_i-a_k|+3\lambda_n}|B(t+a_k)-B(a_k)|\geqq\varepsilon_n/2\right\}$$
$$\leqq\sum_{k=1}^{M} P\left\{\sup_{|t|\leqq 4\lambda_n}|B(t+a_k)-B(a_k)|\geqq\varepsilon_n/2\right\}$$
$$\leqq 2MP\left\{\sup_{0\leqq t\leqq 4\lambda_n}|B(t)|\geqq\varepsilon_n/2\right\}$$
$$\leqq 8MP\{|B(t)|\geqq\varepsilon_n/(4\lambda_n^{1/2})\}$$
$$=o(\varepsilon_n)\,.$$

The relations (4.9), (4.22) and (4.24) implies

$$D_1=o(\varepsilon_n)\,. \tag{4.25}$$

We shall finally estimate D_2 in (4.9). By Lemma 6 we have

$$D_2 \leqq P\left\{\max_{1 \leqq k \leqq M}\left|\sum_{i=1}^{k} \theta_i\right| \geqq \varepsilon_n/4\right\} + P\left\{\max_{1 \leqq k \leqq M}\left|\sum_{i=1}^{k} (g_i - g_{i+1})\right| \geqq \varepsilon_n/4\right\} \tag{4.26}$$

We first estimate the second term on the right hand side of (4.26) by Lemma 6 as follows.

$$\begin{aligned} P\left\{\max_{1 \leqq k \leqq M}\left|\sum_{i=1}^{k} (g_i - g_{i+1})\right| \geqq \varepsilon_n\right\} &\leqq P\left\{\max_{1 \leqq k \leqq M} |g_1 - g_{k+1}| \geqq \varepsilon_n\right\} \\ &\leqq \sum_{k=1}^{M} P\{|g_1 - g_{k+1}| \geqq \varepsilon_n\} \\ &\leqq \sum_{k=1}^{M} \varepsilon_n^{-s} E|g_1 - g_{k+1}|^s \\ &= o(\varepsilon_n) . \end{aligned} \tag{4.27}$$

On the other hand, since $\{\theta_i\}$ is a martingale difference sequence, using the Kolmogorov maximal inequality, the Burkholder inequality for martingale difference sequence and Lemma 6, we have

$$\begin{aligned} P\left\{\max_{1 \leqq k \leqq M}\left|\sum_{i=1}^{k} \theta_i\right| \geqq \varepsilon_n\right\} &\leqq KP\left\{\left|\sum_{i=1}^{M} \theta_i\right| \geqq \varepsilon_n\right\} \\ &\leqq K\varepsilon_n^{-s} E\left|\sum_{i=1}^{M} \theta_i\right|^s \\ &\leqq K\varepsilon_n^{-s} M^{(s-2)/2} \sum_{i=1}^{M} E|\theta_i|^s \\ &\leqq Kn^{-vs/2}\varepsilon_n^{-s} \\ &= o(\varepsilon_n) . \end{aligned} \tag{4.28}$$

From (4.26), (4.27) and (4.28) we have

$$D_2 = o(\varepsilon_n) . \tag{4.29}$$

The proof of Lemma 12 is completed from (4.9), (4.25) and (4.29).

(4.1) follows from Lemmas 10, 11 and 12 and the first part of Theorem 1 is concluded.

We next prove the second part of Theorem 1, where $r > 4$ and (1.2) holds for some s with $4 < s < r$. Let $v = 1/3$, $\varepsilon_n = n^{-2/15-\kappa}$ for any $\kappa < (s-4)/30(1+s)$ and $\lambda_n = n^{-4/15-2\kappa}(\log n)^{-2}$. Then, by the Burkholder inequality, we have

$$\begin{aligned} L_1 &\leqq K\lambda_n^{-s/2} M^{(s-4)/4} \sum_{i=1}^{M} E|T_i - E(T_i|\mathscr{B}_{i-1})|^{s/2} \\ &\leqq Kn^{(v-1)s/4}\lambda_n^{-s/2} \\ &= o(\varepsilon_n) \end{aligned}$$

and

$$L_2 \leqq K\lambda_n^{-s/2} M^{(s-4)/4} \sum_{i=1}^{M} E\,|\,\xi_i^2 - E(\xi_i^2\,|\,\xi_{i-1}, \cdots, \xi_1)\,|^{s/2}$$
$$= o(\varepsilon_n)\,.$$

On the other hand, from the Burkholder inequality, it also follows that

$$\begin{aligned} P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{k}\zeta_i\right| \geqq \lambda_n/6\right\} &\leqq K\lambda_n^{-s/2} E\left|\sum_{i=1}^{M}\zeta_i\right|^{s/2} \\ &\leqq K\lambda_n^{-s/2} M^{(s-4)/4} \sum_{i=1}^{M} E\,|\,\zeta_i\,|^{s/2} \\ &\leqq K n^{(\nu-1)s/4} \lambda_n^{-s/2} \\ &= o(\varepsilon_n) \end{aligned}$$

and

$$\begin{aligned} P\left\{\max_{1\leqq k\leqq M}\left|\sum_{i=1}^{M}\mu_i\right| \geqq \lambda_n\right\} &\leqq K\lambda_n^{-s/2} E\left|\sum_{i=1}^{M}\mu_i\right|^{s/2} \\ &\leqq K\lambda_n^{-s/2} M^{(s-4)/4} \sum_{i=1}^{M} E\,|\,\mu_i\,|^{s/2} \\ &\leqq K n^{-s/4} \lambda_n^{-s/2} + K n^{-(1+\nu)s/4} \lambda_n^{-s/2} \\ &= o(\varepsilon_n)\,. \end{aligned}$$

Thus we have $L_3 = o(\varepsilon_n)$ so that $E_2 = o(\varepsilon_n)$ by (4.11).

The rest of the proof is the same as that of the first part of Theorem 1, and we have

$$\rho(P_n, W) = o(\varepsilon_n) = o(n^{-2/15-\kappa})\,.$$

The theorem is thus completely proved.

5. Proof of Corollary

If

$$\alpha(n) = O(e^{-\gamma n})\,,$$

for some $\gamma > 0$, then (1.2) holds for any $s < r$. Hence the conclusion of Theorem 1 holds for any

$$\delta < r(r-2)/4(r-1)(r+1)\,,$$

in case $r \leqq 4$. The case $r > 4$ is also handled in the same way.

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