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A PRIORI ESTIMATE AND ASYMPTOTIC BEHAVIOR FOR SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

By

KIYOMASA NARITA

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§ 1. Introduction

Here we consider the solution of the d-dimensional stochastic differential equation which is generally defined up to the explosion time. With respect to the explosion problem, it follows from the recent papers [4] and [5] that *the radial unboundedness condition* and *the restriction on the growth* of a Liapunov function are sufficient conditions for the non occurrence of the explosion. But the investigation of the reason why such a Liapunov function should play an effective role on the non occurrence of the explosion is left undone. The first purpose of this paper is to study the relation between the a priori bound for the solution and the property of a Liapunov function. More precisely, in § 2 we estimate the order of the probability of the leaving from a bounded domain for the solution of the stochastic differential equation. From the order of its decay, it will be clarified that the radial unboundedness condition and the restriction on the growth of a Liapunov function work for the non occurrence of the explosion. The second purpose is to give the upper bound of the growth of $U(X(t))/2t\log\log t$ for $t\to\infty$ when $X(t)$ is a solution of (1.1) and U is a certain function, so that the estimate contains the special result of McKean [3, p. 107 (Problem 5)]. In applications, *the second order Ito process* and *the stochastic van der Pol equation* will be taken in § 3.

The precise formulation is as follows. Let R^d denote Euclidean d-space, let $\langle x, y\rangle$ be the inner product of $x\in R^d$ and $y\in R^d$ and let $|x|$ be the Euclidean norm of $x\in R^d$. For a $d\times d$-matrix $M=(m_{ij})$, define

$$|M|=\left(\sum_{i,j=1}^{d} m_{ij}^2\right)^{1/2}.$$

Let $W(t)=(w_i(t))$, $i=1,\cdots,d$, be a d-dimensional Brownian motion process adapted to $\boldsymbol{F}_t$ on the underlying probability space $(\Omega, \boldsymbol{F}, P)$ with an increasing family $\{\boldsymbol{F}_t; t\geq 0\}$ of sub-σ-algebras of $\boldsymbol{F}$. Then we consider the stochastic differential equation

$$(1.1)\qquad dX(t)=b(t, X(t))dt+\sigma(t, X(t))dW(t),$$

where $b(t, x)=(b_i(t, x))$, $i=1,\cdots,d$, is a d-vector function and $\sigma(t, x)=(\sigma_{ij}(t, x))$, $i, j=1,\cdots,d$, is a $d\times d$-matrix function, which are defined on $[0, \infty)\times R^d$ and Borel measurable with respect to the complete set of the variables. Throughout this paper, we assume the following condition:

$b(t, x)$ and $\sigma(t, x)$ are continuous in (t, x), and for any $T>0$ $b(t, x)$ and $\sigma(t, x)$ satisfy the local Lipschitz condition with respect to $x \in R^d$ if $t \leqq T$.

For any natural number n, let $b^{(n)}(t, x)=(b_i^{(n)}(t, x))$ and $\sigma^{(n)}(t, x)=(\sigma_{ij}^{(n)}(t, x))$, $i, j=1, \cdots, d$, be functions which satisfy the following conditions;

(i) $b^{(n)}(t, x)=b(t, x)$ and $\sigma^{(n)}(t, x)=\sigma(t, x)$ for $t \leqq n$ and $|x| \leqq n$,

(ii) $b^{(n)}(t, x)$ and $\sigma^{(n)}(t, x)$ satisfy the global Lipschitz condition with respect to $x \in R^d$ if $t \leqq n$,

(iii) $|b^{(n)}(t, x)|^2+|\sigma^{(n)}(t, x)|^2 \leqq K_n(1+|x|^2)$ for $t \leqq n$, $x \in R^d$ and a constant $K_n>0$ depending only on n. Then, by $X^{(n)}(t)$ we mean the pathwise unique solution of the stochastic differential equation

$$dX^{(n)}(t)=b^{(n)}(t, X^{(n)}(t))dt+\sigma^{(n)}(t, X^{(n)}(t))dW(t) \tag{1.2}$$

which is defined up to $t \leqq n$. For the solution of (1.2) with the initial condition $X^{(n)}(t_0)=x_0 \in R^d$ $(t_0 \geqq 0)$, we set

$$\tau_n(t_0, x_0)=\inf\{t;\ |X^{(n)}(t)| \geqq n\}$$

$$(\tau_n(t_0, x_0)=\infty \quad \text{if} \quad \{\ \}=\varnothing)$$

and

$$e_n(t_0, x_0)=\min\{n, \tau_n(t_0, x_0)\}\ .$$

Then the random process $X(t)$ which is defined by $X(t)=X^{(n)}(t)$ for $t<e_n(t_0, x_0)$ $(n=1, 2, \cdots)$ is called *the solution of* (1.1) *with the initial condition* $X(t_0)=x_0$. A random time $e(t_0, x_0)$ which is defined by $e(t_0, x_0)=\lim_{n\to\infty} e_n(t_0, x_0)$ is called *the explosion time* of $X(t)$ with the initial condition $X(t_0)=x_0$. We introduce the differential generator

$$L=\frac{\partial}{\partial t}+\sum_{i=1}^{d} b_i(t, x)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t, x)\frac{\partial^2}{\partial x_i \partial x_j} \tag{1.3}$$

associated with the stochastic differential equation (1.1), where $a(t, x)=(a_{ij}(t, x))$ is defined by $a(t, x)=\sigma(t, x)\sigma(t, x)^*$ (* means the transpose). By $C^{1,2}([0, \infty)\times R^d)$ we denote the family of scalar functions defined on $[0, \infty)\times R^d$ which are twice continuously differentiable with respect to $x \in R^d$ and once with respect to $t \geqq 0$.

§2. A priori estimate

To begin with, we estimate the probability of the leaving from a bounded domain for the solution of (1.2).

Theorem 2.1. *Suppose that there exists a nonnegative function* $V(t, x) \in C^{1,2}([0, \infty)\times R^d)$ *which satisfies*

$$LV(t, x) \leqq \alpha(t)\beta(V(t, x)) \tag{2.1}$$

for $t \geqq 0$ *and* $x \in R^d$, *where* α: $[0, \infty) \to [0, \infty)$ *is continuous and* β: $[0, \infty) \to [0, \infty)$ *is monotone increasing and differentiable. Further, consider the solution* $X^{(n)}(t)$ *of* (1.2) *with the initial condition* $X^{(n)}(t_0) = x_0 \in R^d$ $(t_0 \geqq 0)$ *and let* $T > t_0$ *be arbitrary and be fixed. Then, for any* $n > \max\{T, |x_0|\}$

$$(2.2)\quad P\Big(\sup_{t_0 \leqq t \leqq T} |X^{(n)}(t)| > n\Big) \leqq \Big\{1 + f\Big(\inf_{\substack{0 \leqq t \leqq T \\ |x| = n}} V(t, x)\Big)\Big\}^{-1} \Big\{1 + f(V(t_0, x_0)) + \int_{t_0}^{T} \alpha(s) ds\Big\},$$

where

$$f(v) = \int_0^v du/(1 + \beta(u)).$$

Proof. We assume that there exist such functions V, α and β in the hypothesis and set

$$U(t, x) = 1 + f(V(t, x)),$$

where

$$f(v) = \int_0^v du/(1 + \beta(u)).$$

By $L^{(n)}$ we denote the differential generator associated with the stochastic differential equation (1.2), namely, the definition of $L^{(n)}$ is the same with that of L in (1.3) where $b(t, x)$ and $\sigma(t, x)$ are replaced by $b^{(n)}(t, x)$ and $\sigma^{(n)}(t, x)$ respectively. Since $b^{(n)}(t, x) = b(t, x)$ and $\sigma^{(n)}(t, x) = \sigma(t, x)$ for $t \leqq n$ and $|x| \leqq n$, we notice that $L^{(n)} V(t, x) = LV(t, x)$ for $t \leqq n$ and $|x| \leqq n$. Now a simple calculation shows that

$$\begin{aligned} LU(t, x) &= L\{f(V(t, x))\} \\ &= \{LV(t, x)\} f'(V(t, x)) + \frac{1}{2} |\sigma(t, x)^* \operatorname{grad} V(t, x)|^2 f''(V(t, x)) \\ &= \frac{LV(t, x)}{1 + \beta(V(t, x))} - \frac{1}{2} |\sigma(t, x)^* \operatorname{grad} V(t, x)|^2 \frac{\beta'(V(t, x))}{(1 + \beta(V(t, x)))^2} \\ &\leqq \alpha(t) \end{aligned}$$

for $t \geqq 0$ and $x \in R^d$ because (2.1) holds by the assumption. Consider the solution $X^{(n)}(t)$ of (1.2) with the initial condition $X^{(n)}(t_0) = x_0 \in R^d$ $(t_0 \geqq 0)$. For notational simplicity we write as $\tau_n = \tau_n(t_0, x_0)$ and $e_n = e_n(t_0, x_0)$, omitting (t_0, x_0). Let $T > t_0$ be arbitrary and be fixed, and then choose n so large that $n > \max\{T, |x_0|\}$. Then Ito's formula concerning stochastic differentials implies that

$$\begin{aligned} E[U(T \wedge e_n, X^{(n)}(T \wedge e_n))] &= U(t_0, x_0) + E\Big[\int_{t_0}^{T \wedge e_n} L^{(n)} U(s, X^{(n)}(s)) ds\Big] \\ &= U(t_0, x_0) + E\Big[\int_{t_0}^{T \wedge e_n} LU(s, X^{(n)}(s)) ds\Big] \end{aligned}$$

$$\leqq 1+f(V(t_0, x_0))+\int_{t_0}^{T} \alpha(s)ds\,,$$

where $u \wedge v$ stands for the smaller of u and v. On the other hand, we see that

$$\begin{aligned} E[U(T \wedge e_n, X^{(n)}(T \wedge e_n)] &\geqq E[U(e_n, X^{(n)}(e_n))\,;\, e_n \leqq T] \\ &= E[U(\tau_n, X^{(n)}(\tau_n))\,;\, \tau_n \leqq T] \\ &= E[1+f(V(\tau_n, X^{(n)}(\tau_n)))\,;\, \tau_n \leqq T] \\ &\geqq \left\{1+f\left(\inf_{\substack{0 \leqq t \leqq T \\ |x|=n}} V(t, x)\right)\right\} P(\tau_n \leqq T) \end{aligned}$$

since $\{e_n \leqq T\}=\{\tau_n \leqq T\}$ and $e_n=\tau_n$ on $\{\tau_n \leqq T\}$ for $n>T$ and since f is strictly monotone increasing. Thus the above inequalities yield that

$$\left\{1+f\left(\inf_{\substack{0 \leqq t \leqq T \\ |x|=n}} V(t, x)\right)\right\} P(\tau_n \leqq T) \leqq 1+f(V(t_0, x_0))+\int_{t_0}^{T} \alpha(s)ds\,,$$

from which follows (2.2) if we note that

$$\{\tau_n \leqq T\}=\left\{\sup_{t_0 \leqq t \leqq T} |X^{(n)}(t)| \geqq n\right\}.$$

Hence the proof is complete.

From the order for which the probability (2.2) decreases, we can obtain a sufficient condition for the non occurrence of the explosion in the following corollary, which is just the same with the result in [4] and [5].

Corollary 2.1. *Under the same assumption as in Theorem* 2.1, *suppose that*

$$\lim_{|x| \to \infty} \inf_{0 \leqq t \leqq T} V(t, x)=0 \qquad \textit{for each} \quad T>0 \tag{2.3}$$

and

$$\int_0^{\infty} du/(1+\beta(u))=\infty\,. \tag{2.4}$$

Further, let $X(t)$ be the solution of (1.1) *with the initial condition $X(t_0)=x_0 \in R^d$. Then, $P(e(t_0, x_0)=\infty)=1$ for all $t_0 \geqq 0$ and $x_0 \in R^d$.*

Proof. Let $T>t_0$ be arbitrary and be fixed. Then we notice that

$$\begin{aligned} \{e_n(t_0, x_0) \leqq T\} &= \{\tau_n(t_0, x_0) \leqq T\} \\ &= \left\{\sup_{t_0 \leqq t \leqq T} |X^{(n)}(t)| \geqq n\right\} \qquad \text{for} \quad n>T \end{aligned}$$

and that $e_n(t_0, x_0) \uparrow e(t_0, x_0)$ for $n \uparrow \infty$. Therefore, by letting n tend to infinity in the both sides of (2.2), we can obtain that $P(e(t_0, x_0) \leqq T) = 0$ for any $T > t_0$, since (2.3) and (2.4) hold by the assumption. Hence the proof is complete.

The condition (2.3) is *the radial unboundedness condition* of $V(t, x)$ and the condition (2.4) is *the restriction on the growth* of $V(t, x)$. The conditions (2.3) and (2.4) work for the non occurrence of the explosion by reason of the decay of the probability of the leaving from a bounded domain for the solution.

§ 3. Asymptotic behavior

For a no explosive solution $X(t)$ of (1.1), we can get the upper bound of the growth of $|X(t)|^2/2t \log\log t$ for $t \to \infty$ if we take $V(t, x) = |x|^2/2$ in the following theorem.

Theorem 3.1. *Let $X(t)$ be the solution of* (1.1) *with the initial condition $X(t_0) = x_0 \in R^d$ such that*

$$P(e(t_0, x_0) = \infty) = 1 .$$

Further, suppose that there exists a nonnegative function $V(t, x) \in C^{1,2}([0, \infty) \times R^d)$ which satisfies the following conditions;

$$LV(t, x) \leqq C(t) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad x \in R^d , \tag{3.1}$$

$$|\sigma(t, x)^* \operatorname{grad} V(t, x)|^2 \leqq D(t) V(t, x) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad x \in R^d , \tag{3.2}$$

where C: $[0, \infty) \to [0, \infty)$ is continuous such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t C(s) ds = \tilde{C} < \infty \tag{3.3}$$

and D: $[0, \infty) \to [0, \infty)$ is continuous such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t D(s) ds = \tilde{D} < \infty . \tag{3.4}$$

Then,

$$P\left(\limsup_{t \to \infty} V(t, X(t)) \Big/ t \log\log t \leqq \exp(\tilde{D}/2) \right) = 1 .$$

Proof. Let $X(t)$ be the solution of (1.1) with the initial condition $X(t_0) = x_0$ such that $P(e(t_0, x_0) = \infty) = 1$ and let $V(t, x)$ be the function in the hypothesis. Then we apply Ito's formula concerning stochastic differentials with the result that

$$V(t, X(t)) = V(t_0, x_0) + \int_{t_0}^t LV(s, X(s)) ds + M(t)$$

for all $t \geqq t_0$, almost surely, where

$$M(t) = \int_{t_0}^t \langle \operatorname{grad} V(s, X(s)), \sigma(s, X(s)) dW(s) \rangle .$$

By the time substitution rule (McKean [3, p. 45 (Problem 1)]), we notice that $M(t)=z(\phi(t))$ for a new Brownian motion process run with the clock

$$\phi(t)=\int_{t_0}^{t}|\sigma(s, X(s))^* \operatorname{grad} V(s, X(s))|^2 ds .$$

Now set $H(t)=V(t, X(t))$ and $e_n=n\wedge\inf\{t; |X(t)|\geqq n\}$ $(n=1, 2, \cdots)$. Then the condition (3.1) yields,

$$H(t)\leqq H(t_0)+\int_{t_0}^{t}C(s)ds+z(\phi(t)) \tag{3.5}$$

for all $t\geqq t_0$, almost surely. Also, the assumption of the non occurrence of the explosion implies,

$$e_n\to\infty \quad \text{as} \quad n\to\infty , \quad \text{almost surely.}$$

Next we set

$$y(t)=\exp\left\{\alpha z(\phi(t))-\frac{1}{2}\alpha^2\phi(t)\right\}$$

with a constant α, so that for fixed n $\{y(t\wedge e_n); t\geqq t_0\}$ is a martingale. Then, using the familiar martingale bound, we get that for any number α, β and $T\geqq t_0$

$$P\left(\sup_{t_0\leqq t\leqq T\wedge e_n} z(\phi(t))-\frac{\alpha}{2}\phi(t)>\beta\right)=P\left(\sup_{t_0\leqq t\leqq T\wedge e_n} y(t)>\exp(\alpha\beta)\right)$$
$$\leqq\exp(-\alpha\beta)$$

(see McKean [3; p. 22, p. 47]). Letting n tend to infinity in the above equation, we have,

$$P\left(\sup_{t_0\leqq t\leqq T} z(\phi(t))-\frac{\alpha}{2}\phi(t)>\beta\right)\leqq\exp(-\alpha\beta),$$

since $e_n\to\infty$ as $n\to\infty$, almost surely. In the following, let $\delta>1$ be arbitrary and be fixed. Then for any natural number n we put

$$T=t_0+\delta^n, \qquad \alpha=\delta^{-n}, \qquad \beta=\delta^{n+1}\log n ,$$

from which follows

$$P\left(\sup_{t_0\leqq t\leqq t_0+\delta^n} z(\phi(t))-\frac{1}{2}\delta^{-n}\phi(t)>\delta^{n+1}\log n\right)\leqq n^{-\delta} .$$

Since $n^{-\delta}$ is the general term of a convergent sum, the Borel-Cantelli lemma implies for the sufficiently large n and for $t_0\leqq t\leqq t_0+\delta^n$

$$z(\phi(t))\leqq\frac{1}{2}\delta^{-n}\phi(t)+\delta^{n+1}\log n$$

almost surely. In the following, let n be sufficiently large and be fixed. On the other hand, consider (3.2) with the result that

$$\phi(t) \leqq \int_{t_0}^{t} D(s)H(s)ds$$

and so

$$z(\phi(t)) \leqq \frac{1}{2}\delta^{-n}\int_{t_0}^{t} D(s)H(s)ds + \delta^{n+1}\log n$$

for all $t_0 \leqq t \leqq t_0 + \delta^n$, almost surely. Combining this with (3.5), we get that

$$H(t) \leqq H(t_0) + \int_{t_0}^{t} C(s)ds + \frac{1}{2}\delta^{-n}\int_{t_0}^{t} D(s)H(s)ds + \delta^{n+1}\log n$$

and hence

$$H(t) \leqq H(t_0) + (p_n + 1)\delta^{n+1}\log n + \frac{1}{2}\delta^{-n}\int_{t_0}^{t} D(s)H(s)ds$$

for all $t_0 \leqq t \leqq t_0 + \delta^n$, almost surely, where

$$p_n = \frac{1}{\delta^{n+1}\log n}\int_{t_0}^{t_0+\delta^n} C(s)ds\,.$$

Accordingly, Gronwall-Bellman inequality yields that

$$H(t) \leqq [H(t_0) + (p_n + 1)\delta^{n+1}\log n]\exp\left[\frac{1}{2}\delta^{-n}\int_{t_0}^{t} D(s)ds\right] \tag{3.6}$$

for all $t_0 \leqq t \leqq t_0 + \delta^n$, almost surely. Now choose t so that

$$t_0 + \delta^{n-1} < t \leqq t_0 + \delta^n$$

and then note that

$$\delta^{n-1}\log\log\delta^{n-1} < t\log\log t$$

and

$$\int_{t_0}^{t} D(s)ds \leqq \int_{t_0}^{t_0+\delta^n} D(s)ds$$

for all $t_0 + \delta^{n-1} < t \leqq t_0 + \delta^n$. Then, dividing the both sides of (3.6) by $t\log\log t$, we obtain that

$$\frac{H(t_0)}{t\log\log t} \leqq \left[\frac{H(t_0)}{\delta^{n-1}\log\log\delta^{n-1}} + (p_n + 1)\delta^2\frac{\log n}{\log\log\delta^{n-1}}\right] \times \exp\left[\frac{1}{2}\delta^{-n}\int_{t_0}^{t_0+\delta^n} D(s)ds\right] \tag{3.7}$$

for all $t_0+\delta^{n-1}<t\leqq t_0+\delta^n$, almost surely. It follows from (3.3) and (3.4) that

$$p_n=\frac{1}{\delta\log n}\frac{1}{\delta^n}\int_{t_0}^{t_0+\delta^n}C(s)ds\to 0 \qquad \text{as} \quad n\to\infty$$

and

$$\delta^{-n}\int_{t_0}^{t_0+\delta^n}D(s)ds\to\tilde{D} \qquad \text{as} \quad n\to\infty\,.$$

Also, it is evident that $\log n/\log\log\delta^{n-1}\to 1$ as $n\to\infty$. Therefore, letting n tend to infinity in the both sides of (3.7), we obtain,

$$\limsup_{t\to\infty}\frac{H(t)}{t\log\log t}\leqq\delta^2\exp(\tilde{D}/2)\,,$$

almost surely. Now make $\delta\downarrow 1$. Then the assertion of the theorem holds, and hence the proof is complete.

Example 3.1. Consider the system of the stochastic differential equations

$$\begin{aligned} &dX_1(t)=X_2(t)dt\,, \\ &dX_2(t)=[-g(X_1(t))X_2(t)-f(X_1(t))]dt+dw(t)\,, \end{aligned} \tag{3.8}$$

where $w(t)$ is a one dimensional Brownian motion process, and f: $(-\infty,\infty)\to(-\infty,\infty)$ is continuously differentiable and g: $(-\infty,\infty)\to(-\infty,\infty)$ is continuously differentiable.

The system (3.8) is one of the formulations such that $X_1(t)$ may correspond to the response of the oscillator

$$\ddot{y}+g(y)\dot{y}+f(y)=\dot{w}$$

with the restoring force f and the damping g to the formal white noise $\dot{w}$, where by we mean the symbolic derivative d/dt. The solution $X_1(t)$ is called *the second order Ito process* by Goldstein [1]. The system (3.8) can be written as a vector stochastic differential equation of the form (1.1), where

$$b=(x_2,\ -g(x_1)x_2-f(x_1))\,, \qquad \sigma=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

for $x=(x_1,x_2)\in R^2$, and $W(t)=(w_0(t),w(t))$ with a (dummy) Brownian motion process $w_0(t)$ which is independent of $w(t)$.

In the following, we assume that

$$x_1f(x_1)>0 \qquad \text{for} \quad x_1\neq 0$$

and

$$g(x_1)\geqq 0 \qquad \text{for} \quad x_1\in(-\infty,\infty)\,.$$

Then, under the above assumption, every solution of (3.8) cannot explode (see [6]).

Now set

$$V(x)=\int_0^{x_1} f(s)ds+\frac{1}{2}x_2^2 \qquad \text{for} \quad x=(x_1, x_2)\in R^2 .$$

Then it is easy to see that

$$LV(x)=-g(x_1)x_2^2+\frac{1}{2}\leqq\frac{1}{2}$$

and

$$|\sigma^* \operatorname{grad} V(x)|^2=x_2^2\leqq 2V(x)$$

for all $x=(x_1, x_2)\in R^2$, where L is the differential generator associated with (3.8). Therefore, if we take $C(t)\equiv\tilde{C}=1/2$ and $D(t)\equiv\tilde{D}=2$, then Theorem 3.1 will apply with the result that

$$P\left(\limsup_{t\to\infty} V(X(t))/t\log\log t\leqq\exp(1)\right)=1 .$$

This is a generalization of McKean's result [3, p. 107 (Problem 5)], where the oscillator $\ddot{y}+f(y)=\dot{w}$ is considered.

Example 3.2. Consider the system of the stochastic differential equations

$$\begin{aligned} &dX_1(t)=(X_2(t)-\varepsilon F(X_1(t)))dt , \\ &dX_2(t)=-X_1(t)dt+h(t, X_1(t), X_2(t))dw(t) , \end{aligned} \tag{3.9}$$

where $w(t)$ is a one dimensional Brownian motion process, and ε is a positive constant (ε may be sufficiently small), $F(s)=s^3/3-s$ and h: $[0, \infty)\times(-\infty, \infty)\times(-\infty, \infty)\to(-\infty, \infty)$ has continuous first partials.

The system (3.9) is one of the formulations such that $X_1(t)$ may correspond to the oscillator

$$\ddot{y}+\varepsilon(y^2-1)\dot{y}+y=h(t, y, \dot{y})\dot{w}$$

to the formal white noise $\dot{w}$. In the deterministic case when $h(t, x_1, x_2)\equiv 0$, (3.9) is equivalent to the *van der Pol equation* $\ddot{y}+\varepsilon(y^2-1)\dot{y}+y=0$ which has a limit cycle (see LaSalle and Lefschetz [2]).

The equation (3.9) can be written as a vector stochastic differential equation of the form (1.1), where

$$b(t, x)=(x_2-\varepsilon F(x_1), -x_1) , \qquad \sigma(t, x)=\begin{pmatrix}0 & 0\\ 0 & h(t, x_1, x_2)\end{pmatrix}$$

for $t\geqq 0$ and $x=(x_1, x_2)\in R^2$, and $W(t)=(w_0(t), w(t))$ with a (dummy) Brownian motion process $w_0(t)$ which is independent of $w(t)$.

In the following, we assume that

$$h^2(t, x_1, x_2) \leqq k(t) \quad \text{for} \quad t \geqq 0 \quad \text{and} \quad x=(x_1, x_2) \in R^2$$

with a continuous function $k(t)$ such that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t k(s)ds = \bar{k} < \infty\,.$$

Now let L be the differential generator associated with (3.9), and set $V(x)=|x|^2/2=(x_1{}^2+x_2{}^2)/2$ and $U(x)=V(x)+1$ for $x=(x_1, x_2)\in R^2$. Then the above assumption yields that

$$\begin{aligned} LU(x) &= LV(x) \\ &= -\varepsilon x_1 F(x_1) + \frac{1}{2}h^2(t, x_1, x_2) \\ &= -\varepsilon x_1{}^2(x_1{}^2/3-1) + \frac{1}{2}h^2(t, x_1, x_2) \\ &\leqq \frac{3}{4}\varepsilon + \frac{1}{2}k(t) \end{aligned}$$

for all $t \geqq 0$ and $x=(x_1, x_2)\in R^2$. Thus, $U(x)$ satisfies that

$$LU(x) \leqq C(t)U(x) \quad \text{and} \quad U(x)\to\infty \quad \text{as} \quad |x|\to\infty\,,$$

where

$$C(t) = \frac{3}{4}\varepsilon + \frac{1}{2}k(t)$$

is continuous. Therefore, Corollary 2.1 implies that every solution of (3.9) cannot explode. Further, we see that $LV(x) \leqq C(t)$ and also

$$\begin{aligned} |\sigma(t, x)^* \operatorname{grad} V(x)|^2 &= x_2{}^2 h^2(t, x_1, x_2) \\ &\leqq D(t)V(x) \end{aligned}$$

for all $t \geqq 0$ and $x=(x_1, x_2)\in R^2$, where $D(t)=2k(t)$. Therefore, since

$$\frac{1}{t}\int_0^t C(s)ds \to \frac{3}{4}\varepsilon + \frac{1}{2}\bar{k} \quad (t\to\infty)$$

and

$$\frac{1}{t}\int_0^t D(s)ds \to 2\bar{k} \quad (t\to\infty)$$

by the assumption, Theorem 3.1 will apply with the result that

$$P\left(\limsup_{t\to\infty} |X(t)|^2/2t \log\log t \leqq \exp(\bar{k})\right) = 1\,.$$

When $h(t, x_1, x_2)=\varepsilon^{1/2}g(t, x_1, x_2)$ in (3.9) for some function $g(t, x_1, x_2)$, it will be important for us to investigate the problem whether the limit process exists or does not.

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Department of Mathematics
Faculty of Technology
Kanagawa University
3–37, Rokkakubashi, Kanagawa-ku
Yokohama 221, Japan