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STABILITY OF CONSTANT MEAN CURVATURE SURFACES IN RIEMANNIAN 3-SPACE FORM

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Let f: $M\to\bar{M}^3(a)$ be a C^∞-immersion of an oriented 2-manifold M into a complete, simply-connected Riemannian 3-manifold $\bar{M}^3(a)$ of constant curvature a. We assume that the mean curvature of the immersion f is constant, say H. In a recent paper [4], Ruchert studied the sufficient condition of positiveness of the second variation of the parameter invariant functional.

$$A(f)=\int_M\left(|f_{x_1}\wedge f_{x_2}|+\frac{2}{3}H(f,f_{x_1}\wedge f_{x_2})\right)dx_1dx_2$$

which was introduced by Heinz for proving the existence of surfaces in $\mathbf{R}^3$ of constant mean curvature H (see [3]).

In this note we study the case the ambient space is a Euclidean 3-sphere $S^3(a)$ of constant positive curvature a or a hyperbolic 3-space $H^3(a)$ of constant negative curvature a. Let F: $(-\varepsilon,\varepsilon)\times M\to\bar{M}^3(a)$ $(\varepsilon>0)$ be a normal variation of f which keeps the boundary ∂D of a domain D fixed. Following Gulliver [2], we consider surfaces of constant curvature H as critical points for the operator

$$A_f(F)=\int_D|F_{x_1}\wedge F_{x_2}|\,dx_1dx_2+2H\int_D\int_0^t(F_{x_1}\wedge F_{x_2}\wedge F_t,d\bar{M})dtdx_1dx_2$$

for all variations F of M as above. Here $d\bar{M}$ is the volume element of $\bar{M}^3(a)$ and $\wedge$ is the exterior product for vectors in $\mathbf{R}^4$ or in the Lorentz space L^4 (depending on a being positive or negative).

We obtain the second variation formula for the above functional. This is a quadratic form I_f that acts on normal vector fields to M that vanish on ∂D. By choosing a unit normal vector field N, I_f can be thought of as an operator acting on the space of C^∞ real functions on D that vanish on ∂D, given by

$$I_f(\Psi)=\int_D(|\nabla\Psi|^2-2(2a+2H^2-K)\Psi^2)dM$$

when $\nabla\Psi$ stands for the gradient of Ψ and K by the Gaussian curvature of M in

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the induced metric. By a domain D on M we mean an open simply connected subset of M whose closure $\bar{D}$ is compact with C^∞ boundary. Then we prove:

Theorem 1. *Let f: $M \to \bar{M}^3(a)$, $a \geq 0$, be a C^∞-immersion. Assume that the mean curvature of the immersion f is the constant H. Let D be a domain on M. Then, if*

$$\int_D (2a + 2H^2 - K)dM < 2\pi,$$

the second variation of the integral A_f is positive for all normal variations with fixed boundary.

Theorem 2. *Let f: $M \to H^3(a)$ be an immersion of M into the hyperbolic 3-space of constant negative curvature a. Assume that the mean curvature of the immersion f is the constant H. Let D be a domain on M. Then the second variation of the integral A_f is positive for all normal variations with fixed boundary if:*

i) $$\int_D (a + 2H^2 - K)dM < 2\pi \quad \text{when } a \geq -4H^2$$

ii) $$\int_D (a + 2H^2 + \delta - K)dM < 2\pi, \ \delta = -\frac{a + 4H^2}{3} \quad \text{when } a \leq -4H^2.$$

Remark. The above results may be regarded as generalization of our previous results in the following sense. When the immersion f is minimal (i.e., $H = 0$), Theorem 1 (resp. Theorem 2) reduces one of results in [1] (resp. in [6]).

1. Preliminaries

Let M be an oriented 2-manifold and f: $M \to S^3(a)$ a C^∞-immersion into an Euclidean 3-sphere $S^3(a)$ of constant positive curvature a. Let $\{x_1, x_2\}$ be local coordinates on M such that $\{\partial/\partial x_1, \partial/\partial x_2\}$ is a positively ordered basis of the tangent plane of M where they can be defined and N the field of unit normal vectors along the immersion f such that $\{\partial/\partial x_1, \partial/\partial x_2, N\}$ is a positively oriented basis of the tangent space of $S^3(a)$ along f where they can be defined.

Note that $S^3(a)$ is realized as a hypersurface of the Euclidean 4-space $\mathbf{R}^4$: $S^3(a) = \{(u_1, u_2, u_3, u_4) \in \mathbf{R}^4; \sum (u_i)^2 = 1/a\}$.

Let D be a domain on M and ψ any element of $C^\infty(\bar{D})$ with $\psi \mid \partial D = 0$. A normal variation with variation vector ψN is given by the equation

$$F(t, x) = \cos(c\psi(x)t)f(x) + \frac{\sin(c\psi(x)t)}{c}N(x), \qquad \text{where} \quad c = \sqrt{a}.$$

Putting $F_j = (\partial/\partial x_j)F$, $N_j = (\partial/\partial x_j)N$, $f_j = (\partial/\partial x_j)f$ and $\psi_j = (\partial/\partial x_j)\psi$, $j = 1, 2$, we get the following exterior product of F_1 and F_2.

$$(1)\qquad F_1 \wedge F_2 = \left(\cos^2(c\psi t) - 2H\frac{\sin(c\psi t)\cos(c\psi t)}{c} + (K-a)\frac{\sin^2(c\psi t)}{c^2}\right) f_1 \wedge f_2$$
$$+ t\cos^2(c\psi t)(\psi_2 f_1 \wedge N + \psi_1 N \wedge f_2)$$
$$- ct\sin(c\psi t)\cos(c\psi t)(\psi_1 f \wedge f_2 + \psi_2 f_1 \wedge f)$$
$$- t\sin^2(c\psi t)(\psi_1 f \wedge N_2 + \psi_2 N_1 \wedge f)$$
$$+ \frac{t\sin(c\psi t)\cos(c\psi t)}{c}(\psi_1 N \wedge N_2 + \psi_2 N_1 \wedge N),$$

where we use the identities

$$N_i = -\sum l_{ij} g^{jk} f_k, \qquad i=1,2,$$
$$K - a = \det(l_{ij})(\det(g_{ij}))^{-1},$$
$$2H = \sum l_{ij} g^{ij},$$

l_{ij} and g_{ij} are coefficients of the second and first fundamental forms of f, and $(g^{ij}) = (g_{ij})^{-1}$. Since f is orthogonal to f_1, f_2 and N, and N is orthogonal to f_1 and f_2, from the above equation we have the following

$$|F_1 \wedge F_2|^2 = \left\{\left(\cos^2(c\psi t) - 2H\frac{\sin(c\psi t)\cos(c\psi t)}{c} + (K-a)\frac{\sin^2(c\psi t)}{c^2}\right)^2 + \left(\sum g^{ij}\psi_i\psi_j\right)t^2\cos^4(c\psi t) + t^3\sigma_1(t,x)\right\} g$$
$$= \left(1 - 4H\psi t + 2(K + 2H^2 - 2a)\psi^2 t^2 + \left(\sum g^{ij}\psi_i\psi_j\right)t^2 + t^3\sigma_2(t,x)\right) g$$

where $g = \det(g_{ij})$ and in what follows we denote by $\sigma_j(t,x)$, $j=1,2,\cdots$, C^∞ functions in t and x. Taking the square root of this equation we get the following

$$|F_1 \wedge F_2| = \left(1 - 2H\psi t + (K-2a)\psi^2 t^2 + \frac{1}{2}\left(\sum g^{ij}\psi_i\psi_j\right)t^2 + t^3\sigma_3(t,x)\right) g^{1/2}$$

Integrating this equation over the domain D we have that

$$(2)\qquad \int_D |F_1 \wedge F_2| dx_1 dx_2 = \text{Area}(D) - 2Ht\int_D \psi dM$$
$$+ t^2\int_D \left(\frac{1}{2}|\nabla\psi|^2 + (K-2a)\psi^2\right) dM + t^3 b(t),$$

where $dM = g^{1/2} dx_1 \wedge dx_2$ is the area element of M, $|\nabla\psi|$ is the norm of the gradient of ψ in the metric on M and $b(t)$ is a C^∞ function in t.

Putting $F_t = \partial F/\partial t$ we get $F_t \wedge F = \psi N \wedge f$, and hence

$$F_1 \wedge F_2 \wedge F_t \wedge cF = \psi\left(\cos^2(c\psi t) - 2H\frac{\sin(c\psi t)\cos(c\psi t)}{c} + (K-a)\frac{\sin^2(c\psi t)}{c^2}\right) \times f_1 \wedge f_2 \wedge N \wedge cf$$

by virtue of (1).

Since the volume element $d\bar{M}$ of $S^3(a)$ is $g^{-1/2}f_1 \wedge f_2 \wedge N$, and cF and cf are unit vectors orthogonal to F_1, F_2, F_t, and f_1, f_2, N, respectively, we get the following integral estimate

$$\int_0^t (F_1 \wedge F_2 \wedge F_t, d\bar{M})dt = \int_0^t (F_1 \wedge F_2 \wedge F_t \wedge cF, g^{-1/2}f_1 \wedge f_2 \wedge N \wedge cf)dt = (t\psi - t^2 H\psi^2 + t^3\sigma_3(t, x))g^{1/2}$$

and hence the following

$$\int_D \int_0^t (F_1 \wedge F_2 \wedge F_t, d\bar{M})dt dx_1 dx_2 = t\int_D \Psi dM - Ht^2 \int_D \psi^2 dM + t^3 d(t) \tag{3}$$

where $d(t)$ is a C^∞ function in t. From (2) and (3) we have

$$A_f(t, \psi N) = A_f(F) = \text{Area}(D) + t^2 \int_D \left(\frac{1}{2}|\nabla\psi|^2 + (K - 2a - 2H^2)\psi^2\right)dM + t^3(b(t) + d(t)).$$

Therefore, the first variation of $A_f(t, \psi N)$ is zero and the second variation of $A_f(t, \psi N)$ is

$$\frac{d^2}{dt^2}A_f(t, \psi N)\bigg|_{t=0} = \int_D (|\nabla\psi|^2 - 2(2a + 2H^2 - K)\psi^2)dM. \tag{4}$$

Now, we want to estimate the Gaussian curvature of the metric which is conformally related to the induced one.

Proposition 1. *Let ds^2 be the induced metric on M. Then the Gaussian curvature $\hat{K}$ of the metric $d\hat{s}^2 = ((1/2)\|B\|^2 + a)ds^2$ satisfies $\hat{K} \le 1$, where $\|B\|$ is the norm of the second fundamental form.*

Proof. For a positive number δ which will be determined later we put $u = (1/2)\|B\|^2 + \delta$. Let's observe that by Gauss equation we obtain

$$\|B\|^2 = 4H^2 - 2K + 2a$$

and

$$H^2 - K + a \ge 0.$$

Therefore

$$u = H^2 + (H^2 - K + a) + \delta \ge H^2 + \delta.$$

The Gaussian curvature $\hat{K}$ of the metric uds^2 (see [1]) is given by:

$$\hat{K}=\frac{K}{u}+\frac{1}{2u^3}(-u\Delta u+|\nabla u|^2)\,. \tag{5}$$

We now adopt the notations in Chern's notes [5]. Let $\{\theta_1, \theta_2\}$ and $\{e_1, e_2\}$ be orthonormal 1-forms and its dual basis and denote by h_{ij}, h_{ijk} the coefficients of the second fundamental form relative to $\{e_1, e_2\}$ and ones of their derivatives. Then from the definition of u

$$du=\sum h_{ij}h_{ijk}\theta_k=\sum u_k\theta_k$$

and

$$Du_k=\sum h_{ijl}h_{ijk}\theta_l+h_{ij}h_{ijkl}\theta_l=\sum_l u_{kl}\theta_l\,,$$

where h_{ijkl} are the second derivatives of h_{ij}. The Laplacian of u is by definition

$$\Delta u=\sum_k u_{kk}=\sum_{i,j,k}h_{ijk}^2+\sum_{i,j,k} h_{ij}h_{ijkk}\,.$$

Putting this equation into (5) we get

$$\hat{K}=\frac{K}{u}-\frac{1}{2u^3}\left\{u\sum_{i,j,k} h_{ij}h_{ijkk}+u\sum_{i,j,k} h_{ijk}^2-\sum_k\Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2\right\}. \tag{6}$$

Now we want to propose some lemmas.

Lemma 1.

$$-u\sum_{i,j,k} h_{ijk}^2+\sum_k\Big(\sum h_{ij}h_{ijk}\Big)^2\le 0\,.$$

Proof. From the definition of u it is sufficient to prove the following inequality.

$$W=-\frac{1}{2}\|B\|^2\sum_{i,j,k} h_{ijk}^2+\sum_k\Big(\sum_{i,j} h_{ij}h_{ijk}\Big)^2\le 0\,.$$

For an arbitrary fixed point $x\in M$ we may choose an orthonormal basis $\{e_1, e_2\}$ of $T_x(M)$, the tangent plane of M at x, relative to which $h_{11}=\lambda$, $h_{22}=2H-\lambda$, $h_{12}=0$. And from the constancy of the mean curvature of the immersion we have $h_{11k}+h_{22k}=0$, $k=1, 2$, and, since the ambient space is of constant sectional curvatures, h_{ijk} are symmetric in i, j and k. Then W can be transformed into the following

$$\begin{aligned}
W&=-\frac{1}{2}(h_{11}^2+h_{22}^2)4(h_{111}^2+h_{112}^2)+(h_{11}h_{111}+h_{22}h_{221})^2+(h_{11}h_{112}+h_{22}h_{222})^2\\
&=-\frac{1}{2}(h_{11}^2+h_{22}^2)4(h_{111}^2+h_{112}^2)+(h_{11}-h_{22})^2(h_{111}^2+h_{112}^2)\\
&=-((\lambda-H)^2+H^2)4(h_{111}^2+h_{112}^2)+4(\lambda-H)^2(h_{111}^2+h_{112}^2)\\
&=-4H^2(h_{111}^2+h_{112}^2)\\
&\le 0\,.
\end{aligned}$$

This completes the proof of Lemma 1.

Lemma 2.

$$\sum h_{ij}h_{ijkk} = -\|B\|^4 + 2a\|B\|^2 + 6H^2\|B\|^2 - 4aH^2 - 8H^4$$

Proof. Since dim $M=2$ and codim $M=1$ the formula (12) in [5, p. 38] becomes,

$$\begin{aligned} h_{ikjl} - h_{iklj} = & -\sum_m h_{mk}h_{ml}h_{ij} + \sum_m h_{mk}h_{mj}h_{il} \\ & -\sum_m h_{im}h_{ml}h_{kj} + \sum_m h_{im}h_{mj}h_{kl} \\ & - ah_{lk}\delta_{ij} + ah_{jl}\delta_{ik} - ah_{il}\delta_{kj} + ah_{ij}\delta_{kl}, \end{aligned}$$

from this and the fact that h_{ijk} are symmetric in i, j and k we have

$$\begin{aligned} \sum_k h_{ijkk} &= \sum_k h_{ikjk} \\ &= \sum h_{ikkj} - \sum_{m,k} h_{mk}^2 h_{ij} + \sum_{m,k} h_{im}h_{mj}h_{kk} - a\sum_k h_{kk}\delta_{ij} + ah_{ij}\sum_k \delta_{kk} \\ &= (2H)_{ij} - \|B\|^2 h_{ij} + 2H\left(\sum_m h_{im}h_{mj}\right) - 2aH\delta_{ij} + 2ah_{ij}. \end{aligned}$$

From this equation together with the constancy of H we get

$$\sum_{i,j,k} h_{ij}h_{ijkk} = -\|B\|^4 - 4aH^2 + 2a\|B\|^2 + 2H \sum_{i,j,m} h_{im}h_{mj}h_{ij}. \tag{7}$$

On the other hand,

$$\begin{aligned} \sum_{i,j,m} (h_{im}h_{mj}h_{ij}) &= \text{trace of } (h_{ij})^3 = \text{trace of} \begin{pmatrix} \lambda & 0 \\ 0 & 2H-\lambda \end{pmatrix}^3 = \text{trace of} \begin{pmatrix} \lambda^3 & 0 \\ 0 & (2H-\lambda)^3 \end{pmatrix} \\ &= 2H(3\lambda^2 - 6\lambda H + 4H^2) \\ &= 2H\left\{\frac{3}{2}(\lambda^2 + (2H-\lambda)^2) - 2H^2\right\}. \end{aligned}$$

Thus we get that

$$\sum_{i,j,m} h_{im}h_{mj}h_{ij} = 3H\|B\|^2 - 4H^3.$$

Putting this equation into (7) we see that the assertion of Lemma 2 is true.

We continue proving Proposition 1. From Lemmas 1, 2 and (6) together with the definition of u and the equation of Gauss we have that

$$\begin{aligned} \hat{K} &\leq \frac{K}{u} + \frac{1}{2u^2}(\|B\|^4 - 2a\|B\|^2 - 6H^2\|B\|^2 + 4aH^2 + 8H^4) \\ &= \frac{2H^2 + a + \delta - u}{u} + \frac{1}{u^2}(2(u-\delta)^2 - 2a(u-\delta) - 6H^2(u-\delta) + 2aH^2 + 4H^4) \end{aligned}$$

$$=1+\frac{1}{u^2}\left(-(a+4H^2+3\delta)u+2\delta^2+2a\delta+2aH^2+6H^2\delta+4H^4\right)$$

Set $g(u)=-(a+4H^2+3\delta)u+2\delta^2+2a\delta+2aH^2+6H^2\delta+4H^4$. Since $u\geq H^2+\delta$, to prove that $\hat{K}\leq 1$ it is sufficient to show that $g(u)\leq 0$ when $u\geq H^2+\delta$. To have this we just choose $\delta\geq 0$ such that $a+4H^2+3\delta>0$ and $g(H^2+\delta)\leq 0$. If $a\geq 0$ then the first inequality holds for all values of $\delta\geq 0$ and the second for the values of $\delta\geq a$. Therefore, by choosing $\delta=a$, as in the statement of Proposition 1 we obtain $\hat{K}\leq 1$.

2. Proof of Theorem 1.

We denote by $\hat{\nabla}$, $d\hat{M}$ the covariant differentiation and the area element of M in the metric

$$d\hat{s}^2=\left(\frac{1}{2}\|B\|^2+a\right)ds^2=(2a+2H^2-K)ds^2\,.$$

Then the second variation of the functional $A_f(t, \psi N)$ satisfies

$$\frac{d^2}{dt^2}A_f(t, \psi N)\bigg|_{t=0}=\int_D(|\hat{\nabla}\psi|^2-2\psi^2)d\hat{M}\,.$$

From this equality and Propositions (3.3) and (3.10) in [1] we see that the assertion of Theorem 1 is true. The proof of Theorem 2 follows the same steps of the proof of Theorem 1.

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