YOKOHAMA MATHEMATICAL
JOURNAL VOL. 30, 1982

A CLASS OF 3-BRIDGE KNOTS II

By

OSAMU MORIKAWA

(Received December 17, 1981)

For integers p, q, and r, we defined a class of 3-bridge knots $K(p, q; r)$ in [6]. We shall show that each $K(p, 1; r)$ $(q=1)$ is a 3-braid knot. And we shall determine the Murasugi signature of $K(p, 1; r)$ by its braid representation (section 1).

Theorem 2.

$$\sigma(K(p, 1; r))=\begin{cases} 2r-2p+2 & r\geq 0,\ p\equiv 1 \pmod 4, \quad or \quad r\geq -2,\ p\equiv 3 \pmod 4, \\ 2r-2p+4 & otherwise. \end{cases}$$

Let $D(p, q; r)$ be the double branched covering space of S^3 branched over $K(p, q; r)$. Then we shall show that for any pair of odd integers (p, q) with $(p, q)=1$, there exists an integer r such that $D(p, q; r)$ and $D(p, q; r+1)$ are homology 3-spheres (Theorem 3). In Theorem 4, we shall show that for any positive integer n and any integer r, $D(4n+1, 2n+1; r)$ and $D(4n+3, 2n+1; r)$ are Seifert manifolds.

In [6], we announced that for $p\leq 17$, every knot $K(p, q; r)$ is not a torus knot except the knots shown in Proposition 2 and 3 in [6], and the bridge index of each knot is three except the knots shown in Proposition 2 and 4 in [6] (Proposition 5). We shall prove these by the help of Theorems 2, 3, and 4.

In this paper we shall use the same notations as [6].

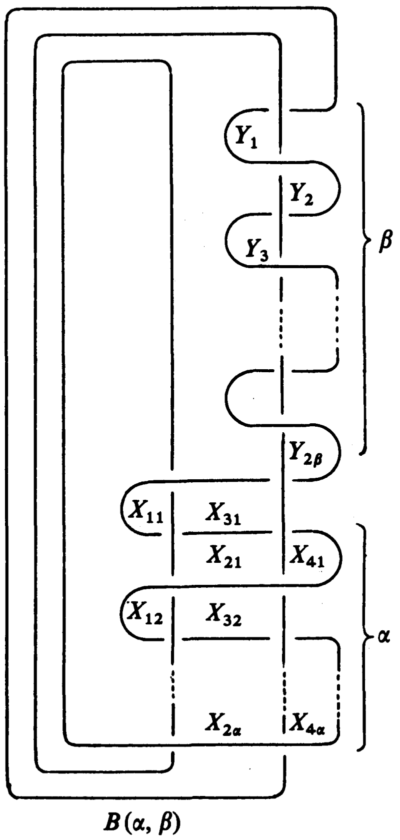
§ 1. Signature of $K(p, 1; r)$

In this section we shall show that for a triplet (p, q, r) of integers, $K(p, q; r)$ possesses the Alexander polynomial equal to that of a torus knot, but that $K(p, q; r)$ is not a torus knot. This will be shown by the help of the Murasugi signature.

For a positive integer α and an integer β, we define a 3-braid knot $B(\alpha, \beta)$ as follows: If $\beta\geq 0$, let $B(\alpha, \beta)$ be the knot as shown in Figure 1. The portion of Figure 1 which is included in the upper bracket consists of β-"full-twists". If $\beta<0$ the knot diagram will be identical except that these particular crossings should be reversed.

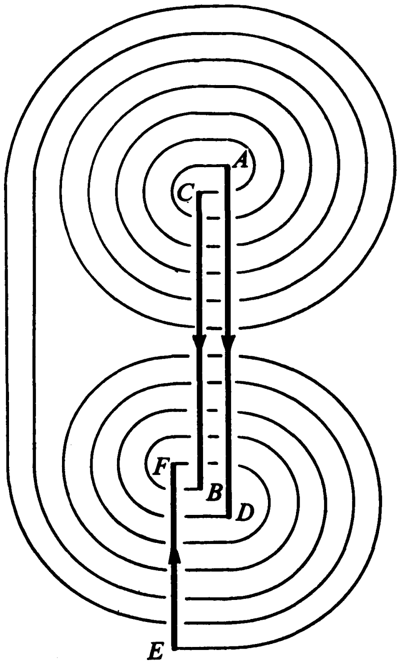
We can get $B(m, r-m)$ by an isotopic transformation of $K(2m+1, 1; r)$ as follows:

Rotate the overpass EF through 180° radian with respect to the line which is perpendicular to the plane R^2 through the point F (Figure 2-1 → Figure 2-2). In Figure 2-2, the underpasses FA and DC satisfy the followings:



$B(\alpha, \beta)$

Figure 1.



$K(4n+1, 1; r)$ $n=1$, $r=3$

Figure 2-1.

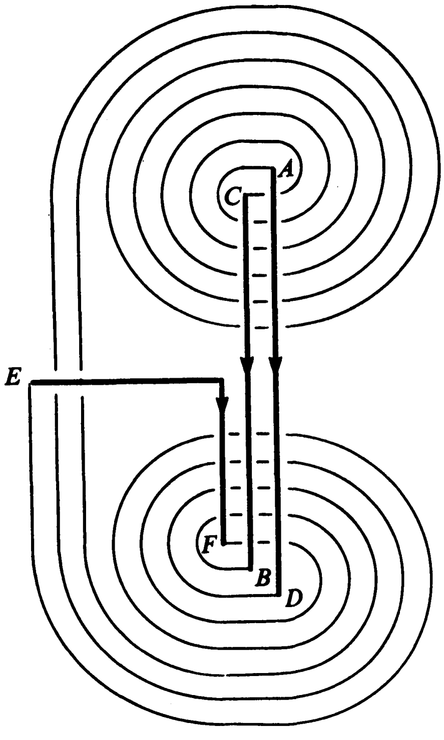


Figure 2-2.

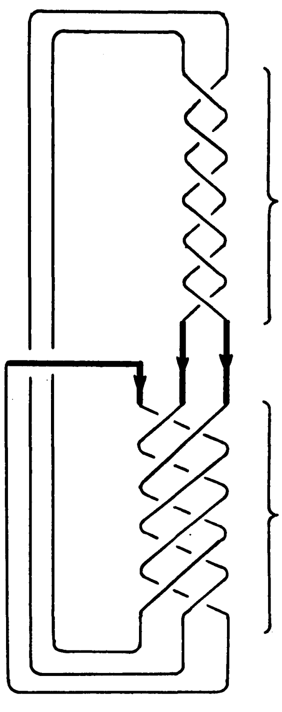


Figure 2-3.

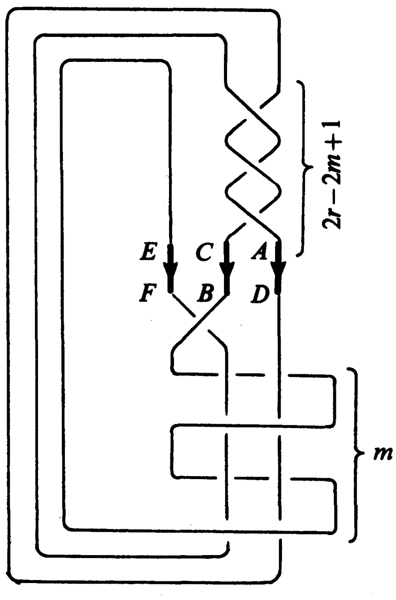


Figure 2-4.

1) each of them lies in R^2,
2) each of them goes around the both points A and C r-times,
3) they are parallel in R^2.

Put the portion into r-"full-twists" as shown in Figure 2-3. Similarly put the portion which is a neighborhood of the points F, B and D into m-"twists" as shown in Figure 2-3. The arcs FA and DC are right-hand-screw twisted $(2m-1)$-times and left-hand-screw twisted $2r$-times. This may be reduced to $(2r-2m+1)$ left- hand-screw twists as shown in Figure 2-4 if $r \geq m$, or $(2m-2r-1)$ right-hand-screw twists if $r<m$ (Figure 2-3 $\rightarrow$ Figure 2-4). Since the $(2r-2m+1)$-twists is decomposed into a twist and $(r-m)$-"full-twists", Figure 2-4 shows $B(m, r-m)$.

The Murasugi matrix $M_{\alpha\beta}$ of a 3-braid knot $B(\alpha, \beta)$ has the form

$$
M_{\alpha\beta}=\left\|\begin{array}{c|c|c|c|c}
X_{11} & X_{21} & X_{31} & X_{41} & Y_1 \\
\hline
\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} & \begin{matrix}0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & 1 & 0\end{matrix} & & & \\
\hline
\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} & \begin{matrix}-1 & 1 & \\ & \ddots & \ddots \\ & & 1 \\ & & -1\end{matrix} & & \\
\hline
 & & \begin{matrix}0 & & \\ & -1 & \\ & & \ddots \\ & & & -1\end{matrix} & \begin{matrix}0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & 1 & 0\end{matrix} & \\
\hline
 & & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & \begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix} & \\
\hline
 & & \begin{matrix} & \\ -1 & \end{matrix} & & \begin{matrix}1 & -1 & \\ & \ddots & \ddots \\ & & -1 \\ & & 1\end{matrix}
\end{array}\right\|
\begin{array}{l} \}\,\alpha \\ \}\,\alpha \\ \}\,\alpha \\ \}\,\alpha \\ \}\,2\beta \end{array}
$$

If $\beta<0$, replace the lower right $2|\beta|\times 2|\beta|$ submatrix of $M_{\alpha|\beta|}$, by the following matrix.

$$
\left\|\begin{matrix}
-1 & 1 & & & \\
 & -1 & 1 & & \\
 & & \ddots & \ddots & \\
 & & & -1 & 1 \\
 & & & & 0
\end{matrix}\right\| \Big\} 2|\beta|
$$

Lemma 1. *For any positive integers α and β, the signature of $M_{\alpha\beta}+M^t_{\alpha\beta}$ is equal to $2\beta-2\alpha$.*

Proof. Let

$$A=\begin{Vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{Vmatrix}, \qquad \text{where} \quad a_{ij}=a_{ji}$$

and

$$P_{Ai}=\begin{Vmatrix} 1 & & & & & \\ & \ddots & & & & \\ \dfrac{a_{i1}}{a_{ii}} & \dfrac{a_{i2}}{a_{ii}} & \cdots & 1 & \cdots & \dfrac{a_{in}}{a_{ii}} \\ & & & & \ddots & \\ & & & & & 1 \end{Vmatrix}$$

Let Pivot (i, A) denote the transformation A into $P^t_{Ai}AP_{Ai}$. In other words, as the result of applying the operation Pivot (i, A), the matrix A is transformed into the matrix such that all the elements in row i or column i are zero except the (i, i) element. Let

$$Q_{ijk}=\begin{Vmatrix} 1 & & \vdots & & \\ & \ddots & \vdots & & \\ \cdots & \cdots & k & \cdots & \cdots \\ & & \vdots & \ddots & \\ & & \vdots & & 1 \end{Vmatrix} \quad \text{(row } (i)\text{, column } (j)\text{)} \qquad \text{and} \qquad R_{ik}=\begin{Vmatrix} 1 & & \vdots & & \\ & \ddots & \vdots & & \\ & & k & & \\ & & \vdots & \ddots & \\ & & \vdots & & 1 \end{Vmatrix} \quad \text{(column } (i)\text{)}$$

We denote the transformation A into $Q^t_{ijk}AQ_{ijk}$ by Add (i, j, k, A), and the transformation A into $R^t_{ik}AR_{ik}$ by Mult (i, k, A). We may write Pivot (i), Add (i, j, k), and Mult (i, k) when A is understood.

Note that these transformations Pivot (i), Add (i, j, k) and Mult (i, k), where $k\neq 0$, unchange the signature of matrices.

We prove the lemma in three steps (see Table 2).

Let $A_0=M_{\alpha\beta}+M^t_{\alpha\beta}$ (see Table 1).

Step 1. We repeat the following transformations (1) and (2) for $i=1, 2, \cdots, \alpha$:

Table 1.

$$A_0=\left\|\begin{array}{c|c|c|c|c}
\begin{matrix}-2&&&\\&\cdot&&\\&&\ddots&\\&&&-2\end{matrix} &
\begin{matrix}1&&&\\1&\cdot&&\\&\ddots&\ddots&\\&&1&1\end{matrix} & & & \\ \hline
\begin{matrix}1&1&&\\&\ddots&\ddots&\\&&\cdot&1\\&&&1\end{matrix} &
\begin{matrix}-2&&&\\&\cdot&&\\&&\ddots&\\&&&-2\end{matrix} &
\begin{matrix}-1&1&&\\&\ddots&\ddots&\\&&\cdot&1\\&&&-1\end{matrix} & & \\ \hline
 & \begin{matrix}-1&&&\\1&\cdot&&\\&\ddots&\ddots&\\&&1&-1\end{matrix} &
\begin{matrix}0&&&\\&-2&&\\&&\ddots&\\&&&-2\end{matrix} &
\begin{matrix}1&&&\\1&\cdot&&\\&\ddots&\ddots&\\&&1&1\end{matrix} &
\begin{matrix}&&-1\\&&\\&&\end{matrix} \\ \hline
 & & \begin{matrix}1&1&&\\&\ddots&\ddots&\\&&\cdot&1\\&&&1\end{matrix} &
\begin{matrix}-2&&&\\&\cdot&&\\&&\ddots&\\&&&-2\end{matrix} & \\ \hline
 & & \begin{matrix}&&\\&&\\-1&&\end{matrix} & &
\begin{matrix}2&-1&&&\\-1&\cdot&\cdot&&\\&\cdot&\cdot&\cdot&\\&&\cdot&\cdot&-1\\&&&-1&2\end{matrix}
\end{array}\right\|
\begin{array}{l}\}\,\alpha\\ \}\,\alpha\\ \}\,\alpha\\ \}\,\alpha\\ \}\,2\beta\end{array}$$

$$A_1=\left\|\begin{array}{c|c|c|c|c}
\begin{matrix}-2&&&\\&\cdot&&\\&&\ddots&\\&&&-2\end{matrix} & & & & \\ \hline
 & \begin{matrix}-4&2&&&\\2&\cdot&\cdot&&\\&\cdot&\cdot&\cdot&\\&&\cdot&-4&2\\&&&2&-6\end{matrix} &
\begin{matrix}-4&4&&\\&\ddots&\ddots&\\&&\cdot&4\\&&&-4\end{matrix} & & \\ \hline
 & \begin{matrix}-4&&&\\4&\cdot&&\\&\ddots&\ddots&\\&&4&-4\end{matrix} &
\begin{matrix}2&2&&&\\2&-4&\cdot&&\\&\cdot&\cdot&\cdot&\\&&\cdot&\cdot&2\\&&&2&-4\end{matrix} & &
\begin{matrix}&&-2\\&&\\&&\end{matrix} \\ \hline
 & & & \begin{matrix}-2&&&\\&\cdot&&\\&&\ddots&\\&&&-2\end{matrix} & \\ \hline
 & & \begin{matrix}&&\\&&\\-2&&\end{matrix} & &
\begin{matrix}2&-1&&&\\-1&\cdot&\cdot&&\\&\cdot&\cdot&\cdot&\\&&\cdot&\cdot&-1\\&&&-1&2\end{matrix}
\end{array}\right\|$$

Table 1. (continued)

$$
A_2'=\left\|\begin{array}{c|c|c|c|c}
\begin{matrix}-2 & & \\ & \ddots & \\ & & -2\end{matrix} & & & & \\
\hline
 & \begin{matrix}-1 & 0 & & & & & \\ 0 & 2 & 0 & -1 & & & \\ & 0 & -1 & 0 & \cdot & & \\ & -1 & 0 & 2 & \cdot & \cdot & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \cdot & 2 & 0 & -1 \\ & & & \cdot & 0 & -1 & 0 \\ & & & -1 & 0 & 7 & 2 \\ & & & & & 2 & -6\end{matrix} & & \begin{matrix}\\ \\ \\ \\ \\ \\ \\ -4 \\ -4\end{matrix} & \begin{matrix}\\ \\ \\ \\ \\ \\ \\ 2 \\ \ \end{matrix} \\
\hline
 & \begin{matrix} & & & & & \\ & & & & & -4 & -4\end{matrix} & \begin{matrix}-2 & & & & & \\ & 2 & & & & \\ & & -2 & & & \\ & & & \ddots & & \\ & & & & 2 & \\ & & & & & 2\end{matrix} & & \begin{matrix}\\ \\ \\ \\ \\ -2\end{matrix} \\
\hline
 & & & \begin{matrix}-2 & & \\ & \ddots & \\ & & -2\end{matrix} & \\
\hline
 & \begin{matrix}\\ \\ \\ 2\end{matrix} & \begin{matrix}\\ \\ \\ -2\end{matrix} & & \begin{matrix}2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{matrix}
\end{array}\right\|
\begin{matrix}\}\ \alpha \\ \}\ \alpha \\ \}\ \alpha \\ \}\ \alpha \\ \}\ 2\beta\end{matrix}
$$

$$
A_2=\left\|\begin{array}{c|c|c}
\begin{matrix}2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{matrix} & \begin{matrix}\\ \\ \\ -1\end{matrix} & \\
\hline
\begin{matrix}-1 \\ \\ \ \end{matrix} & \begin{matrix}7 & 2 & -4 \\ 2 & -6 & -4 \\ -4 & -4 & 2\end{matrix} & \begin{matrix}2 \\ 0 \\ -2\end{matrix} \\
\hline
 & \begin{matrix}\\ \\ \\ 2 & 0 & -2\end{matrix} & \begin{matrix}2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{matrix}
\end{array}\right\|
\begin{matrix}\Big\}\ \dfrac{\alpha-3}{2} \\ \\ \\ \Big\}\ 2\beta\end{matrix}
$$

$$
A_3=\left\|\begin{matrix}6+\dfrac{2}{\alpha-1} & 2 & -4 & 2 \\ 2 & -6 & -4 & 0 \\ -4 & -4 & 2 & -2 \\ 2 & 0 & -2 & \dfrac{2\beta+1}{2\beta}\end{matrix}\right\|
$$

$$
A_4=\left\|\begin{array}{c|c}
\begin{matrix}2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 5\end{matrix} & \begin{matrix}\\ \\ \\ \\ 2\end{matrix} \\
\hline
\begin{matrix}\\ \\ \\ & & & & 2\end{matrix} & \begin{matrix}2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2\end{matrix}
\end{array}\right\|
\begin{matrix}\Big\}\ \dfrac{\alpha}{2} \\ \\ \Big\}\ 2\beta\end{matrix}
$$

$$
A_5=\left\|\begin{matrix}4+\dfrac{2}{\alpha} & 2 \\ 2 & 1+\dfrac{1}{2\beta}\end{matrix}\right\|
$$

Table 1. (continued)

$$X=\left\|\begin{matrix} 2 & -1 & & & \\ -1 & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & -1 \\ & & & -1 & 2 \end{matrix}\right\| \Bigg\} 2\beta \qquad Y=\left\|\begin{matrix} -2 & 1 & & & \\ 1 & \cdot & & & \\ & & \cdot & & \\ & & & -2 & 1 \\ & & & 1 & 0 \end{matrix}\right\| \Bigg\} 2|\beta|$$

$$A_7=\left\|\begin{matrix} 6+\frac{2}{\alpha-1} & 2 & -4 & 2 \\ 2 & -6 & -4 & 0 \\ -4 & -4 & 2 & -2 \\ 2 & 0 & -2 & \frac{2\beta-1}{2\beta} \end{matrix}\right\| \qquad A_8=\left\|\begin{matrix} 4+\frac{2}{\alpha} & 2 \\ 2 & 1-\frac{1}{2\beta} \end{matrix}\right\|$$

(1) Mult $(\alpha+i, 2)$,
(2) Pivot (i).

And we repeat the transformations (3) and (4) for $i=\alpha, \alpha-1, \cdots, 1$:

(3) Mult $(2\alpha+i, 2)$
(4) Pivot $(3\alpha+i)$.

Performing the above transformations on the matrix A_0 yields the matrix A_1 as shown in Table 1.

Step 2. For each i, the operations (5), (6), (7), and (8) do not change elements of the matrix A_1 except the elements of row and column $\alpha+2i-2$, $\alpha+2i-1$, $\alpha+2i$, $2\alpha+2i-1$, $2\alpha+2i$, $2\alpha+2i+1$, and $4\alpha+2\beta$. And we have the followings:

1) $(2\alpha+2i+1, 2\alpha+2i+1)$ element of the new matrix= $(2\alpha+2i-1, 2\alpha+2i-1)$ element of the old matrix, and
2) $(2\alpha+2i+2, 2\alpha+2i+2)$ element of the new matrix= $(2\alpha+2i, 2\alpha+2i)$ element of the old matrix.

We repeat these four operations for $i=1, 2, \cdots, [\alpha/2]$, where $[x]$ means the greatest integer which is not greater than x.

(5) Pivot $(\alpha+2i-1)$,
(6) Add $(2\alpha+2i, 2\alpha+2i-1, 1)$,
(7) Pivot $(2\alpha+2i)$,
(8) Pivot $(2\alpha+2i-1)$.

Performing the above transformations on the matrix A_1 yields the matrix A_2' (if α is odd) or A_4' (if α is even).

Step 3. There are two cases.

Case 1: The number α is odd.

Let A_2 be the matrix as shown in Table 1. Then $\sigma(A_0)=-2\alpha-(\alpha-1)/2+\sigma(A_2)$. We repeat the transformation (9) for $i=1, 2, \cdots, (\alpha-3)/2$:

(9) Pivot (i).

And then we repeat the transformation (10) for $i=1, 2, \cdots, 2\beta-1$:

Table 2.

```
procedure shrink (var sign : integer);
(* obtain submatrix and calculate signature *)
begin
...
end;

begin
        sign := 0;
(* A0→A1 *)
        for i := 1 to α do begin
                Mult (α+i, 2);
                Pivot (i);
                end;
        for i := α down to 1 do begin
                Mult (2*α+i, 2);
                Pivot (3*α+i):
                end;
(* A1→A2′ or A4′ *)
        for i := 1 to (α div 2) do begin
                Pivot (α+2*i);
                Add (2*α+2*i, 2*α+2*i−1, 1);
                Pivot (2*α+2*i);
                Pivot (2*α+2*i−1);
                end;
(* A2′→A2 or A4′→A4 *)
        shrink (sign);
        if odd (α) then begin
                (* A2→A3′ *)
                for i := 1 to (α−3) div 2 do
                        Pivot (i);
                for i := 1 to 2*β−1 do
                        Pivot (((α+3) div 2) +i);
                shrink (sign);          (* A3′→A3 *)
                for i := 1 to 3 do      (* calculate σ(A3) *)
                        Pivot (i);
                shrink (sign);
                end
        else begin
                (* A4→A5′ *)
                for i := 1 to (α div 2)+1 do
                        Pivot (i);
                for i := 1 to 2*β−1 do
                        Pivot ((α div 2) +i);
                shrink (sign);          (* A5′→A5 *)
                Pivot (1);              (* calculate σ(A5) *)
                shrink (sign);
                end;
end
```

(10) Pivot $(i+(\alpha+3)/2)$.

Performing the above transformations on A_2 yields A_3'. Let A_3 be the matrix as shown in Table 1. Then $\sigma(A_2)=(\alpha-3)/2+2\beta-1+\sigma(A_3)$. It is easy to see that $\sigma(A_3)=2$. Therefore $\sigma(A_0)=2\beta-2\alpha$.

Case 2: The number α is even.

Let A_4 be the matrix as shown in Table 1. Then $\sigma(A_0)=-2\alpha-\alpha/2+\sigma(A_4)$. We repeat the transformation (11) for $i=1, 2, \cdots, \alpha/2-1$ and we repeat the transformation (12) for $i=1, 2, \cdots, 2\beta-1$:

(11) Pivot (i),

(12) Pivot $(i+\alpha/2)$.

Performing the above transformations on A_4 yields A_5'. If we define A_5 as in Table 1, $\sigma(A_0)=-2\alpha+2\beta-2+\sigma(A_5)$. It is easy to see that $\sigma(A_5)=2$. Therefore $\sigma(A_0)=2\beta-2\alpha$. The proof is completed.

Lemma 2. *For integers $\alpha>0>\beta$,*

$$\sigma(M_{\alpha\beta}+M^t_{\alpha\beta})=\begin{cases}2\beta-2\alpha & \alpha\text{: } \textit{odd}, \ \alpha+\beta+2\geq 0, \quad \textit{or} \quad \alpha\text{: } \textit{even}, \ \alpha+\beta\geq 0,\\ 2\beta-2\alpha+2 & \textit{otherwise}.\end{cases}$$

Proof. Let X and Y be the matrices as shown in Table 1. Then the matrix X is the lower right $2|\beta|\times 2|\beta|$ submatrix of A_0. Thus $M_{\alpha\beta}+M^t_{\alpha\beta}$ is the matrix obtained from A_0 by replacing the submatrix X of A_0 by Y. Set $A_6=M_{\alpha\beta}+M^t_{\alpha\beta}$. If we define the matrices A_7 and A_8 as shown in Table 1, it is easy to see that:

$$\sigma(A_6)=\begin{cases}2\beta-2\alpha+\sigma(A_7) & \text{if } \alpha \text{ is odd},\\ 2\beta-2\alpha+\sigma(A_8) & \text{if } \alpha \text{ is even},\end{cases}$$

$$\sigma(A_7)=1+\operatorname{sign}(1/(2\alpha+5)+1/2\beta)=\begin{cases}0 & \alpha+2\geq-\beta,\\ 2 & \text{otherwise},\end{cases}$$

$$\sigma(A_8)=1+\operatorname{sign}(1/(2\alpha+1)+1/2\beta)=\begin{cases}0 & \alpha\geq-\beta,\\ 2 & \text{otherwise}.\end{cases}$$

Therefore

$$\sigma(A_6)=\begin{cases}2\beta-2\alpha & \alpha\text{: odd}, \ \alpha+\beta+2\geq 0, \quad \text{or} \quad \alpha\text{: even}, \ \alpha+\beta\geq 0,\\ 2\beta-2\alpha+2 & \text{otherwise}.\end{cases}$$

The proof is completed.

Theorem 1.

$$\sigma(B(\alpha, \beta))=\begin{cases}2\beta-2\alpha & \alpha\text{: } \textit{odd}, \ \alpha+\beta+2\geq 0, \quad \textit{or} \quad \alpha\text{: } \textit{even}, \ \alpha+\beta\geq 0,\\ 2\beta-2\alpha+2 & \textit{otherwise}.\end{cases}$$

Proof. The matrix $M_{\alpha\beta}$ is the Murasugi matrix of the 3-braid knot $B(\alpha, \beta)$ (see Figure 1 and [7]). The Murasugi signature of $B(\alpha, \beta)$ is the signature of the matrix $M_{\alpha\beta}+M^t_{\alpha\beta}$. By Lemmas 1 and 2, the signature of the above matrix is equal to $2\beta-2\alpha$ or $2\beta-2\alpha+2$. Therefore we have the result.

Theorem 2.

$$\sigma(K(p, 1; r))=\begin{cases} 2r-2p+2 & r\geq 0, p\equiv 1 \pmod 4, \quad \textit{or} \quad r\geq -2, p\equiv 3 \pmod 4, \\ 2r-2p+4 & \textit{otherwise}. \end{cases}$$

Proof. We have shown that $K(p, 1; r)$ and $B((p-1)/2, r-(p-1)/2)$ have the same knot type. By Theorem 1, we have the result.

Proposition 1. *Given a positive integer n, the knot* $K(4n+1, 1; 3n)$ *possesses the Alexander polynomial equal to the* 2*-bridge torus knot* $K(6n+1, 1)$, *and has bridge index* 3. (*Hence it is not a torus knot*). *Similarly* $K(4n+3, 1; 3n)$ *does not have the same knot type as the* 2*-bridge torus knot* $K(6n+5, 1)$.

Proof. By the under presentation, the knot group πk of the knot $K(4n+1, 1; 3n)$ can be represented by the generators a, b, and c corresponding to the underpasses BE, FA, and DC respectively (Figure 3), and the relators corresponding to the overpasses EF and CB. Let W be the word obtained by reading off the generators by going along the overpasses EF in the positive direction, and V the word along the overpass BC in the positive direction in the knot $K(4n+1, 1; 0)$, respectively, as shown in Figure 3. (We use the same notations as [6]). Then $W=(bca)^{2n}$ and $V=(cab)^{1-2n}b^{-1}a^{-1}$. Thus $\pi k=\langle a, b, c \mid aWb^{-1}W^{-1}, aV(bc)^{3n}c^{-1}(bc)^{-3n}V^{-1}\rangle$. The Alexander polynomial of $K(4n+1, 1; 3n)$ is equal to

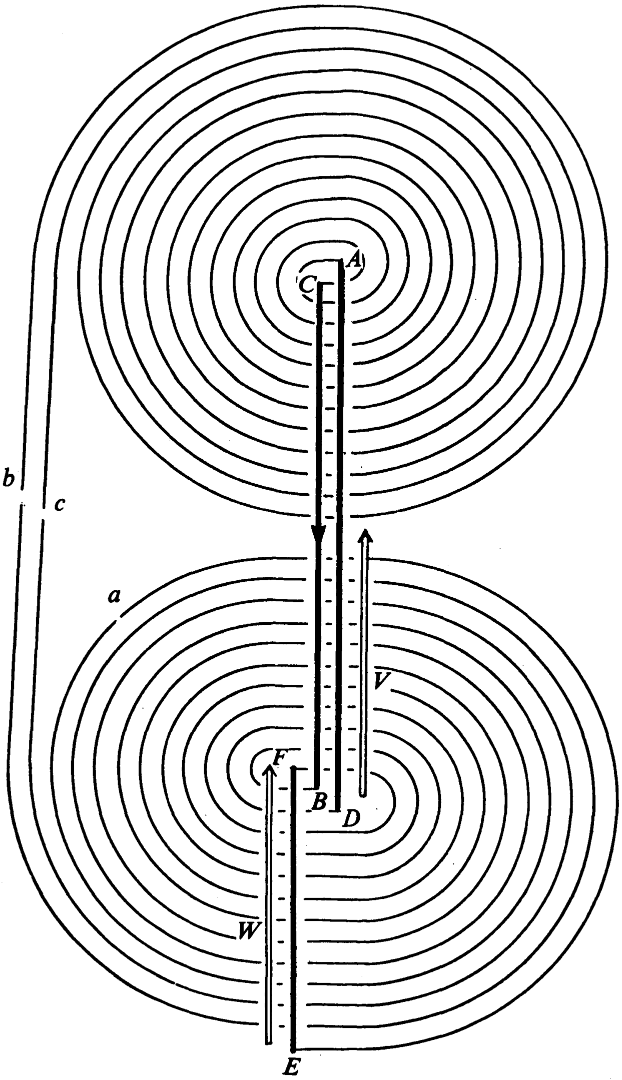
$$\Delta(x)=\frac{x^{6n+1}+1}{x+1}$$

It is well known that the polynomial is equal to the Alexander polynomial of the 2-bridge knot $K(6n+1, 1)$.

We shall use contradiction to prove our proposition. Suppose that $K(4n+1, 1; 3n)$ is a 2-bridge knot. Since $\Delta(x)=(x^{6n+1}+1)/(x+1)$, $K(4n+1, 1; 3n)$ must have the same knot type as $K(6n+1, \pm 1)$ [2], [8]. Since $K(6n+1, \pm 1)$ is the torus knot of type $(6n+1, \pm 2)$, the signature $\sigma(K(6n+1, \pm 1))=\pm 6n$ [7]. On the other hand, by Theorem 2 the signature $\sigma(K(4n+1, 1; 3n))=-2n$. This is contradiction. Then $K(4n+1, 1; 3n)$ has bridge index 3.

Similarly suppose that $K(4n+1, 1; 3n)$ is a torus knot. Then $K(4n+1, 1; 3n)$ must have the same knot type as the torus knot $T(6n+1, \pm 2)$, since the Alexander polynomial $\Delta(x)=(x^{6n+1}+1)/(x+1)$. Thus $K(4n+1, 1; 3n)$ must have bridge index 2, since $T(6n+1, \pm 2)$ and $K(6n+1, \pm 1)$ have the same knot type. This is contradiction. Therefore $K(4n+1, 1; 3n)$ is not a torus knot.

We can show that the Alexander polynomial of $K(4n+3, 1; 3n)$ is $\Delta(x)=(x^{6n+5}+1)/(x+1)$, and its signature is equal to $-2n-4$. Suppose that $K(4n+3, 1; 3n)$



$K(4n+1, 1; 3n)$ $(n=2)$

$W = bcabcabcabca$
$V = b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}c^{-1}b^{-1}a^{-1}$

Figure 3.

is a 2-bridge knot or a torus knot. Then $K(4n+3, 1; 3n)$ must have the same knot type as the 2-bridge torus knot $K(6n+5, \pm 1)$. Thus the signature of $K(4n+3, 1; 3n)$ must be equal to $\pm(6n+4)$. This is contradiction. Therefore $K(4n+3, 1; 3n)$ has bridge index 3 and is not a torus knot. The proof is completed.

Proposition 2. *Each of the following pairs of knots has the same Alexander polynomial. But we can distinguish them by the Murasugi signatures.*

(1) $K(4n+1, 1; 3n+3k) \not\approx K(4k+1, 1; 3n+3k)$ $n > k \geq 1$, *where* $K_1 \approx K_2$ *means that* K_1 *and* K_2 *have the same knot type.*

(2) $K(4n-1, 1; 3n+3k-3) \not\approx K(4k-1, 1; 3n+3k-3)$ $n>k\geq 1$,

(3) $K(4n+1, 1; 3n-3k) \not\approx K(4k-1, 1; -3n+3k-3)$ $n>k\geq 1$, *or* $k>n\geq 1$.

Proof. (1) By the under presentation, the knot group πk of the knot $K(2m+1, 1; r)$ has the following representation:

$\pi k = \langle a, b, c \mid aWb^{-1}W^{-1}, aV(bc)^r c^{-1}(bc)^{-r}V^{-1} \rangle$, where $W=(bca)^m$ and $V=(cab)^{1-m}b^{-1}a^{-1}$, as the proof of Proposition 1. Then we can show that $K(4n+1, 1; 3n+3k)$ has the same Alexander polynomial as $K(4k+1, 1; 3n+3k)$. By Theorem 1,

$$\sigma(K(4n+1, 1; 3n+3k)) = 6k-2n+2 \neq 6n-2k+2$$
$$= \sigma(K(4k+1, 1; 3n+3k)).$$

Therefore $K(4n+1, 1; 3n+3k) \not\approx K(4k+1, 1; 3n+3k)$.

We can show (2) and (3) by the similar way. The proof is completed.

Remark. When $n=k$ in (3) of Proposition 2, we know that $K(4n+1, 1; 0)$ and $K(4n-1, 1; -3)$ have the same knot type.

§2. Double branched covering of S^3

There is a knot $K(p, q; r)$ with bridge index 3 whose Alexander polynomial that is equal to that of a 2-bridge knot. But we can distinguish it from the 2-bridge knot by examining that the 2-fold branched covering spaces of S^3 are different from each other.

Let ρ: $S^3-\{\infty\}=R^3\to R^2\times 0=R^2$ be a regular projection $\rho(x, y, z)=(x, y, 0)$. We may assume that the knot $K(p, q; r)$ is the union of six arcs FA, AD, DC, CB, BE and EF such that

1) the arcs FA, DC and BE, which are called underpasses, are contained in R^2,

2) the arcs AD, CB and EF, which are called overpasses, are contained in $R_+{}^3=\{(x, y, z) \mid z\geq 0\}$.

Let $D(p, q; r)$ be the double branched covering space of S^3 branched over a knot $K(p, q; r)$ and P: $D(p, q; r)\to S^3$ the covering map.

Proposition 3. *Every manifold $D(p, q; r)$ is a homology lens space.*

Proof. By Birman-Hilden [1] and Takahashi [12], the 3-manifold $D(p, q; r)$ has a Heegaard diagram of genus 2. Let l_a, l_b, m_1, m_2 and m_3 be simple closed curves $P^{-1}(BE)$, $P^{-1}(FA)$, $P^{-1}(\rho(EF))$, $P^{-1}(\rho(AD))$ and $P^{-1}(\rho(CB))$ respectively. Then the union $m_1\cup m_2\cup m_3$ is a complete meridian system of the Heegaard diagram of $D(p, q; r)$.

Let $B_0=P^{-1}(B)$, B_1, B_2, $\cdots$, B_{2n-1} be all the points of the intersection of l_a with $m_1\cup m_2\cup m_3$ in this order on l_a. Then $P(B_n)=E$ and $P(B_i)=P(B_{2n-i})$ for $i=1, 2, \cdots, n$. If $P(B_i)$ lies in $\rho(EF)$ and $i>3$, then $P(B_{i-3})$ lies in $\rho(EF)$, by the definition of $K(p, q; r)$. Since $P(B_1)$ lies in $\rho(EF)$, the arc $P(B_{3i+1})$ lies in $\rho(EF)$ for $i=0, 1, \cdots, (2n-2)/3$. Thus B_{3i+1} lies in $m_1=P^{-1}(\rho(EF))$.

Orient l_a, l_b, m_1, m_2 and m_3 as shown in Figure 4. For each j, at the point B_{2j}, the

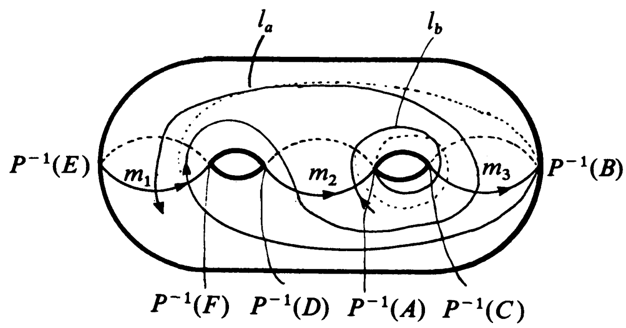


Figure 4.

intersection number of l_a and the complete meridian system $\{m_1 \cup m_2 \cup m_3\}$ is equal to -1, and at B_{2j+1}, it is equal to $+1$. Then l_a crosses m_1 at B_{6i+1} with intersection number $+1$, for $i=0, 1, \cdots, (n-1)/3$. And l_a crosses m_1 at B_{6i+4} with intersection number -1 for $i=0, 1, \cdots, (n-4)/3$. Therefore the intersection number $m_1 \cdot l_a$ is equal to $+1$. Similarly $m_1 \cdot l_b$ is equal to $+1$ or -1 if the intersection number at $P^{-1}(E)$ is equal to $+1$ or -1 respectively. Let α and β be the intersection numbers $m_2 \cdot l_a$ and $m_2 \cdot l_b$ of $D(p, q; 0)$ respectively. Note that α is even but β is odd because $m_2 \cdot l_a$ has no branch points but $m_2 \cdot l_b$ has a branch point $P^{-1}(A)$. Clearly, the intersection number $m_2 \cdot l_a$ of $D(p, q; r)$ is equal to α, and $m_2 \cdot l_b$ of $D(p, q; r)$ is equal to $\beta+2r$. Then the homology group of $D(p, q; r)$ is

$$H(D(p, q; r)) = \begin{cases} \langle l_a, l_b \mid l_a + l_b = 0,\ \alpha l_a + (\beta+2r) l_b = 0 \rangle & \text{if } m_1 \cdot l_a = m_1 \cdot l_b, \\ \langle l_a, l_b \mid l_a - l_b = 0,\ \alpha l_a + (\beta+2r) l_b = 0 \rangle & \text{if } m_1 \cdot l_a = -m_1 \cdot l_b \end{cases}$$

Therefore the homology group is $Z_{\alpha+(\beta+2r)}$ if $m_1 \cdot l_a = m_1 \cdot l_b$ or $Z_{\alpha-(\beta+2r)}$ if $m_1 \cdot l_a = -m_1 \cdot l_b$. This completes the proof of Proposition 3.

Theorem 3. *For any pair of integers (p, q) which yields the 2-bridge knot $K(p, q)$, there exists an integer r such that $D(p, q; r)$ and $D(p, q; r+1)$ are homology 3-spheres.*

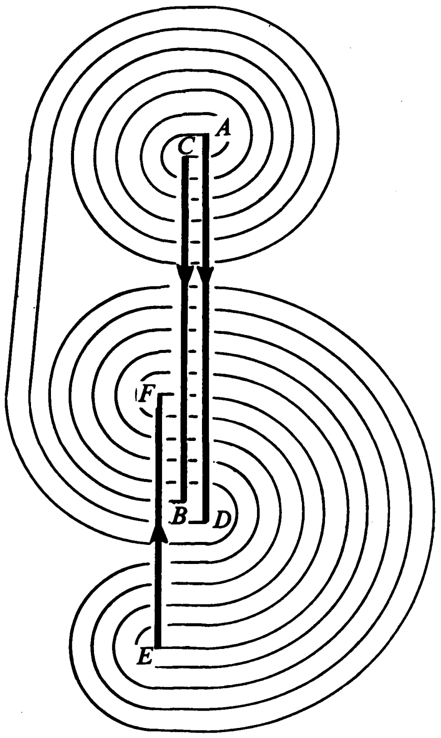
Proof. We have shown in the proof of Proposition 3 that α is even and β is odd. Therefore we choose $r = -(\alpha+\beta+1)/2$ if $m_1 \cdot l_a = m_1 \cdot l_b$ or $r = (\alpha-\beta+1)/2$ if $m_1 \cdot l_a = -m_1 \cdot l_b$. The proof is completed.

A portion of a knot which is a set of just two arcs embedded in 3-ball is called a *tangle*.

Theorem 4. *For any positive integer n and any integer r, $D(4n+1, 2n+1; r)$ and $D(4n+3, 2n+1; r)$ are Seifert manifolds.*

Proof. We shall show that $K(4n+1, 2n+1; r)$ and $K(4n+3, 2n+1; r)$ belong to a class of knots studied by Montesinos in [5]. Note that the pair of integers (s, t) of each of these knots, which is defined in Lemma 1 in [6], is $(2, 1)$.

For $K(4n+3, 2n+1; r)$: By isotopic transformation of $K(4n+3, 2n+1; r)$ we successively have Figures 5-2, 5-3, 5-4, and 5-5:



$K(11, 5; 3)$

Figure 5-1.

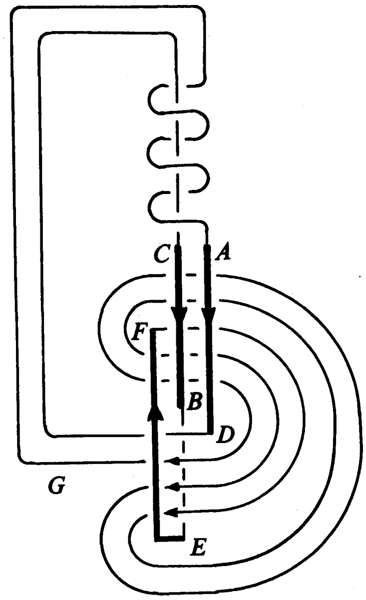


Figure 5-2.

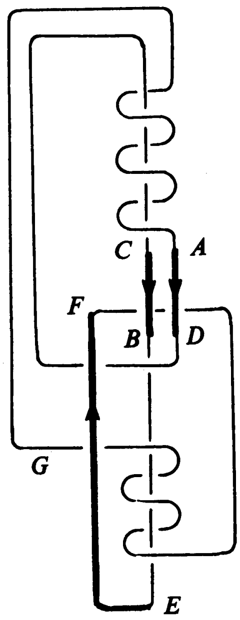


Figure 5-3.

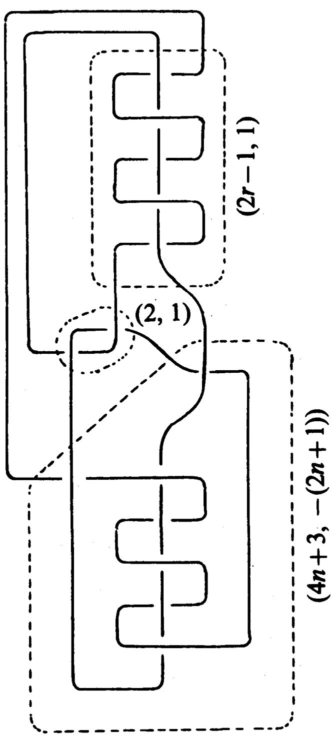


Figure 5-4.

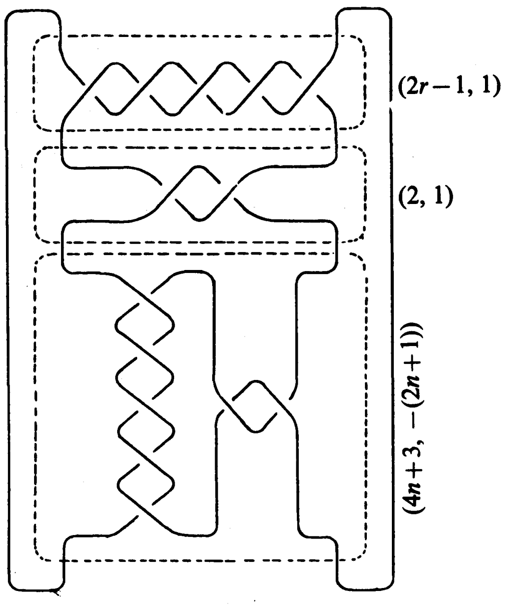


Figure 5-5.

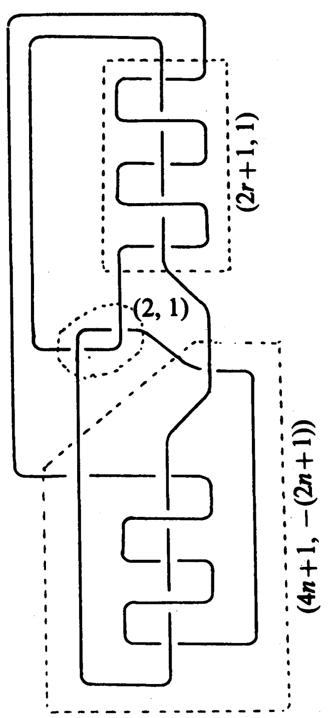


Figure 5-6.

The underpasses FA and DC satisfy the followings:

1) each of them lies in R^2,
2) each of them goes around the both points A and C r-times,
3) they are parallel in neighbourhood of the both points A and C in R^2.

Put the neighbourhood of $A \cup C$ in 3) into r-"full twists" as shown in Figure 5-2. Push the underpass BE in $R_-{}^3 = \{(x, y, z) \mid z \leq 0\}$, and redraw it by a straight line (Figure 5-1 → Figure 5-2).

Let G be a point in the arc FA as shown in Figure 5-2. The arc FG over-crosses the straight line BE at $(n+1)/2$ double points, and the other double points of FG are under-crossing points. Then redraw the arc FG and get the Figure 5-3.

Make the arc AD stride over across the arc CB. And redraw the diagram as shown in Figure 5-5. The portions of Figure 5-5 which are bounded by the dotted circles are tangles labelled $(2r-1, 1)$, $(2, 1)$ and $(4n+3, -(2n+1))$ (see [5]). Since each $D(4n+3, 2n+1; r)$ belongs to a class of manifolds studied by Montesinos [5], it is the Seifert fiber space $(O \circ 0 \mid 0; (2r-1, 1), (2, 1), (4n+3, -(2n+1)))$ in the notation of [10].

For $K(4n+1, 2n+1; r)$: By the same manner of the previous case, we prove this case. By isotopic transformation we get Figure 5-6 from $K(4n+1, 2n+1; r)$. Therefore $D(4n+1, 2n+1; r)$ is described by the symbol $(O \circ 0 \mid 0; (2r+1, 1), (2, 1), (4n+1, -(2n+1))$. The proof is completed.

Corollary 1. *For any positive integer n and any integers r,*

(1) $K(4n-1, 2n-1; r)$ *has bridge index* 3 *iff* $r \neq 0$ *or* $+1$,

(2) $K(4n+1, 2n+1; r)$ *has bridge index* 3 *iff* $r \neq 0$ *or* -1.

Proof. In Proposition 4 of [6], we have shown that $K(4n+1, 2n+1; 0)$, $K(4n+1, 2n+1; -1)$, $K(4n-1, 2n-1, 0)$ and $K(4n-1, 2n-1; 1)$ have bridge index 2. The Seifert manifold $(O \circ 0 \mid 0; (2r+1, 1), (2, 1), (4n+1, -(2n+1)))$ has three exceptional fibers if and only if $2r+1 \neq \pm 1$. Then for r with $2r+1 \neq \pm 1$, $D(4n+1, 2n+1; r)$ is not a lens space. On the other hand, the 2-fold branched covering space of S^3 branched over a 2-bridge knot is a lens space. Therefore $K(4n+1, 2n+1; r)$ has bridge index 3 if and only if $r \neq 0$ or -1.

Similarly, since the Seifert manifold $(O \circ 0 \mid 0; (2r-1, 1), (2, 1), (4n-1, -(2n-1)))$ is a lens space if and only if $2r-1 \neq \pm 1$, $K(4n-1, 2n-1; r)$ has bridge index 3 if and only if $r \neq 0$ or 1. The proof is completed.

Proposition 4. *For any positive integer n,* $K(4n+3, 4n+1; 1)$ *has unknotting number* 1.

Proof. Push the underpass BE slightly into $R_-{}^3$ and redraw it by a straight line (Figure 6-1 → Figure 6-2). We apply the unknotting operation (see [4], [7]) at the crossing point in the portion of Figure 6-2 which is bounded by the dotted circle (Figure 6-2 → Figure 6-3).

Now we shall show that the resulting knot is unknotted. The arc FA into Figure 6-3 is an underpass. Then push FA into $R_-{}^3$ and redraw it by an arc without double points (Figure 6-4). Since the arc *EFAD* can be considered as an overpass, push it into R_+^3 and redraw it by a straight line (Figure 6-5). Since the arc *BEDC* can be considered as an underpass, we get a 1-bridge knot, that is a trivial knot. Therefore $K(4n+3, 4n+1; 1)$ has unknotting number 1 or 0. On the other hand, we can show that the Alexander polynomial of $K(4n+3, 4n+1; 1)$ is equal to $x^{n+1}-x^n+x-1+x^{-1}-x^{-n}+x^{-n-1}$. Thus $K(4n+3, 4n+1; 1)$ is not a trivial knot. Therefore the unknotting number of $K(4n+3, 4n+1; 1)$ is equal to 1. The proof is completed.

The following Theorem 5 was proved in [6].

Theorem 5. *For any pair of odd integers (p, q) with $0<|q|<p$ and $(p, q)=1$, there exist integers $\Gamma(p, q)=(r_0, r_1, n; \alpha_1, \alpha_2, \cdots, \alpha_n)$ such that*

1) $r_0<r_1$,

2) *for any $r\geq r_1$, the sequence of coefficients of the Alexander polynomial of $K(p, q; r)$ is*

$$\alpha_1, \alpha_2, \cdots, \alpha_n, \underline{1, -1, \cdots, 1, -1,} 1, \alpha_n, \cdots, \alpha_2, \alpha_1,$$

where the pair "1, −1" appears on the underlined part $(r-r_1)$ times repeatedly,

3) *for any $r\leq r_0$, the sequence of coefficients of the one of $K(p, q; r)$ is*

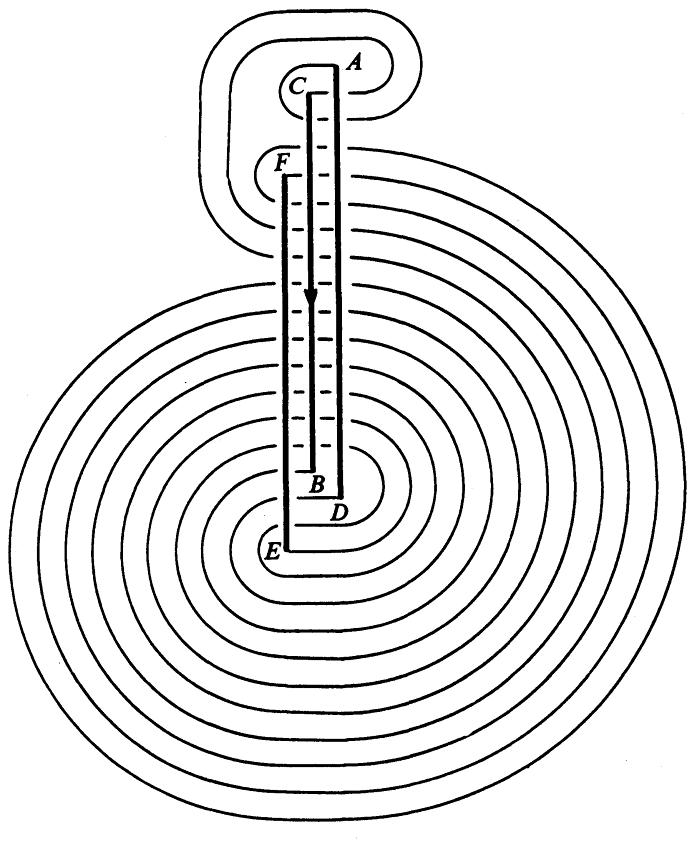


Figure 6-1. $K(11, 9; 1)$

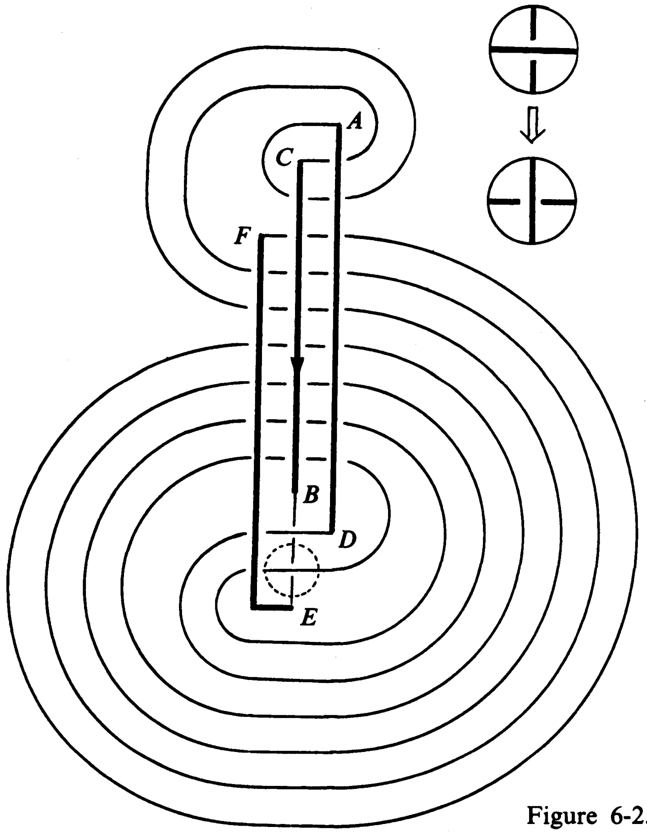


Figure 6-2.

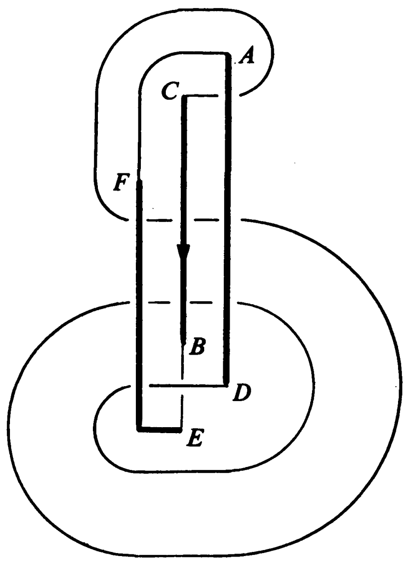


Figure 6-4.

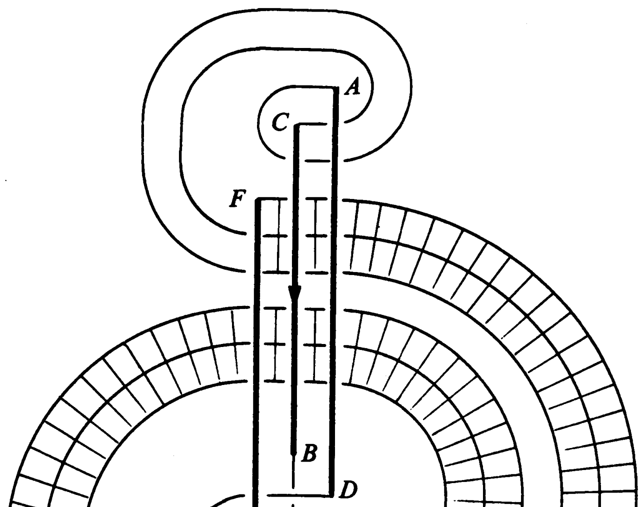
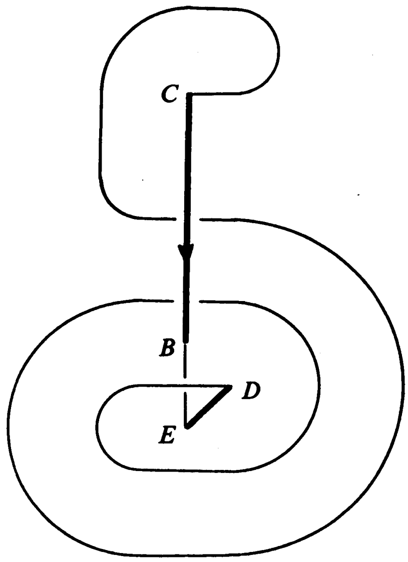


Figure 6-3.



trivial knot

Figure 6-5.

$\alpha_n+1, \alpha_{n+1}-1, \cdots, \alpha_2+1, \alpha_1-1, \underline{1, -1, \cdots, 1, -1}, 1, \alpha_1-1,$
$\alpha_2+1, \cdots, \alpha_{n-1}-1, \alpha_n+1,$ *if n is even, or*

$\alpha_n+1, \alpha_{n+1}-1, \cdots, \alpha_2-1, \alpha_1+1, -1, \underline{1, \cdots, -1, 1, -1}, \alpha_1+1,$
$\alpha_2-1, \cdots, \alpha_{n-1}-1, \alpha_n+1,$ *if n is odd*

where the pair "1, −1" appears on the underlined part (r_0-r) *times repeatedly.*

We shall prove the following result announced in [6] by the help of Theorem 5, Proposition 1 and Corollary 1.

Proposition 5.
1) $K(3, 1; 0) \approx K(5, 1) \approx T(5, 2)$
2) $K(2n+1, 1; -1) \approx T(3n+1, 3)$ $(n=1, 2, \cdots)$
3) $K(2n+1, 1; -2) \approx T(3n+2, 3)$ $(n=1, 2, \cdots)$
4) $K(4n+1, 2n+1; 0) \approx K(8n+1, 6n+1)$ $(n=1, 2, \cdots)$
5) $K(4n+1, 2n+1; -1) \approx K(8n+3, 2n+1)$ $(n=1, 2, \cdots)$
6) $K(4n+3, 2n+1; 0) \approx K(8n+5, 2n+1)$ $(n=0, 1, \cdots)$
7) $K(4n+3, 2n+1; 1) \approx K(8n+7, 6n+5)$ $(n=0, 1, \cdots)$
8) *for* $p \leq 17$, $K(p, q; r)$ *is not a torus knot except the knots in* 1), 2), *and* 3),
9) *for* $p \leq 17$, $K(p, q; r)$ *has bridge index* 3 *except the knots in* 1), 4), 5), 6) *and* 7).

Proof.
For 1): See Proposition 2 in [6].
For 2), 3): See Proposition 3 in [6].
For 4), 5), 6) and 7): See Proposition 4 in [6].
For 8): It is well known that the Alexander polynomial of a torus knot $T(p, q)$ is $\Delta(x)=(x^{pq}-1)(x-1)/(x^p-1)(x^q-1)$, and $T(p, q)$ has bridge index min (p, q). We shall use the same notation as the proof of Proposition 1, that is, a, b and c are generators of the knot group and W and V are words. In the case of $K(3, 1; r)$, the words W and V are written by bca and $b^{-1}a^{-1}$ respectively. Then we have $\Gamma(3, 1)=(-1, 4, 4; 1, -2, 2, -1)$. Thus we have all the Alexander polynomials of $K(3, 1; r)$ $(r \leq -1,\ 4 \leq r)$, by Theorem 5. For $-1<r<4$, the Alexander polynomial Δ_r of $K(3, 1; r)$ is:

$$\Delta_0(x)=x^2-x+1-x^{-1}+x^{-2}\,,$$
$$\Delta_1(x)=2x-3-2x^{-1}\,,$$
$$\Delta_2(x)=x^2-2x+3-2x^{-1}+x^{-2}\,,$$
$$\Delta_3(x)=x^3-2x^2+2x-1+2x^{-1}-2x^{-2}+x^{-3}\,.$$

Thus for each r $(r \neq 0, -1, -2)$, the polynomial $\Delta_r(x)$ is not an Alexander polynomial of a torus knot. Therefore $K(3, 1; r)$ is not a torus knot except the knots in 1), 2) and 3).

For the other (p, q), we can also calculate $\Gamma(p, q)=(r_0, r_1, n; \alpha_1, \alpha_2, \cdots, \alpha_n)$ and the Alexander polynomials of $K(p, q; r)$ $(r=r_0+1, r_0+2, \cdots, r_1-1)$. Thus we can obtain the following complete list of (p, q, r) such that

(1) $p \leq 17$, and

(2) there exists a torus knot T for which $\Delta_{K(p,q;r)}(x)=\Delta_T(x)$:

(3, 1, 0), (3, 1, −1), (3, 1, −2), (5, 1, 3), (5, 1, −1), (5, 1, −2), (7, 1, 3), (7, 1, −1), (7, 1, −2), (9, 1, 6), (9, 1, −1), (9, 1, −2), (11, 1, 6), (11, 1, −1), (11, 1, −2), (11, 9, 1), (13, 1, 9), (13, 1, −1), (13, 1, −2), (15, 1, 9), (15, 1, −1), (15, 1, −2), (15, 13, 1), (17, 1, 12), (17, 1, −1) and (17, 1, −2).

By Proposition 1, the knots $K(5, 1; 3)$, $K(7, 1; 3)$, $K(9, 1; 6)$, $K(11, 1; 6)$, $K(13, 1; 9)$, $K(15, 1; 9)$ and $K(17, 1; 12)$ are not torus knots.

The knots $K(11, 9; 1)$ and $K(15, 13; 1)$ have the Alexander polynomials of the torus knots $T(7, 2)$ and $T(8, 3)$ respectively. By Proposition 4, the unknotting number of $K(11, 9; 1)$ and $K(15, 11; 1)$ are equal to 1. On the other hand, the unknotting number of $T(7, 2)$ is equal to 3 ([7]). Thus $K(11, 9; 1)$ is not a torus knot. The unknotting number of $T(8, 3)$ is greater than 4 ([7]). Thus $K(15, 13; 1)$ is not a torus knot.

Therefore for $p \leq 17$, $K(p, q; r)$ is not a torus knot except the knots in 1), 2) and 3).

For 9): Hartley proved in [2] the followings:

The Alexander polynomial $\Delta(x)$ of a 2-bridge knot satisfies:

1) $\Delta(-x)=\sum_{i=-k}^{k} \beta_i x^i$, and

2) for some integer s,

$$\beta_{-k}<\beta_{-k+1}<\cdots<\beta_{-s}=\beta_{-s+1}=\cdots=\beta_s>\beta_{s+1}>\cdots>\beta_k.$$

Thus for each r $(r \neq 0, 1, 2)$, $K(3, 1; r)$ has bridge index 3. By Corollary 1, the knot $K(3, 1; 2)$ has bridge index 3. Therefore $K(3, 1; r)$ has bridge index 3 except the knots in 6) and 7).

In this manner we have all the Alexander polynomials of $K(p, q; r)$ $(p \leq 17)$ by finitely many calculations. So, we can obtain the following complete list of (p, q, r) such that

(1) $p \leq 17$, and

(2) the Alexander polynomial of $K(p, q; r)$ satisfies Hartley's conditions:

(3, 1, 0), (3, 1, 1), (3, 1, 2), (5, 1, 3), (5, 3, −1), (5, 3, 0), (7, 1, 3), (7, 3, 0), (7, 3, 1), (7, 3, 2), (9, 1, 6), (9, 5, −1), (9, 5, 0), (11, 1, 6), (11, 5, 0), (11, 5, 1), (11, 5, 2), (11, 9, 1), (13, 1, 9), (13, 7, −1), (13, 7, 0), (15, 1, 9), (15, 7, 0), (15, 7, 1), (15, 7, 2), (17, 1, 12), (17, 9, −1) and (17, 9, 0).

By Proposition 1, each of the knots $K(5, 1; 3)$, $K(7, 1; 3)$, $K(11, 1; 6)$, $K(13, 1; 6)$, $K(15, 1; 9)$ and $K(17, 1; 12)$ has bridge index 3.

By Corollary 1, each of the knots $K(7, 3; 2)$, $K(11, 5; 2)$ and $K(15, 7; 2)$ has bridge index 3.

We have shown that $K(11,9;1)$ has bridge index 3. Therefore for $p\leq 17$, $K(p, q; r)$ has bridge index 3 except the knots in 1), 4), 5), 6) and 7). The proof is completed.

References

[1] J. S. Birman and H. M. Hilden: *Heegaard splitting of branched coverings of* S^3, Trans. Amer. Math. Soc. 213 (1975), 315–352.
[2] R. I. Hartley: *On two bridge knot polynomials*, J. Austral. Math. Soc. 28 (1979), 241–249.
[3] R. H. Crowell and R. H. Fox: *An Introduction to Knot Theory*, New York, 1963.
[4] H. Kondo: *Knots of unknotting number* 1 *and their Alexander polynomials*, Osaka J. Math. 16 (1979), 551–559.
[5] J. M. Montesinos: *Variedades de Seifert que son recubridores circlicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana 18 (1973), 1–32.
[6] O. Morikawa: *A class of* 3*-bridge knots I*, Math. Semi. notes 9 (1981), 349–369.
[7] K. Murasugi: *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. 117 (1965), 387–442.
[8] D. Rolfsen: *Knots and Links*, Publish or Perish Inc., Berkeley, 1976.
[9] H. Schubert: *Knoten mit zwei Brucken*, Math. Z. 65 (1956), 133–170.
[10] H. Seifert: *Topologie dreidimensionaler gefaserter Raume*, Acta Math. 60 (1933), 147–238.
[11] Y. Shinohara: *On the signature of knots and links*, Ph. D., Florida, 1969.
[12] M. Takahashi: *An alternative proof of Birman-Hilden-Viro's theorem*, Tsukuba J. Math. 2 (1978), 27–34.

Department of Information Science
Tokyo Institute of Technology
Ookayama, Meguro-ku
Tokyo 152, Japan