YOKOHAMA MATHEMATICAL
JOURNAL VOL. 30, 1982

A NOTE ON THE LOCAL TRIVIALITY OF G-VECTOR BUNDLES

By

MICHIKAZU FUJII and SUSUMU KÔNO

(Received November 15, 1981)

The purpose of this note is to discuss the local triviality of G-vector bundles. Throughout this note we assume that G is a compact Lie group and consider real or complex vector bundles.

1. A *G-space* is a Hausdorff space X together with a given action of G on X and a *G-map* between two G-spaces is a continuous map which commutes with the action of G.

A *G-vector bundle* over a G-space X is a vector bundle p: $E\to X$, together with a G-space structure on E, such that

i) p is a G-map,

ii) if $g\in G$ then g: $p^{-1}(x)\to p^{-1}(gx)$ is a linear map.

Two G-vector bundles E and F over a G-space X are *G-isomorphic*, if there exists an isomorphism f: $E\to F$ of vector bundles such that f is also a G-map. If X is a G-space and V is a G-vector space then we call the G-vector bundle p_1: $X\times V\to X$ a *G-product bundle with the diagonal G-action* over the G-space X.

Since any G-vector bundle E over a G-space X is a vector bundle, E is locally trivial as a vector bundle, that is, E is locally isomorphic to a product bundle as a vector bundle. But E is not always locally G-isomorphic to a G-product bundle with the diagonal G-action. However, we have the following

Theorem. *If, for any closed subgroup H of G, every representation of H with degree n can be extended to a representation of G, then any n-dimensional G-vector bundle E over a compact G-space X is locally G-isomorphic to a G-product bundle with the diagonal G-action over X. That is, for any x in X, there is a G-invariant open neighbourhood U of x and an n-dimensional G-vector space V such that $E|U$ is G-isomorphic to $U\times V$.*

If a group G is abelian, every complex irreducible representation of G is of the first degree, that is, it essentially coincides with the character of the representation. And every character of any subgroup H of G can be extended to a character of G (cf. [3], Theorem 35). Hence, as a corollary of the above theorem we obtain the following

Corollary 1. *If G is an abelian group, then any finite dimensional complex G-*

vector bundle E over a compact G-space X is locally G-isomorphic to a G-product bundle with the diagonal G-action over X.

Remark 1.[1] Let G be the subgroup $Q_8=\{\pm 1, \pm i, \pm j, \pm k\}$ of $Sp(1)$ and H the subgroup $Z_4=\{\pm 1, \pm i\}$ of Q_8. Then the representation ρ: $H\to GL(1, C)$ defined by the natural inclusion can not be extended to any representation of G. This shows that there is a complex G-line bundle over a compact G-space which is never locally G-isomorphic to any complex G-product bundle with the diagonal G-action over the G-space (cf. Lemma 2).

Furthermore, as for real vector bundles we have

Corollary 2. *Let p be a prime number and G a cyclic group of order p. Then any finite dimensional real G-vector bundle E over a compact G-space X is locally G-isomorphic to a G-product bundle with the diagonal G-action over X.*

Remark 2. Let G be the subgroup $Z_4=\{\pm 1, \pm i\}$ of $U(1)$ and H the subgroup $A_2=\{\pm 1\}$ of Z_4. Then the representation ρ: $H\to GL(1, R)$ defined by the natural inclusion can not be extended to any representation of G. This shows that there is a real G-line bundle over a compact G-space which is never locally G-isomorphic to any real G-product bundle with the diagonal G-action over the G-space (cf. Lemma 2).

2. Let H be a subgroup of G, and V an n-dimensional H-vector space. Then let $G\times_H V$ denote the identification space obtained from $G\times V$ by the equivalence relation: $(g_1, v_1)\sim(g_2, v_2)$ if and only if $g_2=g_1h^{-1}$, $v_2=hv_1$ for some $h\in H$. The space $G\times_H V$ admits a G-space structure given by $g[g_1, v]=[gg_1, v]$ for $[g_1, v]\in G\times_H V$ and $g\in G$, and the map

$$p\colon G\times_H V\to G/H\,, \qquad [g, v]\longmapsto [g],$$

is a G-vector bundle over the G-space G/H (left coset space).

For a G-space X and $x\in X$, let G_x denote the isotropy group of x and let $G(x)$ denote the orbit of x. Then the left coset space G/G_x can be identified with $G(x)$ under the G-homeomorphism given by the map $[g]\longmapsto gx$. And we obtain

Lemma 1. *Let p: E→X be a G-vector bundle. Then, for any x∈X there is a G-isomorphism*

$$p^{-1}(G(x))\cong G\times_{G_x}E_x$$

of G-vector bundles over the G-space $G(x)\equiv G/G_x$, *where* $E_x=p^{-1}(x)$.

Proof. The G-isomorphism

$$f\colon G\times_{G_x}E_x\to p^{-1}(G(x))$$

of G-vector bundles over $G/G_x\equiv G(x)$ is given by $f([g, v])=gv$ for $g\in G$ and $v\in E_x$.

q.e.d.

---

1) We would like to appreciate Professor Peter S. Landweber for his suggestions.

Lemma 2. *Let H be a closed subgroup of G and ρ: $H \rightarrow GL(n, K)$ a representation, where $K=R$ or C. And let $V=(K^n, \rho)$ be an n-dimensional H-vector space. Then the G-vector bundle $G \times_H V$ over the G-space G/H is G-isomorphic to a G-product bundle with the diagonal G-action over G/H iff the representation ρ can be extended to a representation of G.*

Proof. Let a representation μ: $G \rightarrow GL(n, K)$ be an extension of ρ and $W=(K^n, \mu)$ a G-vector space. Then a map f': $G \times V \rightarrow G/H \times W$ given by $f'(g, v)=([g], \mu(g)v)$ for $(g, v) \in G \times K^n$ induces a G-isomorphism

$$f\colon G \times_H V \rightarrow G/H \times W$$

of G-vector bundles over the G-space G/H.

Conversely, we assume that for a G-vector space $W=(K^n, \nu)$ there is a G-isomorphism

$$f\colon G \times_H V \rightarrow G/H \times W$$

of G-vector bundles over the G-space G/H.

Let us consider the composition of the linear maps

$$K^n \xrightarrow{k} p^{-1}(H) \xrightarrow{f \mid H} H \times K^n \xrightarrow{p_2} K^n$$

where $k(v)=[e, v]$ for $v \in K^n$ and e the unit in H. Then there is an element $B \in GL(n, K)$ such that

$$p_2 f \mid Hk(v)=Bv\,.$$

And we have

$$f([e, v])=(H, Bv) \qquad \text{for} \quad v \in K^n\,.$$

Therefore, for any $h \in H$ and $v \in K^n$,

$$\begin{aligned}(H, B\rho(h)v) &= f([e, \rho(h)v]) \\ &= f([h, v]) \\ &= f(h[e, v]) \\ &= h(H, Bv) \\ &= (H, \nu(h)Bv)\,.\end{aligned}$$

Hence we have

$$B\rho(h)=\nu(h)B \qquad \text{for any} \quad h \in H\,.$$

A representation μ: $G \rightarrow GL(n, K)$ defined by $\mu(g)=B^{-1}\nu(g)B$ for $g \in G$ is an extension of ρ. q.e.d.

It is easy to see that the following lemma stated in [2] only for complex vector bundles holds for real vector bundles, too.

Lemma 3 ([2], Lemma 2.2.1). *Let E and F be n-dimensional G-vector bundles over a compact G-space X, and let $f: E|A \to F|A$ be a G-isomorphism of G-vector bundles over a closed G-invariant subspace A of X. Then there is a G-invariant open neighbourhood U of A and a G-isomorphism $g: E|U \to F|U$ of G-vector bundles over U such that $g|A = f$.*

3. Now we prove the theorem. Let x be any point of X. In virtue of Lemma 1, there is the G-isomorphism

$$p^{-1}(G(x)) \cong G \times_{G_x} E_x$$

of G-vector bundles. Furthermore, in virtue of Lemma 2, there is the G-isomorphism

$$\begin{aligned} G \times_{G_x} E_x &\cong G/G_x \times E_x \\ &\cong G(x) \times E_x \end{aligned}$$

of G-vector bundles, because the representation of G_x can be extended to a representation of G by the assumption of the theorem and E_x can be regarded as a G-vector space.

Let $X \times E_x$ be the G-product bundle with the diagonal G-action over the G-space X. Then Lemma 3 shows that there exists a G-invariant open neighbourhood U of $G(x)$ and a G-isomorphism

$$p^{-1}(U) \cong U \times E_x .$$

This completes the proof of the theorem.

References

[1] M. F. Atiyah: *K-Theory*, Benjamin, 1976.
[2] M. F. Atiyah and G. Segal: *Equivariant K-Theory*, mimeographed note, Warwick, 1965.
[3] L. Pontrjagin: *Topological Groups*, Princeton University Press, 1946.
[4] G. B. Segal: *Equivariant K-Theory*, Inst. Hautes Études Sci., Publ. Math. 34 (1968), 129–151.

Department of Mathematics
Okayama University
Okayama 700, Japan