FREELY ACTING AUTOMORPHISMS
OF C^*-ALGEBRAS

By

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0. Introduction

An automorphism α of a von Neumann algebra M is called freely acting if for any non-zero central projection e of M with $\alpha(e)=e$, α is outer on Me [4]. We extend this notion to C^*-algebras, i.e., we call an automorphism α of a C^*-algebra A freely acting if for any closed ideal I of A with $\alpha(I)=I$, the Borchers spectrum $T_B(\alpha\,|\,I)$ of $\alpha\,|\,I$ is not trivial. Our main result is a C^*-analogue of [2, 1.2.1] by Connes and improvement of [7, 1.1], which says that α is freely acting if and only if for any $\varepsilon>0$ and for any non-zero hereditary C^*-subalgebra B of A, there exists $x\in B$ such that $x\geqslant 0$, $\|x\|=1$ and $\|x\alpha(x)\|<\varepsilon$. (This result has applications as in [7].)

If A is separable, we can avoid using the Borchers spectrum in the definition of free action; in this case α is freely acting if and only if for any closed ideal I of A with $\alpha(I)=I$, $\alpha\,|\,I$ is not universally weakly inner.

In Section 1 we give a preliminary consideration concerning the Borchers spectrum and in Section 2 we give the main theorem.

In Section 3, for any pair (A,α) we construct a sequence of mutually orthogonal closed ideals (I_k) indexed by $\mathbf{N}\cup\{\infty\}$, some of which may be (0), such that (I_k) generates an essential ideal of A and $T(\alpha\,|\,I_k)=T_B(\alpha\,|\,I_k)$ is the subgroup of order k of $\mathbf{T}$ ($\mathbf{T}$ itself if $k=\infty$) for non-zero I_k. We show a property of α on I_k in 3.3 (cf. [1]).

In Section 4 we make a simple remark on the strong Connes spectrum [5].

1. Borchers spectrum

Let A be a C^*-algebra and G a locally compact abelian group with action α on A. Let $\mathscr{H}^\alpha(A)$ denote the set of α-invariant, hereditary, non-zero C^*-subalgebras of A, and let $\mathscr{H}_B^\alpha(A)$ denote the subset consisting of algebras B in $\mathscr{H}^\alpha(A)$ such that the closed ideal of A generated by B is essential in A. The Borchers spectrum of α is defined by

$$\Gamma_B(\alpha)=\cap\operatorname{Sp}(\alpha\,|\,B)\,;\qquad B\in\mathscr{H}_B^\alpha(A)$$

whereas the Connes spectrum $\Gamma(\alpha)$ is defined by the same formula without subscript B (see [9, 8.8]). We characterize $\Gamma_B(\alpha)$ using ideals of the crossed product $A\times_\alpha G$ and

the dual action $\hat{\alpha}$ of $\Gamma = \hat{G}$ on $A \times_\alpha G$; similar results are known for the Connes spectrum [9, 8.11.8] and its variant [5, 3.4]. We adopt the notation in Pedersen's book [9, 7.8] throughout the following. In particular $\lambda(t)$, $t \in G$ is the canonical unitary group in the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$ and $\lambda(f)$, for $f \in L^1(G)$, means

$$\lambda(f) = \int_G f(t)\lambda(t)dt$$

which is also a multiplier.

1.1. Proposition. *In the above situation, let n be a positive integer. Then $\sigma \in \Gamma$ belongs to $\Gamma_B(\alpha)$ if and only if for any neighbourhood Ω of σ, and for any closed ideal J of $A \times_\alpha G$ such that J generates an essential $\hat{\alpha}$-invariant closed ideal of $A \times_\alpha G$, there exist $\sigma_k \in \Omega + \cdots + \Omega$ (k terms), $k = 1, \cdots, n$, such that*

$$J \cap \hat{\alpha}_{\sigma_1}(J) \cap \cdots \cap \hat{\alpha}_{\sigma_n}(J) \neq (0) .$$

For the proof we prepare

1.2. Lemma. *Let $x \in A$ and let $f, g \in L^1(G)$ be non-zero such that $\mathrm{supp}\,\hat{f}$ and $\mathrm{supp}\,\hat{g}$ are compact. Suppose that $\lambda(f) \cdot x\lambda(g) \neq 0$ (in $A \times_\alpha G$). Then $\mathrm{Sp}_\alpha x \cap (\mathrm{supp}\,\hat{f} - \mathrm{supp}\,\hat{g}) \neq \varnothing$.*

We omit the proof; this can be shown by standard techniques of spectral theory (cf. [9, chap. 8]).

Proof of 1.1. Let $\sigma \notin \Gamma_B(\alpha)$ and let $B \in \mathscr{H}_B^\alpha(A)$ satisfy $\mathrm{Sp}(\alpha | B) \not\ni \sigma$. Then there exist a neighbourhood Ω of σ and a non-zero $f \in L^1(G)$ such that $\lambda(f_\tau)^* B\lambda(f) = (0)$ for $\tau \in \Omega$, where $f_\tau(t) = \langle t, \tau \rangle f(t)$. Let J be the ideal of $A \times_\alpha G$ generated by $B\lambda(f)$. Then the $\hat{\alpha}$-invariant ideal $\hat{J}$ of $A \times_\alpha G$ generated by J is essential in $A \times_\alpha G$ because B is essential in A. Since $\lambda(f_\tau)^* B\lambda(f) = (0)$ for $\tau \in \Omega$, it follows that

$$J \cap \hat{\alpha}_\tau(J) = (0) .$$

Let $\sigma \in \Gamma_B(\alpha)$. Let J be a closed ideal of $A \times_\alpha G$ such that $\hat{J}$ is essential. It follows from [6] that there are $B_i \in \mathscr{H}^\alpha(A)$ and non-zero $f_i \in L^1(G)$ such that $B_i\lambda(f_i) \subset J$ and the family $\{B_i\lambda(f_i)\}$ generates J. Let $\{B_i\lambda(f_i)\}$ be a maximal family of these with $B_i A B_j = (0)$ for $i \neq j$. Then $\{B_i\}$ generates an essential ideal of A. For, otherwise, there is an ideal I of $\mathscr{H}^\alpha(A)$ such that $IB_i = (0)$ for any i. The ideal I_1 of $A \times_\alpha G$ generated by I is $\hat{\alpha}$-invariant and so we obtain $I_1 \cap \hat{J} \neq (0)$ which implies $I_1 \cap J \neq (0)$. Then there is $B_0 \in \mathscr{H}^\alpha(A)$ and non-zero $f_0 \in L^1(G)$ such that $B_0\lambda(f_0) \subset I_1 \cap J$. Since $I_1 B_i = (0)$, B_0 satisfies that $B_0 A B_i = (0)$ which contradicts the maximality of $\{B_i\lambda(f_i)\}$.

Let B be the hereditary C^*-subalgebra of A generated by $\{B_i\}$. Then $B \in \mathscr{H}_B^\alpha(A)$. Since $\sigma \in \Gamma_B(\alpha)$, it follows from the proof of [9, 8.8.5] that for any compact neighbourhood Ω of σ there exist $x_1, \cdots, x_{n+1}$ in B such that $\mathrm{Sp}_\alpha x_k \subset \Omega$ for k and the product $x_{n+1} \cdots x_1$ is non-zero. We may assume that all x_k belong to some B_i, and we let f be a non-zero element of $L^1(G)$ such that $\mathrm{supp}\,\hat{f}$ is a small compact subset of

supp $\hat{f}_i$, and $\hat{f}$ is positive. We assert that there exist $\sigma_1, \cdots, \sigma_{n+1}$ in Γ such that

$$\lambda(f_{\sigma_{n+1}})x_{n+1}\lambda(f_{\sigma_n})x_n \cdots \lambda(f_{\sigma_1})x_1 \neq 0 .$$

Otherwise the integrations over σ_k (in $M(A \times_\alpha G)$) would yield

$$x_{n+1} \cdots x_1 f(0)^n = 0$$

which is a contradiction. Hence, since we may suppose $\sigma_1 = 0$, we obtain $\sigma_2, \cdots, \sigma_{n+1}$ such that

$$\lambda(f_{\sigma_{n+1}})x_{n+1} \cdots \lambda(f_{\sigma_2})x_2\lambda(f)x_1 \neq 0 .$$

Then by Lemma 1.2,

$$\sigma_k - \sigma_{k-1} \in \Omega + \operatorname{supp} \hat{f} - \operatorname{supp} \hat{f} \equiv \Omega_1 .$$

Hence $\sigma_k \in \Omega_1 + \cdots + \Omega_1$ $(k-1$ terms). Since $\lambda(f_{\sigma_k})x_k \in \hat{\alpha}_{\sigma_k}(J)$ due to $\lambda(f)x_k \in \lambda(f)B_i \subset J$, we obtain

$$J \cap \hat{\alpha}_{\sigma_2}(J) \cdots \cap \hat{\alpha}_{\sigma_{n+1}}(J) \neq (0) .$$

This completes the proof since Ω_1 can be an arbitrarily small neighbourhood of σ.

1.3. Corollary. *Let α and β be actions of a locally compact abelian group G on a C^*-algebra A. If α and β are exterior equivalent, then $\Gamma_B(\alpha) = \Gamma_B(\beta)$.*

Proof. Since $(A \times_\alpha G, \Gamma, \hat{\alpha})$ and $(A \times_\beta G, \Gamma, \hat{\beta})$ are equivalent (cf. [9, 8.11]), this follows from 1.1.

2. Free action

Let A be a C^*-algebra and α an automorphism of A. We say that α is *freely acting* on A if for any non-zero α-invariant closed ideal J of A, the Borchers spectrum $T_B(\alpha|J)$ does not equal $\{1\} \subset \mathbf{T} = \hat{\mathbf{Z}}$. When A is a W^*-algebra, this definition coincides with that given by Kalleman [4, 1.8] (see [9, 8.8.3, 8.9.3]).

The following is an improvement of [7, 1.1]; a similar result was obtained by Connes [2, 1.2.1] in the W^*-case (where 'freely acting' is called 'properly outer').

2.1. Theorem. *Let A be a C^*-algebra and α an automorphism of A. Then α is freely acting on A if and only if for any non-zero hereditary C^*-subalgebra B of A and for any multiplier a of A (or for $a=1$)*

$$(*) \qquad \inf\{\|xa\alpha(x)\|: 0 \leqslant x \in B, \|x\|=1\} = 0 .$$

Proof. If α is not freely acting, there is a non-zero α-invariant closed ideal J of A with $T_B(\alpha|J) = \{1\}$. Then by [9, 8.8.7] for any $\varepsilon > 0$ there is $B \in \mathcal{H}_B^\alpha(J)$ such that

$$\operatorname{Sp}(\alpha|B) \subset \{e^{i\theta}: |\theta| < \varepsilon\} .$$

It follows from [9, 8.7.10] that $\|\alpha(a) - x\| < \varepsilon\|x\|$ for $x \in B$. Hence, for $x \in B$ with $0 \leqslant x$, $\|x\| = 1$,

$$2\|\alpha(x)x\| \geq \|\alpha(x)x + x\alpha(x)\|$$

$$\geq \|\alpha(x^2) + x^2\| - \varepsilon^2$$

$$\geq 2\|x^2\| - \|\alpha(x^2) - x^2\| - \varepsilon^2$$

$$\geq 2 - \varepsilon - \varepsilon^2 \ .$$

Thus the infimum in $(*)$ is not zero.

Suppose that the infimum in $(*)$ is positive, say δ. We use the proof of [7, 1.1] without any alteration until the very last stage (a is allowed to be a multiplier as remarked after the proof of [7, 1.1]).

Let ϕ be a pure state of B, which has a unique extension to a state of A, denoted by ϕ again. In the GNS representation space $\mathscr{H}_\phi$ we construct a unitary V_ϕ such that

$(**)$ $\qquad\qquad V_\phi \pi_\phi(x) V_\phi^* = \pi_\phi \circ \alpha(x) , \qquad x \in A ,$

$$\mathrm{Re}\,(\bar\pi_\phi(ea) V_\phi \bar\pi_\phi(e)\Phi, \Phi) \geq \delta$$

for any $\Phi \in \bar\pi_\phi(e)\mathscr{H}_\phi$ with $\|\Phi\| = 1$, where e denotes the identity of B^{**} in A^{**} and $\bar\pi$ is the unique extension of π to a representation of A^{**}. Define π as the direct sum of π_ϕ with all pure states ϕ of B. In the representation space of π define V as the direct sum of all V_ϕ corresponding to π_ϕ. Then (π, V) gives a covariant representation of (A, α) and satisfies the properties analogous to $(**)$.

Let N be the numerical range of $\bar\pi(ea) V \bar\pi(e)$ on the range of $\bar\pi(e)$, and let $r = \sup\{|\sigma|; \sigma \in N\}$. Then $r \geq \delta$ (> 0). Let $\theta_0 = \arcsin \delta/r$. Let (ρ, U) be the direct sum of $(\pi, e^{i\theta}V)$ with $|\theta| < (\pi + \theta_0)/2$. Then the numerical range N_1 of $\bar\rho(ea) U \bar\rho(e)$ on the range of $\bar\rho(e)$ is the convex hull of $e^{i\theta}N$ with $|\theta| < (\pi + \theta_0)/2$. In particular N_1 is contained in

$$\{\lambda: \mathrm{Re}\,\lambda \geq -r\cos\theta_0/2, |\lambda| \leq r\} \ .$$

Further the closure $\bar N_1$ of N_1 contains r, since $\bar N$ contains $re^{i\theta}$ for some θ with $|\theta| \leq \pi/2 - \theta_0$. Hence the norm of $T \equiv 2^{-1}(\bar\rho(ea) U \bar\rho(e) + \bar\rho(e) U^* \bar\rho(a^*e))$ is equal to r, since the closure of the numerical range of T contains r and is included in $[-r\cos\theta_0/2, r]$, and T is self-adjoint.

Let $\rho \times U$ be the representation of the crossed product $A \times_\alpha \mathbf{Z}$ corresponding to (ρ, U). From the above fact on T we can conclude that $\rho \times U$ is not faithful, as follows.

For any $b \in B$ we obtain

$$\rho(b^*a) U \rho(b) + \rho(b^*) U^* \rho(a^*b) \geq -2r\cos\theta_0/2\rho(b^*b) \ .$$

If $\rho \times U$ were faithful, the dual automorphism $\hat\alpha$ would induce automorphisms of $(\rho \times U)(A \times_\alpha \mathbf{Z})$. By applying $\hat\alpha_{-1}$, we obtain

$$-\rho(b^*a) U \rho(b) - \rho(b^*) U^* \rho(a^*b) \geq -2r\cos\theta_0/2\rho(b^*b) \ .$$

Since the above inequalities are valid for any $b \in B$, they are valid for $b = e$ with $\bar\rho$ in

place of ρ. Thus

$$-r\cos\theta_0/2\bar{\rho}(e) \leqslant T \leqslant r\cos\theta_0/2\bar{\rho}(e) .$$

which implies that $\|T\| \leqslant r\cos\theta_0/2$, a contradiction.

Let I be the α-invariant closed ideal of A generated by B. Since $\ker\rho \cap B = (0)$, it follows that $\ker\rho \cap I = (0)$. Let $J = \ker\rho \times U \,|\, I \times_\alpha \mathbf{Z}$. Then the $\hat{\alpha}$-invariant closed ideal $\hat{J}$ of $I \times_\alpha \mathbf{Z}$ generated by J is essential in $I \times_\alpha \mathbf{Z}$. Because otherwise there is a non-zero α-invariant closed ideal K of I with $\hat{J} \cap K = (0)$. Then, since $K \cap B \neq (0)$, we can repeat the argument for $K \cap B$ instead of B to yield that $\rho \times U$ is not faithful on $K \times_\alpha \mathbf{Z}$, i.e., $J \cap K \times_\alpha \mathbf{Z} \neq 0$, a contradiction. Thus $\hat{J}$ is essential in $I \times_\alpha \mathbf{Z}$. Let $\lambda \in \mathbf{T}_B(\alpha \,|\, I)$ with $\lambda \neq 1$. There exist $\varepsilon > 0$ and $n \in \mathbf{N}$ such that

$$\{\lambda_k^k e^{i\theta} : |\theta| < (\pi + \theta_0)/2, \ k = 0, 1, \cdots, n\} = \mathbf{T}$$

for any $\lambda_k \in \mathbf{T}$ with $|\lambda_k - \lambda| < \varepsilon$. Then

$$\bigcap_{k=0}^{n} \hat{\alpha}_{\lambda_k^k}(J)$$

is $\hat{\alpha}$-invariant and so is (0) since $J \cap I = (0)$. By 1.1, this contradicts $\lambda \in \mathbf{T}_B(\alpha \,|\, I)$.

As a corollary to the above theorem, we obtain [2, 1.2.1] due to Connes:

2.2. Corollary. *Let M be a von Neumann algebra and α an automorphism of M. Then α is freely acting if and only if for any projection e of M and for any $a \in M$ (or for $a = 1$)*

$$\inf\{\|pa\alpha(p)\| : p \in M, p = p^* = p^2 \neq 0, p \leqslant e\} = 0 .$$

We omit the proof (cf. [9, 8.8.3]).

2.3. Remark. Theorem 2.1 can be used to strengthen 3.1 in [7] in an obvious way.

2.4. Remark. From the proof of [7, 2.1] we obtain: If α is a freely acting automorphism of a separable C^*-algebra, then there exist uncountably many equivalence classes of pure states ϕ of A such that $\phi \circ \alpha$ is disjoint from ϕ. To prove this it is enough to make uncountably many pure states ϕ of A such that the support projections of ϕ are mutually orthogonal and $\phi \circ \alpha$ is disjoint from ϕ, since A is separable.

For this purpose, on each induction step in the proof of [7, 2.1], we split $a_{n-1}Aa_{n-1}$ into two non-zero orthogonal hereditary C^*-subalgebras (this is possible because A does not have a minimal projection) and construct a pair (e_n, a_n) for each.

2.5. Remark. The notation of properly outer was defined by Elliott [3]. It easily follows that a freely acting automorphism is properly outer (cf. [7]).

2.6. Remark. Let α be an element of the connected component $\mathrm{Aut}_0(A)$ of the identity in the automorphism group of the C^*-algebra A equipped with the uniform

topology. Then $T(\alpha) = \{1\}$. To prove this, it suffices to show that if $\alpha \in \mathrm{Aut}_0(A)$ satisfies that $T(\alpha \mid J) = \{1\}$ for any closed ideal J, then $T(e^\delta \circ \alpha \mid J) = \{1\}$ for any such J for any *-derivation δ of A with $e^{\|\delta\|} - 1 < \frac{1}{2}$ (noting that any closed ideal is left invariant under $\alpha \in \mathrm{Aut}_0(A)$) ([9, 8.7.7–8]). By the assumption on α, we have $T_B(\alpha) = \{1\}$ (cf. 3.1 below) and so $B \in \mathscr{H}_B^\alpha(A)$ such that $\|(\alpha - 1) \mid B\| < \frac{1}{2}$. Then

$$\|e^\delta \circ \alpha(x) - x\| \leqslant e^{\|\delta\|} - 1 + \|(\alpha - 1) \mid B\| < 1$$

for $x \in B$ with $\|x\| = 1$. It follows from the first part of the proof of 2.1 that the infimum in $(*)$ is positive for $a = 1$. Hence $T(e^\delta \circ \alpha) = \{1\}$. Since this reasoning applies for any closed ideal of A, we get the conclusion.

3. Universally weakly inner automorphisms

Let A be a C^*-algebra and α an automorphism of A. For $k \in \mathbf{N}$ let T_k be the subgroup of T of order k and let $T_\infty = T$. Let F_k be the set of all α-invariant closed ideals I of A such that $T(\alpha \mid I) = T_B(\alpha \mid I) = T_k$. If $I \in F_k$ and J is a non-zero α-invariant closed ideal of I, then $J \in F_k$. Let I_k be the closed ideal generated by all $I \in F_k$. Then we shall show that $I_k \in F_k$. It is obvious that $T(\alpha \mid I_k) \supset T_k$. To prove that $T_B(\alpha \mid I_k) \subset T_k$, let (I_i) be a maximal family in F_k such that $I_i \cap I_j = (0)$ for $i \neq j$. Then (I_i) generates an essential ideal of I_k. For any neighbourhood Ω of 1 in T there exist $B_i \in \mathscr{H}_B^\alpha(I_i)$ such that $\mathrm{Sp}(\alpha \mid B_i) \subset T_k \cdot \Omega$ [9, 8.8.7]. Let B be the direct sum of (B_i). Then $B \in \mathscr{H}_B^\alpha(I_k)$ and $\mathrm{Sp}(\alpha \mid B) \subset T_k \cdot \Omega$. This completes the proof.

3.1. Proposition. *Let A be a C^*-algebra and α an automorphism of A. For each $k \in \mathbf{N} \cup \{\infty\}$ let I_k be the maximal α-invariant closed ideal of I such that $T(\alpha \mid I_k) = T_B(\alpha \mid I_k) = T_k$ (if there are no such ideals, set $I_k = (0)$). Then (I_k) are mutally orthogonal and generate an essential ideal of A.*

Proof. The orthogonality is trivial.

Let I be the closed ideal generated by $\{I_k\}$. Suppose that the ideal $J \equiv \{x \in A : xI = (0)\}$ is non-zero. Then $T(\alpha \mid J) \subsetneqq T_B(\alpha \mid J)$. Hence in particular $T(\alpha \mid J) \neq T$ and $T_B(\alpha \mid J) \neq \{1\}$. If $T_B(\alpha \mid J) = T$, choose $B \in \mathscr{H}^\alpha(I)$ such that $\mathrm{Sp}(\alpha \mid B) \neq T$ and let J' be the ideal generated by B. Then since $B \in \mathscr{H}_B^\alpha(J')$, $T_B(\alpha \mid J') \neq T$. Since $T_B(\alpha \mid J')$ is a closed subset of T and satisfies that if $\lambda \in T_B(\alpha \mid J')$ and $n \in \mathbf{Z}$, then $\lambda^n \in T_B(\alpha \mid J')$ [9, 8.8.5] $T_B(\alpha \mid J')$ has the following form:

$$(*) \qquad\qquad T_B(\alpha \mid J') = T_{k_1} \cup \cdots \cup T_{k_n}.$$

where $k_1 < k_2 < \cdots < k_n$ and no k_i divides k_j for $i < j$.

Let $J_1 = J$ or J' and suppose that $T_B(\alpha \mid J_1)$ is of the form $(*)$. If $T(\alpha \mid J_1) = T_B(\alpha \mid J_1)$, then $n = 1$ and $J_1 \in F_{k_1}$, a contradiction. If $T(\alpha \mid J_1) \subsetneqq T_B(\alpha \mid J_1)$, then there exist $B \in \mathscr{H}^\alpha(J_1)$ and i such that $\mathrm{Sp}(\alpha \mid B)$ does not contain T_{k_i}. Let J_2 be the ideal generated by B. Then, since $T_B(\alpha \mid J_2)$ does not contain T_{k_i}, the total number of subgroups contained in $T_B(\alpha \mid J_2)$ is smaller than that for $T_B(\alpha \mid J_1)$. After a finite number of steps we find a non-zero α-invariant closed ideal J_m such that $T(\alpha \mid J_m) =$

$\mathbf{T}_B(\alpha \mid J_m)$. Hence J_m should have been in some F_k, a contradiction.

3.2. Remark. An automorphism α on the C^*-algebra A is freely acting if and only if $I_1 = (0)$, i.e., if and only if the set of α-invariant closed ideals I with $\mathbf{T}(\alpha \mid I) \neq \{1\}$ generates an essential ideal.

3.3. Proposition. *Let A be a separable C^*-algebra and α an automorphism of A. If $\mathbf{T}(\alpha) = \mathbf{T}_B(\alpha) = \mathbf{T}_k$, then there exists an α-invariant essential closed ideal I of A such that $\alpha^n \mid I$ is implemented by a unitary in the α^{**}-fixed point algebra of I^{**} if and only if $n \equiv 0 \bmod k$ (when $k = \infty$, if and only if $n = 0$).*

Proof. Let Ω be a sufficiently small neighbourhood of $1 \in \mathbf{T}$ and let $B \in \mathscr{H}_B^\alpha(A)$ satisfy $\mathrm{Sp}(\alpha \mid B) \subset \mathbf{T}_k \cdot \Omega$. Let I be the closed ideal generated by B. Then, when k is finite, $\alpha^{nk} \mid I$ satisfies the required property (cf. [9, 8.9]).

Assume that $n \not\equiv 0 \bmod k$ and $\alpha^n \mid I$ is universally weakly inner. Then $\alpha^n \mid J$ is not freely acting (cf. [7, 2.1], 2.4), for any non-zero (α^n-invariant) closed ideal J of A. Hence $\mathbf{T}_B(\alpha^n \mid I) = \{1\}$ and so for any $\varepsilon > 0$ there is $B \in \mathscr{H}_B^{\alpha^n}(I)$ such that

$$\mathrm{Sp}(\alpha^n \mid B) \subset \{e^{i\theta} : |\theta| < \varepsilon\} .$$

Then there is a unitary u in I^{**} such that $\alpha^n = \mathrm{Ad}\, u$ on I, $ue = eu$, and

$$\mathrm{Sp}(ue) \subset \{e^{i\theta} : |\theta| < \varepsilon/2\}$$

where e is the identity of B^{**} in I^{**} [9, 8.7.9, 8.9.1]. Let B_1 be the α-invariant hereditary C^*-subalgebra of I generated by B, which is a closed linear span of

$$\alpha^i(B) I \alpha^j(B) , \quad 0 \leqslant i, j \leqslant n-1 .$$

Then

$$\mathrm{Sp}(\alpha^n \mid B_1) \supset \{\lambda^n ; \lambda \in \mathbf{T}_k\} ,$$

which is not a trivial group. Suppose that $\bar{\alpha}(u) = u$, where $\bar{\alpha} = \alpha^{**}$. Then for $x \in \alpha^i(B) A \alpha^j(B)$

$$\alpha^n(x) = uxu^* = \bar{\alpha}^i(ue) x \bar{\alpha}^j(eu^*) .$$

Since $\mathrm{Sp}\, \bar{\alpha}^i(ue) = \mathrm{Sp}\, ue$, we obtain

$$\mathrm{Sp}\, \alpha^n(x) \subset \{\exp i\theta : |\theta| < \varepsilon\} .$$

Hence we are led to a contradiction:

$$\mathrm{Sp}(\alpha^n \mid B_1) \subset \{\exp i\theta : |\theta| \leqslant \varepsilon\} .$$

Thus $\bar{\alpha}(u) \neq u$. For similar results, see [1], [8].

4. A Remark concerning $\hat{\mathbf{T}}$.

In a previous paper [7], we used the invariant $\hat{\mathbf{T}}$ to obtain a result similar to 2.1. The condition was in fact quite strong as shown by the following:

4.1. Proposition. *Let (A, G, α) be a C^*-dynamical system with a discrete abelian group G. Then the strong Connes spectrum is given by*

$$\tilde{\Gamma}(\alpha) = \bigcap_I \Gamma(\dot{\alpha} \mid A/I)$$

where I runs over the set of proper α-invariant closed ideals of A and $\dot{\alpha}$ is the induced action on the quotient A/I.

Proof. By the characterization of $\tilde{\Gamma}(\alpha)$ in terms of covariant representations of (A, G, α) [5] it follows that $\tilde{\Gamma}(\alpha) \subset \Gamma(\dot{\alpha} \mid A/I)$ for such I ((0) included). Hence

$$\tilde{\Gamma}(\alpha) \subset \bigcap_I \Gamma(\dot{\alpha} \mid A/I) .$$

To show the converse we use the characterization of $\tilde{\Gamma}(\alpha)$ in terms of the dual action on the ideal space of $A \times_\alpha G$ [5].

Let J be a primitive ideal of $A \times_\alpha G$, and let

$$J_0 = \bigcap_{\sigma \in \Gamma} \hat{a}_\sigma(J), \quad \text{and} \quad I_0 = J_0 \cap A .$$

Then I_0 generates J_0 and

$$A/I_0 \times_{\dot{\alpha}} G \simeq (A \times_\alpha G)/J_0 .$$

Let $\dot{J}$ be the image of J in $A/I_0 \times_{\dot{\alpha}} G$. Then $\dot{J}$ is a primitive ideal of $A/I_0 \times_\alpha G$ and

$$\bigcap_{\sigma \in \Gamma} \hat{a}_\sigma(\dot{J}) = (0) .$$

Let

$$\Gamma_J = \{\sigma \in \Gamma; \hat{\alpha}_\sigma(J) = J\} .$$

which is equal to $\{\sigma \in \Gamma: \hat{\alpha}_\sigma(\dot{J}) = \dot{J}\}$.

Then the family $\{\hat{\alpha}_\sigma(\dot{J}); \sigma \in \Gamma\}$ is isomorphic to Γ/Γ_J (with the isomorphism which preserves topologies and actions of Γ) as shown in [5, 3.7]. Hence

$$\Gamma(\alpha \mid A/I_0) \subset \Gamma_J .$$

Since $\tilde{\Gamma}(\alpha) = \cap_J \Gamma_J$, the result follows.

4.2. Remark. If A is the gauge-invariant CAR algebra and α is an automorphism which preserves ideals, then $\tilde{\Gamma}(\alpha \mid I) = \{1\}$ for any ideal I of A. (Because, there is an ideal J of I such that $I/J \simeq \mathbf{C}$.) But the Connes spectrum $\Gamma(\alpha)$ can be non-trivial (e.g. if α is a certain quasi-free automorphism).

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