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FOURIER SERIES FOR A GENERAL LINEAR STOCHASTIC PROCESS

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ABSTRACT. We shall study about two kinds of Fourier series for a general linear process (GLP) defined by the author motivated by a work of Lugannani on pulse train processes. First we consider the Fourier series of a GLP truncated at $\pm T/2$ $(T>0)$. Our main concern with this is to study the asymptotic behaviors of Fourier coefficients when T goes to infinity. Corrections and generalizations of some results obtained or announced before will be made among other results. Secondly the approximate Fourier series representation of a GLP will be given and as a consequence of it, the existence of a sample continuous version of the process is shown.

1. Introduction

Let $X(t, \omega)$, $-\infty<t<\infty$, be a second order process with

(1.1) $$\int_I E|X(t, \omega)|^2 dt<\infty ,$$

for every finite interval I. $X(t, \omega)$ is called a general linear process (GLP) if it is defined by

(1.2) $$\int_I E\left|\int_\alpha^\beta a(t-\lambda)\eta(d\lambda, \omega)-X(t, \omega)\right|^2 dt \to 0 ,$$

as $\beta\to\infty$, $\alpha\to-\infty$, for every finite interval I, in which

(1.3) $$\eta(S, \omega)=\xi(S, \omega)+m(S) ,$$

where $\xi(S, \omega)$ is a random measure not necessarily bounded with $E\xi(S, \omega)=0$, $m(S)$ is a nonrandom signed measure, S being Borel sets on R^1 and $a(t)$ is a nonrandom function such that

(1.4) $$\int_I |a(t-\lambda)|^2 dF(t)<\infty , \quad \int_I |a(t-\lambda)|^2 |dm(\lambda)|<\infty ,$$

for every t and for every finite interval I, where $F(t)$ and $m(t)$ are point functions associated with ξ and m respectively:

$$E|\xi([s, t), \omega)|^2=F(t)-F(s) , \quad s<t .$$

Such a GLP was defined by the author [10] motivated by a Lugannani's work [14] on pulse train processes. If $m(t)$ is a constant and $F(t)$ is ct, c being a constant up to additive constants, then it is seen that $a(t)$ should belong to $L^2(-\infty, \infty)$ and $X(t, \omega)$ is $\int_{-\infty}^{\infty} a(t-\lambda)\xi(d\lambda, \omega)$ for almost all (t, ω) in $(-\infty, \infty)\times\Omega$ that is, it is a linear process in the ordinary sense and then a stationary process. If $F(t)$ is not ct up to additive constants, $X(t, \omega)$ is not even a weakly stationary process (WSP) in general and in fact, in order for a GLP with $m(t)$ constant, to be a WSP for every $a(t)\in L^2(-\infty, \infty)$ it is necessary and sufficient that $F(t)$ is ct up to additive constants.

There are several examples of GLP which are not WSP and of interest, for instance, pulse train processes studied by Lugannani mentioned above and GLP with $\eta([0, t), \omega)-m(t)$ which is nonhomogeneous compound Poisson processes studied by Endow [3], [4]. Some results such as those on weak or strong laws of large numbers still hold for these processes if

$$\Phi(\lambda)=\sup_{-\infty<t<\infty}|F(\lambda+t)-F(t)-v_0\lambda| \tag{1.5}$$

with a certain constant v_0, is assumed to be small when λ is large. For the strong law of large numbers to hold, beside (1.5) some additional condition on the covariance function

$$\rho(s, t)=E[X(s, \omega)-EX(s, \omega)][\overline{X(t, \omega)}-\overline{EX(t, \omega)}] \tag{1.6}$$

which is somewhat analogous to the condition required for WSP (see [11], Theorem 7.1) is needed. For further results on GLP, see Honda [5], [6], [7], Butzer-Gather [2]. (1.5) is thought of as a quantity which measures the closeness of a GLP to a WSP.

2. The Fourier series of a truncated GLP

Let $X(t, \omega)$ be a GLP and let us consider the Fourier series

$$\sum_{n=-\infty}^{\infty} C_n(T, \omega)e^{2n\pi it/T} \tag{2.1}$$

of $X(t, \omega)$ truncated at $\pm T/2$, where

$$C_n(t, \omega)=\frac{1}{T}\int_{-T/2}^{T/2} X(t, \omega)e^{-2n\pi it/T}dt\,. \tag{2.2}$$

It is almost obvious that the Fourier series (2.1) is $(C, 1)$ summable to $X(t, \omega)$ in $(-T/2, T/2)$ almost surely and it is not so difficult to show

Theorem 1. (2.1) *is* $(C, 1)$ *summable to* $X(t, \omega)$ *in the metric*

$$\frac{1}{T}\int_{-T/2}^{T/2} E|\cdot|^2dt\,,$$

if $a(t)\in L^2(-\infty, \infty)$ *and*

$$\Psi(\lambda)=\sup_{-\infty<t<\infty}|F(\lambda+t)-F(t)|=O(|\lambda|)\,. \tag{2.3}$$

for large $|\lambda|$.

Actually this is proved by a known result on $(C, 1)$ summability of a Fourier series (Katznelson [8], Theorem 2.3) with some manipulations. The details are omitted here.

It is well recognized that the role of the Fourier series of a truncated ordinary linear process is of importance particularly for random noise processes and so it is for a WSP. Actually Root and Pitcher [16], the author [9], [10, I] and Arimoto [1] studied about the asymptotic behaviors of covariances of Fourier coefficients of a WSP truncated at $\pm T/2$, when T goes to infinity. It is thus of some interest to work with the same kind of problems for a GLP as an extension to a non-stationary process.

3. Asymptotic behavior of the mean of a Fourier coefficient

The author has shown (Theorem 3.1 of [11]) that $EC_0(T, \omega)$ converges as $T \to \infty$, under some condition on $m(t)$. In there, however, the condition that

$$M(\lambda) = \sup_{t \in (-\infty, \infty)} |m(t+\lambda) - m(t) - m_0 \lambda|$$

for some constant m_0, is monotone, was used without being explicitly stated, so that this condition must be placed in the statement of Theorem 3.1 of [10].

Unfortunately this condition seems to be not quite suitable and restricts processes too much. Also in [12] the author announced a result which dealt with $EC_n(T, \omega)$ but the condition placed there seems unreasonable because we stuck to the mere condition for $a(t)$ that $a(t) \in L^2(-\infty, \infty)$. We now assume that $a(t) \in L^1(-\infty, \infty)$ and improve the result.

We begin with the following simple

Lemma 1. *If* $X(t, \omega)$ *is a* GLP, *then*

$$\int_I dt \left| \int_\alpha^\beta a(t-\lambda) dm(\lambda) - EX(t, \omega) \right|^2 dt \to 0, \tag{3.1}$$

as $\beta \to \infty$, $\alpha \to -\infty$, *for every finite interval* I.

For, the integral in (3.1) is

$$\int_I dt \left| E \int_\alpha^\beta a(t-\lambda) \eta(d\lambda, \omega) - EX(t, \omega) \right|^2 \leqq \int_I dt E \left| \int_\alpha^\beta a(t-\lambda) \eta(d\lambda, \omega) - X(t, \omega) \right|^2$$

which converges to zero as $\beta \to \infty$, $\alpha \to \infty$ by the definition of a GLP.

We shall prove

Theorem 2. *Suppose*

$$a(t) \in L^1(-\infty, \infty) \tag{3.2}$$

and

$$M(\lambda) = o(\lambda), \quad |\lambda| \to \infty \tag{3.3}$$

for some constant m_0. Then

$$EC_n(T, \omega)\to 0\,, \qquad for \quad n\neq 0\,, \tag{3.4}$$

as $T\to\infty$, and

$$EC_0(T, \omega)\to m_0\int_{-\infty}^{\infty} a(t)dt\,. \tag{3.5}$$

Proof. From Lemma 1, we see that

$$\begin{aligned}
EC_n(T, \omega) &= \lim_{\alpha\to-\infty,\beta\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} e^{-2n\pi it/T}dt \int_{\alpha}^{\beta} a(t-\lambda)dm(\lambda) \\
&= \lim_{\alpha\to-\infty,\beta\to\infty} \Bigg[\frac{1}{T}\int_{-T/2}^{T/2} e^{-2n\pi it/T}dt\int_{T}^{\beta} a(t-\lambda)dm(\lambda) \\
&\qquad +\frac{1}{T}\int_{-T/2}^{T/2}\cdot dt\int_{\alpha}^{-T}\cdot dm(\lambda)+\frac{1}{T}\int_{-T/2}^{T/2}\cdot dt\int_{-T}^{T}\cdot dm(\lambda)\Bigg] \\
&= \lim_{\alpha\to-\infty,\beta\to\infty} [K_{n1}(T, \beta)+K_{n2}(T, \alpha)+K_{n3}(T)] \qquad \text{say}.
\end{aligned} \tag{3.6}$$

We may suppose $\alpha<-T$, $\beta>T$. We first prove

$$\lim_{\beta\to\infty} K_{n1}(T, \beta)=o(1)\,, \qquad \text{as} \quad T\to\infty\,, \tag{3.7}$$

$$\lim_{\alpha\to-\infty} K_{n2}(T, \alpha)=o(1)\,, \qquad \text{as} \quad T\to\infty \tag{3.8}$$

and hence

$$EC_n(T, \omega)=K_{n3}(T)+o(1)\,, \qquad \text{as} \quad T\to\infty\,. \tag{3.9}$$

We give the proof of (3.7). The proof of (3.8) is carried out in a similar way.

$$K_{n1}(T, \beta)=\frac{1}{T}\int_{T}^{\beta} dm(\lambda)\int_{-T/2-\lambda}^{T/2-\lambda} e^{-2n\pi i(s+\lambda)/T}a(s)ds$$

which is by the interchange of the order of integrations

$$\begin{aligned}
&=\frac{1}{T}\int_{-\beta-T/2}^{-\beta+T/2} a(s)e^{-2n\pi is/T}ds\int_{-T/2-s}^{\beta} e^{-2n\pi i\lambda/T}dm(\lambda) \\
&\quad +\frac{1}{T}\int_{-\beta+T/2}^{-3T/2}\cdot ds\int_{-T/2-s}^{T/2-s}\cdot dm(\lambda)+\frac{1}{T}\int_{-3T/2}^{-T/2}\cdot ds\int_{-T}^{-T/2-s}\cdot dm(\lambda) \\
&=K_{n1}^{(1)}(T, \beta)+K_{n1}^{(2)}(T, \beta)+K_{n1}^{(3)}(T)\,, \qquad \text{say}.
\end{aligned} \tag{3.10}$$

Now

$$K_{n1}^{(1)}(T, \beta)=\frac{1}{T}\int_{-\beta-T/2}^{-\beta+T/2} a(s)e^{-2n\pi is/T}ds\int_{-T/2}^{\beta} e^{-2n\pi i\lambda/T}d[m(\lambda)-m(-T/2-s)]$$

$$=\frac{1}{T}\int_{-\beta-T/2}^{-\beta+T/2} a(s)e^{-2n\pi is/T}\left\{[m(\lambda)-m(-T/2-s)]e^{-2n\pi i\lambda/T}\Big|_{-T/2-s}^{\beta}\right.$$

$$\left.+(2n\pi i/T)\int_{-T/2-s}^{\beta} e^{-2n\pi i\lambda/T}[m(\lambda)-m(-T/2-s)]d\lambda\right\}ds\,.$$

Thus we have

$$|K_{n1}^{(1)}(T,\beta)|\leqq\frac{1}{T}\int_{-\beta-T/2}^{-\beta+T/2}|a(s)|\,ds\left[|m(\beta)-m(-T/2-s)|\right.$$

$$\left.+(2n\pi/T)\int_{-T/2-s}^{\beta}|m(\lambda)-m(-T/2-s)|\,d\lambda\right].$$

Noting that $0\leqq\beta-(-T/2-s)\leqq T$, $0\leqq\lambda-(-T/2-s)\leqq T$ and that

$$\sup_{|t|<T,\,-\infty<u<\infty}|m(t+u)-m(u)|\leqq CT,$$

for some constant C which is a consequence of (3.3), we see that

$$|K_{n1}^{(1)}(T,\beta)|\leqq\int_{-\beta-T/2}^{-\beta+T/2}|a(s)|\,ds\cdot C(1+2n\pi)\,.$$

Hence in view of (3.2) the integral on the right hand side is $o(1)$ when $\beta\to\infty$, which proves (3.7).

$K_{n1}^{(2)}(t,\beta)$ is handled in a similar way. In fact

$$|K_{n1}^{(2)}(T,\beta)|\leqq\frac{1}{T}\int_{-\beta+T/2}^{-3T/2}|a(s)|\,ds\left[|m(T/2-s)-m(-T/2-s)|\right.$$

$$\left.+(2n\pi/T)\int_{-T/2-s}^{T/2-s}|m(\lambda)-m(-T/2-s)|\,d\lambda\right]$$

which is, because of $0\leqq\lambda-(-T/2-s)\leqq T$,

$$\leqq\frac{1}{T}\int_{-\beta+T/2}^{-3T/2}|a(s)|\,ds\cdot M(T)+\frac{2n\pi}{T^2}\int_{-\beta+T/2}^{-3T/2}|a(s)|\,ds\cdot M(T)T$$

$$\leqq\int_{-\infty}^{-3T/2}|a(s)|\,ds\cdot C(1+2n\pi)=o(1)\,,$$

as $T\to\infty$.

In a quite similar way, we see that

$$|K_{n1}^{(3)}(T)|\leqq\int_{-\infty}^{-T/2}|a(s)|\,ds\cdot C(1+2n\pi)=o(1)\,.$$

We thus have shown (3.7) and hence (3.9).

We now handle $K_{n3}(T)$, which can be written by

$$K_{n3}(T)=\frac{1}{T}\int_{-T/2}^{T/2}e^{-2n\pi it/T}dt\int_{-T}^{T}a(t-\lambda)d[m(\lambda)-m_0\lambda]$$
$$+m_0\frac{1}{T}\int_{-T/2}^{T/2}e^{-2n\pi it/T}dt\int_{-T}^{T}a(t-\lambda)d\lambda$$
$$=K_{n3}^{(1)}(T)+K_{n3}^{(2)}(T),\qquad \text{say}.$$

Writing $m_1(\lambda)=m(\lambda)-m_0\lambda$,

$$K_{n3}^{(1)}(T)=\frac{1}{T}\int_{-T}^{T}dm_1(\lambda)\int_{-T/2}^{T/2}e^{-2n\pi it/T}a(t-\lambda)dt$$
$$=\frac{1}{T}\int_{-T}^{T}dm_1(\lambda)\int_{-T/2-\lambda}^{T/2-\lambda}e^{-2n\pi i(s+\lambda)/T}a(s)ds$$

which turns out to be, as in (3.10),

$$\text{(3.11)}\qquad \frac{1}{T}\int_{-3T/2}^{-T/2}a(s)e^{-2n\pi is/T}ds\int_{-T/2-s}^{T}e^{-2n\pi i\lambda/T}dm_1(\lambda)$$
$$+\frac{1}{T}\int_{-T/2}^{3T/2}\cdot ds\int_{-T}^{T/2-s}\cdot dm_1(\lambda)+\frac{1}{T}\int_{-T/2}^{T/2}\cdot ds\int_{-T/2-s}^{T/2-s}\cdot dm_1(\lambda).$$

By the argument similar to the one in handling $K_{n1}^{(1)}(T,\beta)$, we can show that the first and second integrals in (3.11) converge to zero as $T\to\infty$. The last integral in (3.11) is

$$\text{(3.12)}\qquad \frac{1}{T}\int_{-T/2}^{T/2}a(s)e^{-2n\pi is/T}ds\left\{e^{-2n\pi i(T/2-s)/T}[m_1(T/2-s)-m_1(-T/2-s)]\right.$$
$$\left.+\frac{2n\pi i}{T}\int_{-T/2-s}^{T/2-s}[m_1(\lambda)-m_1(-T/2-s)]e^{-2n\pi i\lambda/T}d\lambda\right\},$$

in which $|m_1(T/2-s)-m_1(-T/2-s)|\leqq M(T)=o(T)$ and $|m_1(\lambda)-m_1(-T/2-s)|\leqq M(\lambda+T/2+s)=o(T)$, because of (3.3). Hence (3.12) is

$$\frac{1}{T}\int_{-T/2}^{T/2}|a(s)|ds\cdot o(T)=o(1),$$

as $T\to\infty$. We therefore have shown that $K_{n3}^{(1)}(T)=o(1)$, as $T\to\infty$.

Finally noting that

$$\left|\frac{1}{T}\int_{-T/2}^{T/2}e^{-2n\pi it/T}dt\int_{|\lambda|>T}a(t-\lambda)d\lambda\right|\leqq\frac{1}{T}\int_{-T/2}^{T/2}dt\int_{|u|>T/2}|a(u)|du$$
$$=\int_{|u|>T/2}|a(u)|du=o(1),$$

as $T\to\infty$. We thus have

$$K_{n3}^{(2)} = \frac{m_0}{T}\int_{-T/2}^{T/2} e^{-2n\pi i t/T}dt \cdot \int_{-\infty}^{\infty} a(u)du + o(1)$$
$$= m_0 \int_{-\infty}^{\infty} a(u)du + o(1)\,, \qquad \text{for} \quad n=0\,,$$
$$= o(1)\,, \qquad \text{for} \quad n \neq 0\,.$$

This completes the proof of Theorem 2.

4. Asymptotic behavior of the covariance of Fourier coefficients

In this section we shall study the limit behavior of

$$L_{mn}(T) = E[C_m(T,\omega) - EC_m(T,\omega)][\overline{C_n(T,\omega)} - E\overline{C_n(T,\omega)}]\,, \tag{4.1}$$

the covariance of the Fourier coefficients $C_m(T,\omega)$ and $C_n(T,\omega)$, when T goes to infinity, m and n being kept fixed. A result concerning this, was announced in [13]. The following theorem is a generalization of that as far as the integrability condition (3.2) on $a(t)$ is assumed.

Theorem 3. *Suppose* (3.2).

(i) *If, for some* v_0,

$$\Phi(\lambda) = o(\lambda)\,, \qquad as \quad |\lambda| \to \infty\,, \tag{4.2}$$

then

$$\lim_{T\to\infty} L_{mn}(T) = 0\,, \qquad \textit{for all } m \textit{ and } n. \tag{4.3}$$

(ii) *If, for some* v_0,

$$\Phi(\lambda) = o(1) \tag{4.4}$$

then

$$\lim_{T\to\infty} TL_{mn}(T) = 0\,, \qquad for \quad m \neq n\,, \tag{4.5}$$
$$= v_0 \left| \int_{-\infty}^{\infty} a(t)dt \right|^2, \qquad for \quad m = n\,.$$

We shall prove (ii) only. Actually the proof of (i) will be found in the course of the proof of (ii).

Proof. As we easily see,

$$L_{mn}(T) = \frac{1}{T^2} E\left\{ \int_{-T/2}^{T/2} e^{-2m\pi i s/T}[X(s,\omega) - EX(s,\omega)]ds \right. \tag{4.6}$$
$$\left. \times \int_{-T/2}^{T/2} e^{2n\pi i t/T}[\overline{X(t,\omega)} - E\overline{X(t,\omega)}]dt \right\}$$

$$=\lim_{\alpha\to-\infty,\beta\to\infty}\frac{1}{T^2}\int_{-T/2}^{T/2}ds\int_{-T/2}^{T/2}e^{-2\pi i(ms-nt)T}dt\int_{\alpha}^{\beta}a(s-\lambda)\overline{a(t-\lambda)}dF(\lambda)$$

$$=\lim_{\alpha\to-\infty,\beta\to\infty}\left[\frac{1}{T^2}\int_{-T/2}^{T/2}ds\int_{-T/2}^{T/2}e^{-2\pi i(ms-nt)/T}dt\int_{T}^{\beta}a(s-\lambda)\overline{a(t-\lambda)}dF(\lambda)\right.$$

$$\left.+\frac{1}{T^2}\int_{-T/2}^{T/2}ds\int_{-T/2}^{T/2}\cdot dt\int_{\alpha}^{-T}\cdot dF(\lambda)+\frac{1}{T^2}\int_{-T/2}^{T/2}ds\int_{-T/2}^{T/2}\cdot dt\int_{-T}^{T}\cdot dF(\lambda)\right]$$

$$=\lim_{\alpha\to-\infty,\beta\to\infty}[L_{mn1}(T,\beta)+L_{mn2}(T,\alpha)+L_{mn3}(T)]\,,\qquad \text{say}.$$

In what follows the computations of these quantities are made in a way similar to those of $K_{n1}(T,\beta)$, $K_{n2}(T,\alpha)$ and $K_{n3}(T)$ in the preceding section. Appearance of triple integral makes the manipulation more complicated or tedious. We carry out the proof in three stages.

(i) First we show that

$$\lim_{\beta\to\infty}L_{mn1}(T,\beta)=o(T^{-1})\,,\qquad \text{as}\quad T\to\infty \tag{4.7}$$

and

$$\lim_{\alpha\to-\infty}L_{mn2}(T,\alpha)=o(T^{-1})\,,\qquad \text{as}\quad T\to\infty \tag{4.8}$$

hold with the condition

$$\Psi(\lambda)=O(\lambda) \tag{4.9}$$

in place of (4.2) or (4.4), either of which is much stronger than (4.9).

(4.7) and (4.8) are handled in a quite similar way and we give only the proof of (4.7).

$$|L_{mn1}(T,\beta)|\leq\frac{1}{T^2}\int_{T}^{\beta}dF(\lambda)\int_{-T/2}^{T/2}ds\int_{-T/2}^{T/2}|a(s-\lambda)a(t-\lambda)|\,dt$$

$$=\frac{1}{T^2}\int_{T}^{\beta}dF(\lambda)\left[\int_{-T/2-\lambda}^{T/2-\lambda}|a(s)|\,ds\right]^2$$

Since

$$\int_{-T/2-\lambda}^{T/2-\lambda}|a(s)|\,ds\leq\int_{-\infty}^{-T/2}|a(s)|\,ds=o(1)\,,\qquad T\to\infty\,,$$

$$L_{mn1}(T,\beta)=o(1)\cdot\frac{1}{T^2}\int_{T}^{\beta}dF(\lambda)\int_{-T/2-\lambda}^{T/2-\lambda}|a(s)|\,ds\,,$$

where $o(1)$ is uniform in β $(>T)$. The last integral is

$$\frac{1}{T^2}\int_{-\beta-T/2}^{-\beta+T/2}|a(s)|\,ds\int_{-T/2-s}^{\beta}dF(\lambda)+\frac{1}{T^2}\int_{-\beta+T/2}^{-3T/2}|a(s)|\,ds \tag{4.10}$$

$$\int_{-T/2-s}^{T/2-s} dF(\lambda)+\frac{1}{T^2}\int_{-3T/2}^{-T/2} |a(s)|\,ds\int_{T}^{T/2-s} dF(\lambda)\,.$$

$L_{mn1}(T,\beta)$ is an analogue of $K_{n1}(T,\beta)$, but the estimation of it is much simpler because $F(\lambda)$ is nondecreasing. The integral ranges of the inner integrals of three double integrals of (4.10) are intervals of length not greater than T and hence keeping (4.9) in mind, we see that (4.10) is not greater than

$$\frac{\Psi(T)}{T^2}\left[\int_{-\beta-T/2}^{-\beta+T/2}+\int_{-\beta+T/2}^{-3T/2}+\int_{-3T/2}^{-T/2}\right]|a(s)|\,ds=o(T^{-1})\,.$$

This gives us (4.7).

We now proceed to estimate $L_{mn3}(T)$ which is written by

$$\begin{aligned}(4.11)\qquad L_{mn3}(T)&=\frac{1}{T^2}\int_{-T/2}^{T/2} ds\int_{-T/2}^{T/2} e^{-2\pi i(ms-nt)/T}dt\int_{-T}^{T} a(s-\lambda)\overline{a(t-\lambda)}dG(\lambda)\\&\quad+\frac{v_0}{T}\int_{-T/2}^{T/2} ds\int_{-T/2}^{T/2} e^{-2\pi i(ms-nt)/T}dt\int_{-T}^{T} a(s-\lambda)\overline{a(t-\lambda)}d\lambda\\&=L_{mn3}^{(1)}(T)+L_{mn3}^{(2)}(T)\,,\qquad \text{say}\,,\end{aligned}$$

where $G(\lambda)=F(\lambda)-F(0)-v_0\lambda$.

(ii) As the second stage of the proof, we shall show that

$$\begin{aligned}(4.12)\qquad \lim_{T\to\infty} TL_{mn3}^{(2)}(T)&=v_0\left|\int_{-\infty}^{\infty} a(t)dt\right|^2, &&\text{for}\quad m=n\,,\\&=0\,, &&\text{for}\quad m\neq n\,.\end{aligned}$$

Since

$$\begin{aligned}&\left|\frac{1}{T^2}\int_{-T/2}^{T/2} ds\int_{-T/2}^{T/2} e^{-2\pi i(ms-nt)/T}dt\int_{|\lambda|>T} a(s-\lambda)\overline{a(t-\lambda)}d\lambda\right|\\&\leqq\frac{1}{T^2}\int_{|\lambda|>T} d\lambda\int_{-T/2}^{T/2}|a(s-\lambda)|\,ds\int_{-\infty}^{\infty}|a(u)|\,du\\&\leqq\frac{1}{T^2}\int_{-\infty}^{\infty}|a(u)|\,du\cdot\int_{-T/2}^{T/2} ds\int_{|\lambda|>T/2}|a(v)|\,dv\\&=o(T^{-1})\,,\end{aligned}$$

we have

$$(4.13)\qquad L_{mn3}^{(2)}(T)=o(T^{-1})+\frac{v_0}{T^2}\int_{-T/2}^{T/2} ds\int_{-T/2}^{T/2} e^{-2\pi i(ms-nt)}\phi(t-s)dt\,,$$

where

(4.14) $$\phi(u)=\int_{-\infty}^{\infty} a(v)\overline{a(u+v)}dv\,.$$

$\phi(u)$ is of $L^1(-\infty, \infty)$. Thus we may write

(4.15) $$L_{mn3}^{(2)}(T)=o(T^{-1})+\frac{v_0}{T^2}\int_{-T/2}^{T/2} e^{-2\pi i(m-n)s/T}ds\int_{-T/2-s}^{T/2-s} e^{2n\pi iu/T}\phi(u)du$$
$$=o(T^{-1})+\frac{v_0}{T^2}\left[\int_{-T}^{0}\phi(u)e^{2n\pi iu/T}du\int_{-T/2-u}^{T/2} e^{-2\pi i(m-n)s/T}ds\right.$$
$$\left.+\int_0^T\phi(u)e^{2n\pi iu/T}du\int_{-T/2}^{T/2-u} e^{-2\pi i(m-n)s/T}ds\right].$$

If $m\neq n$, then, $\phi(u)=\overline{\phi(-u)}$ being kept in mind, the last expression is easily seen to be

$$o(T^{-1})+\frac{v_0}{T^2}\int_0^T \mathrm{Re}\,[\phi(u)e^{-\pi i(m+n)u/T}]\frac{\sin\pi(m-n)(1-u/T)}{\pi(m-n)/T}du$$

and hence

$$|L_{mn3}^{(2)}(T)|\leqq o(T^{-1})+\frac{v_0}{T}\int_0^T|\phi(u)|\frac{|\sin\pi(m-n)u/T|}{\pi|m-n|}du\,.$$

Take A arbitrarily large and fix it. For $T>A$, we write

$$|L_{mn3}^{(2)}(T)|\leqq o(T^{-1})+\frac{v_0}{\pi T}\int_0^A|\phi(u)|\left|\frac{\sin\pi(m-n)u/T}{m-n}\right|du+\frac{v_0}{\pi T}\int_A^T|\phi(u)|du\frac{1}{|m-n|}$$

and see that

$$\limsup_{T\to\infty} T|L_{mn3}^{(2)}(T)|\leqq o(1)+\frac{v_0}{\pi|m-n|}\int_A^\infty|\phi(u)|du\,.$$

Since A is arbitrarily large, this shows that

(4.16) $$L_{mn3}^{(2)}(T)=o(T^{-1})\,.$$

Now we suppose $m=n$. We have, from (4.15),

(4.17) $$L_{mn3}^{(2)}(T)=o(T^{-1})+\frac{v_0}{T^2}\left[\int_{-T}^{0}\phi(u)e^{2m\pi iu/T}(T+u)du+\int_0^T\phi(u)e^{2m\pi iu/T}(T-u)du\right]$$
$$=o(T^{-1})+\frac{v_0}{T^2}\int_0^T du\int_{-u}^{u}\phi(v)e^{2m\pi iv/T}dv\,.$$

Let ε be any positive small number and write the last integral as

$$\frac{v_0}{T^2}\int_0^T du\left[\int_{u\geqq|v|\geqq\varepsilon u}+\int_{|v|<\varepsilon u}\right]\phi(v)e^{2m\pi iv/T}dv\,.$$

Then

$$\left|\frac{1}{T^2}\int_0^T du\int_{u\geqq|v|\geqq\varepsilon u}\phi(v)e^{2m\pi iv/T}dv\right|\leqq\frac{1}{T^2}\int_0^T du\int_{|v|\geqq\varepsilon u}|\phi(v)|dv=o(T^{-1}),$$

since $\int_{|v|>\varepsilon u}|\phi(v)|dv\to 0$, as $u\to\infty$. On the other hand,

$$\frac{1}{T^2}\int_0^T du\int_{|v|<\varepsilon u}\phi(v)e^{2m\pi iv/T}dv=\frac{1}{T^2}\int_0^T du\int_{|v|<\varepsilon u}\phi(u)(e^{2m\pi iv/T}-1)dv$$
$$+\frac{1}{T^2}\int_0^T du\int_{|v|<\varepsilon u}\phi(v)dv,$$

in which the first integral of the right hand side does not exceed in absolute value

$$\frac{1}{T^2}\int_0^T du\int_{|v|<\varepsilon u}|\phi(v)|dv\cdot 2|m|\pi\varepsilon\leqq 2|m|\pi\varepsilon T^{-1}\int_{-\infty}^{\infty}|\phi(v)|dv$$

and for the second integral we see

$$\frac{1}{T}\int_0^T du\int_{|v|<\varepsilon u}\phi(v)dv\to\int_{-\infty}^{\infty}\phi(v)dv.$$

Therefore inserting above estimates into (4.17), we have

$$\left|L_{mn3}^{(2)}(T)-\frac{v_0}{T}\int_{-\infty}^{\infty}\phi(u)du\right|\leqq o(T^{-1})+2|m|\pi\varepsilon T^{-1}\int_{-\infty}^{\infty}|\phi(v)|dv.$$

Since ε is arbitrary, this gives us (4.12), in view of

$$\int_{-\infty}^{\infty}\phi(u)du=\left|\int_{-\infty}^{\infty}a(t)dt\right|^2.$$

(iii) As the final step, we shall show

$$TL_{mn3}^{(1)}(T)=o(1),\qquad\text{as}\quad T\to\infty.\tag{4.18}$$

This is the part of the proof of the theorem in which the condition (4.4) is needed. $L_{mn3}^{(1)}(T)$ can be written by

$$\frac{1}{T^2}\int_{-T}^{T}dG(\lambda)\psi_n(T,\lambda)\int_{-T/2}^{T/2}e^{-2m\pi is/T}a(s-\lambda)ds,$$

where

$$\psi_n(T,\lambda)=\int_{-T/2}^{T/2}e^{2n\pi it/T}\overline{a(t-\lambda)}dt.$$

Thus

$$L_{mn3}^{(1)}(T)=\frac{1}{T^2}\int_{-T}^{T}\psi_n(T,\lambda)e^{-2m\pi i\lambda/T}dG(\lambda)\int_{-T/2-\lambda}^{T/2-\lambda}e^{-2m\pi iu/T}a(u)du$$

which is, as in (3.11),

$$\frac{1}{T^2}\int_{3T/2}^{-T/2} a(u)e^{-2m\pi iu/T}du\int_{-T/2-u}^{T} e^{-2m\pi i\lambda/T}\psi_n(T,\lambda)dG(\lambda)$$
$$+\frac{1}{T^2}\int_{T/2}^{3T/2}\cdot du\int_{-T}^{T/2-u}\cdot dG(\lambda)+\frac{1}{T^2}\int_{-T/2}^{T/2}\cdot du\int_{-T/2-u}^{T/2-u}\cdot dG(\lambda)$$
$$=M_{mn}^{(1)}(T)+M_{mn}^{(2)}(T)+M_{mn}^{(3)}(T),\qquad \text{say}.$$

Now

(4.19)
$$M_{mn}^{(1)}(T)=\frac{1}{T^2}\int_{-3T/2}^{-T/2} a(u)e^{-2m\pi iu/T}du\int_{-T/2-u}^{T} e^{-3m\pi i\lambda/T}dG(\lambda)\int_{-T/2}^{T/2} e^{2n\pi i(\lambda+v)/T}\overline{a(v)}dv$$
$$=\frac{1}{T^2}\int_{-3T/2}^{-T/2} a(u)e^{-2m\pi iu/T}du\left[\int_{-3T/2}^{-u} e^{2n\pi iv/T}\overline{a(v)}dv\int_{-T/2-v}^{T} e^{-2(m-n)\pi i\lambda/T}dG(\lambda)\right.$$
$$\left.+\int_{u}^{-T/2}\cdot dv\int_{-T/2}^{T}\cdot dG(\lambda)+\int_{-T/2}^{T+u}dv\int_{-T/2-u}^{T/2-v}dG(\lambda)\right].$$

The first integral in the bracket in (4.19) with $G(\lambda)-G(-T/2-v)$ in place of $G(\lambda)$ without changing the value of inner integral, turns out, after integration by parts, to be

$$\int_{-3T/2}^{-u} e^{2n\pi iv/T}\overline{a(v)}dv\Big\{[G(T)-G(-T/2-v)]e^{-2(m-n)\pi i}$$
$$+2(m-n)\pi i/T\int_{-T/2-v}^{T}[G(\lambda)-G(-T/2-v)]e^{-2(m-n)\pi i\lambda/T}d\lambda\Big\}$$

which is, in view of $G(\lambda)-G(-T/2-v)=O(T)$, not greater than in absolute value

$$\int_{-3T/2}^{3T/2}|a(v)|\,dv\cdot O(T)=O(T).$$

The second integral in the bracket in (4.19) is seen, in the same way, to be, in absolute value,

$$\leqq\int_{-3T/2}^{3T/2}|a(v)|\,dv\cdot O(T)=o(T).$$

The last double integral in (4.19) is in absolute value

$$\leqq\int_{-T/2}^{T+u}|a(v)|\,dv\cdot O(T)=O(T).$$

Thus we have

$$M_{mn}^{(1)}(T)=O\left(\frac{1}{T^2}\int_{-3T/2}^{-T/2}a(u)du\right)O(T)=o(T^{-1}). \tag{4.20}$$

In the quite same way, we also get

$$M_{mn}^{(2)}(T)=o(T^{-1})\,. \tag{4.21}$$

Finally we deal with

$$M_{mn}^{(3)}(T)=\frac{1}{T^2}\int_{-T/2}^{T/2} a(u)e^{-2m\pi iu/T}du\int_{-T/2-u}^{T/2-u} e^{-2(m-n)\pi i\lambda/T}dG(\lambda)\int_{-T/2-\lambda}^{T/2-\lambda} e^{2n\pi iv/T}\overline{a(v)}dv\,.$$

This is written, by the interchange of the order of integrations, as

$$\frac{1}{T^2}\int_{-T/2}^{T/2} a(u)e^{-2m\pi iu/T}du\left[\int_{-T+u}^{u} e^{2n\pi iv/T}\overline{a(v)}dv\int_{-T/2-v}^{T/2-u} e^{2(m-n)\pi i\lambda/T}dG(\lambda)\right.$$
$$\left.+\int_{u}^{T+u}\cdot dv\int_{-T/2-u}^{T/2-v}\cdot dG(\lambda)\right]. \tag{4.22}$$

The first integral in the bracket can be written, as before, by

$$\int_{-T+u}^{u} e^{2n\pi iv/T}\overline{a(v)}dv\left\{[G(T/2-u)-G(T/2-v)]e^{-(m-n)\pi i(1-2u/T)}\right.$$
$$\left.+2\pi(m-n)/T\int_{-T/2-v}^{T/2-u}[G(\lambda)-G(-T/2-v)]e^{-2(m-n)\pi i\lambda/T}d\lambda\right\}.$$

Hence (4.22) does not exceed in absolute value

$$\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|\,du\int_{-T-u}^{u}|a(v)|\,|G(T/2-u)-G(-T/2-v)|\,dv$$
$$+\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|\,du\int_{-T+u}^{u}|a(v)|\,dv\int_{-T/2-v}^{T/2-u}|G(\lambda)-G(-T/2-v)|\,d\lambda$$
$$=N_1(T)+N_2(T)\,,\qquad \text{say}.$$

Note that by the assumption (4.4)

$$G(T/2)-G(-T/2-v)=F(T/2-u)-F(-T/2-v)-v_0(T-u+v)$$
$$=o(1)\,,$$

if $T-u+v$ is large. Hence for any small $\varepsilon>0$,

$$N_1(T)\leqq\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|\,du\left[\int_{|v|\leqq 3T/2,|T-u+v|>\varepsilon T}|a(v)|\right.$$
$$\left.\times|G(T/2-u)-G(-T/2-v)|\,dv+\int_{|v|\leqq 3T/2,|T-u+v|\leqq\varepsilon T}\cdot dv\right]$$
$$\leqq\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|\,du\left[\int_{-3T/2}^{3T/2}|a(v)|\,dv\cdot o(1)+\int_{|T-u+v|\leqq\varepsilon T}|a(v)|\,dv\cdot O(T)\right] \tag{4.23}$$

as $T\to\infty$, ε being fixed, and $G(T/2-u)-G(-T/2-u)$ being $O(T)$ uniformly for u. The above is

$$=o(T^{-2})+O(T^{-2})\int_{-3T/2}^{-T/2}|a(w+T)|dw\int_{w-\varepsilon T}^{w+\varepsilon T}|a(v)|dv$$

$$=o(T^{-2})+O(T^{-1})\int_{-3T/2}^{-T/2}|a(w+T)|dw\int_{-3T/2-\varepsilon T}^{-T/2+\varepsilon T}|a(v)|dv$$

$$=o(T^{-2})+O(T^{-1})\int_{-\infty}^{\infty}|a(u)|du\cdot o(1)=o(T^{-1}),$$

that is,

(4.24) $$N_1(T)=o(T^{-1}).$$

Finally

$$|N_2(T)|\leqq\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|du\int_{-T+u}^{u}|a(v)|dv$$

$$\times\left[\int_{|\lambda+T/2-v|>\varepsilon T,\,-T/2-v<\lambda<T/2-u}|G(\lambda)-G(-T/2-v)|d\lambda\right.$$

$$\left.+\int_{|\lambda+T/2-v|\leqq\varepsilon T}\cdot d\lambda\right]$$

which is, as in (4.23),

$$\leqq\frac{1}{T^2}\int_{-T/2}^{T/2}|a(u)|du\int_{-T+w}^{u}|a(v)|dv\left[o(1)\int_{-T/2-v}^{T/2-u}d\lambda+\int_{-T/2-v-\varepsilon T}^{-T/2-v+\varepsilon T}d\lambda\right]$$

$$=o(T^{-1})\left(\int_{-\infty}^{\infty}|a(u)|du\right)^2+O(T^{-2})\int_{-\infty}^{\infty}|a(u)|^2du\cdot\varepsilon T$$

$$=o(T^{-1})+O(T^{-1})\cdot\varepsilon,$$

which gives us

$$TN_2(T)\to 0,\qquad T\to\infty.$$

This, with (4.24), shows

$$M_{mn}^{(3)}(T)=o(T^{-1}).$$

Altogether we have shown (4.18). The proof of Theorem 3 (ii) is now complete.

It is easily seen that for the estimate $L_{mn3}^{(1)}(T)=o(1)$, the condition (4.2) is sufficient. We have seen that for $\limsup_{\alpha\to-\infty} L_{mn1}(T,\alpha)$, $\limsup_{\beta\to\infty} L_{mn2}(T,\beta)$ and $L_{mn3}^{(2)}(T)$ to be $o(1)$, the condition $\Phi(\lambda)=O(\lambda)$, $|\lambda|\to\infty$ is sufficient. These observations give us the proof of (i) of the theorem.

5. Approximate Fourier series of a GLP

For a WSP represented by

(5.1) $$X(t, \omega)=\int_{-\infty}^{\infty} e^{it\lambda}\xi(d\lambda, \omega)$$

$\xi(S, \omega)$ being the spectral random measure, the series defined by

(5.2) $$\sum_{n=-\infty}^{\infty} e^{2n\pi it/T}\xi_n(\omega),$$

where $T>0$ and

(5.3) $$\xi_n(\omega)=\xi_n(T, \omega)=\int_{2n\pi/T}^{2(n+1)\pi/T} \xi(d\lambda, \omega),$$

plays a particular role in the theory of approximation by a T-periodic Fourier series. See [15] and [10, II]. For the study of the sample continuity of a WSP, the series (5.2) provided a useful tool, see [10, II]. (5.2) is called the approximate Fourier series of $X(t, \omega)$. Actually it is the Fourier series of some T-periodic WSP $\tilde{X}(t, \omega)$ which approximates $X(t, \omega)$ in $|t|\leqq A$, when T is much larger than A.

This situation is true even when we deal with a GLP, which is the aim of this section to study about.

Throughout this section we consider a GLP with mean 0, that is a GLP $X(t, \omega)$ defined by (1.2) with

(5.4) $$m(S)=0.$$

Again the quantity

(5.5) $$G(\lambda)=F(\lambda)-F(0)-v_0\lambda$$

plays a good deal.

It will be convenient to set

Condition A. Let $a(t)$ be the Fourier transform of a function $b(u)$:

$$a(t)=\hat{b}(t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty} b(u)e^{itu}du,$$

which $b(u)$ is supposed to be absolutely continuous and of $L^1(-\infty, \infty)$.

Lemma 2. *Condition A is assumed. Suppose either of the following conditions is satisfied.*

(i) *For some constant v_0 and for some $0\leqq p\leqq 1$,*

(5.6) $$\int_{-\infty}^{\infty} \frac{|dG(\lambda)|}{1+|\lambda|^p}<\infty,$$

(ii) *For some $1<p<2$,*

(5.7) $$\int_{-\infty}^{\infty} \frac{dF(\lambda)}{1+|\lambda|^p}<\infty.$$

Then

$$A_n(T,\omega)=\underset{\alpha\to-\infty,\beta\to\infty}{\text{l.i.m.}}(2\pi)^{-1/2}\int_{2n\pi/T}^{2(n+1)\pi/T}b(u)du\int_\alpha^\beta e^{-i\lambda u}\xi(d\lambda,\omega) \tag{5.8}$$

exists, where l.i.m. *means the limit in* $L^2(\Omega)$.

Note that if $1<p<2$, then (5.6) is equivalent to (5.7) and that if (2.3) is assumed (5.7) is always true. We remark also that if $F(\lambda)$ is absolutely continuous and $F'(\lambda)=f(\lambda)$, then the condition (5.6) is obviously equivalent to

$$\int_{-\infty}^{\infty}\frac{|f(\lambda)-v_0|}{1+|\lambda|^p}d\lambda<\infty\,. \tag{5.9}$$

We call, for each $T>0$, the formal series

$$\sum_{n=-\infty}^{\infty}A_n(T,\omega)e^{2n\pi it/T}\,, \tag{5.10}$$

the approximate Fourier series of a GLP $X(t,\omega)$.

Before proving Lemma 2, we give one more lemma

Lemma 3. *Write*

$$S=\int_a^b f(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda\,. \tag{5.11}$$

Under the condition that $f(u)$ is absolutely continuous around the origin, we have

$$\begin{aligned}\lim_{\alpha\to-\infty,\beta\to\infty}S&=2\pi f(0)\,, &&\text{if } a<0<b\,,\\ &=\pi f(0)\,, &&\text{if } a=0 \text{ or } b=0\,,\\ &=0\,, &&\text{otherwise}\end{aligned} \tag{5.12}$$

and

$$\lim_{\beta\to\infty,\alpha\to\infty}S=\lim_{\alpha\to-\infty,\beta\to-\infty}S=0\,. \tag{5.13}$$

Proof.

$$S=\int_a^b f(u)\frac{e^{-iu\beta}-e^{-iu\alpha}}{-iu}du\,.$$

Let $a<0<b$. Choose ε such that $a<-\varepsilon<0<\varepsilon<b$. Write

$$S=\int_{|u|<\varepsilon}+\int_{(a,-\varepsilon)\cup(\varepsilon,b)} \tag{5.14}$$

The second integral converges to zero as $|\alpha|\to\infty$, $|\beta|\to\infty$ by Riemann-Lebesgue lemma.

The first integral in (5.13) can be written by

$$f(0)\int_{|u|<\varepsilon}\frac{e^{-iu\beta}-e^{-iu\alpha}}{-iu}du+\int_{|u|<\varepsilon}\frac{f(u)-f(0)}{-iu}(e^{-iu\beta}-e^{-iu\alpha})du$$

in which the second integral converges to zero as $|\alpha|\to\infty$, $|\beta|\to\infty$ again by Riemann-Lebesgue lemma, since $f(u)$ is absolutely continuous near the origin, while the first integral is

$$f(0)\int_{|u|<\varepsilon}du\int_\alpha^\beta e^{-iu\lambda}d\lambda=f(0)\int_\alpha^\beta d\lambda\int_{-\varepsilon}^{\varepsilon}e^{-iu\lambda}du$$
$$=2f(0)\int_\alpha^\beta\frac{\sin\varepsilon\lambda}{\lambda}d\lambda$$

which converges to $2\pi f(0)$ as $\alpha\to-\infty$, $\beta\to\infty$ and to zero as $\alpha\to-\infty$, $\beta\to-\infty$ or $\beta\to\infty$, $\alpha\to\infty$. This shows the first and third cases of (5.12) and also (5.13). The second case of (5.12) is shown in a similar way.

We now prove Lemma 2. We have to show

$$E\left|\int_a^b b(u)du\int_\alpha^\beta e^{-i\lambda u}\xi(d\lambda,\omega)\right|^2\to 0 \tag{5.15}$$

with $a=2n\pi/T$, $b=2(n+1)\pi/T$, as $\beta\to\infty$, $\alpha\to\infty$, and as $\alpha\to-\infty$, $\beta\to-\infty$.

We first note that under the conditions in Lemma 2

$$\left|\int_a^b b(u)e^{-iu\lambda}du\right|^2\leqq C|\lambda|^{-p}\left(\int_a^b|b(u)|du\right)^{2-p},\quad 0\leqq p<2, \tag{5.16}$$

for large $|\lambda|$ and for $b-a\leqq 1$, C being a constant independent of λ. When $p=0$ this is trivial and it is easily seen even for $0<p<2$ from

$$\left|\int_a^b b(u)e^{-iu\lambda}du\right|\leqq\int_a^b|b(u)|du,$$

and

$$\left|\int_a^b b(u)e^{-iu\lambda}du\right|=\left|\frac{-1}{i\lambda}e^{-iu\lambda}b(u)\Big|_{u=a}^{b}+\frac{1}{i\lambda}\int_a^b b'(u)e^{-iu\lambda}du\right|$$
$$\leqq\frac{C}{|\lambda|}+\frac{1}{\lambda}\int_a^b|b'(u)|du\leqq\frac{C}{|\lambda|},$$

where C is constants independent of λ which may be different on each occurrence. The above two estimates give us (5.16).

Now we shall prove (5.15). We may suppose $T>2\pi$.

$$J=E\left|\int_a^b b(u)du\int_\alpha^\beta e^{-i\lambda u}\xi(d\lambda,\omega)\right|^2$$
$$=E\left[\int_a^b b(u)du\int_\alpha^\beta e^{-i\lambda u}\xi(d\lambda,\omega)\int_a^b\overline{b(v)}dv\int_\alpha^\beta e^{i\mu v}\overline{\xi(d\mu,\omega)}\right]$$

$$=\int_\alpha^\beta dF(\lambda)\int_a^b b(u)e^{-i\lambda u}du\int_a^b \overline{b(v)}e^{i\lambda v}dv$$

$$=\int_\alpha^\beta dF(\lambda)\left|\int_a^b b(u)e^{-i\lambda u}du\right|^2$$

$$=\int_a^b dG(\lambda)\left|\int_a^b b(u)e^{-i\lambda u}du\right|^2+v_0\int_\alpha^\beta d\lambda\left|\int_a^b b(u)e^{-i\lambda u}du\right|^2$$

$$=J_1+J_2, \qquad \text{say}, \tag{5.17}$$

where $G(\lambda)=F(\lambda)-F(0)-v_0\lambda$ as before. From (5.16) we have

$$|J_1|\leqq C\int_\alpha^\beta \frac{|dG(\lambda)|}{1+|\lambda|^p}\left(\int_a^b |b(u)|\,du\right)^{2-p} \tag{5.18}$$

which converges to zero as $\beta, \alpha\to\infty$ or $\alpha, \beta\to-\infty$.

$$J_2=v_0\int_a^b b(u)du\int_a^b \overline{b(v)}dv\int_\alpha^\beta e^{-i\lambda(u-v)}d\lambda \tag{5.19}$$

$$=v_0\left|\int_a^b b(u)du\int_\alpha^\beta e^{-i\lambda u}du\right|^2$$

which converges to zero as $\beta, \alpha\to\infty$ or $\alpha, \beta\to-\infty$, because of (5.13).

We shall prove the following

Theorem 4. *Assume Condition A.*

(i) *Let* $0\leqq p\leqq 1$. *If* (5.6) *is satisfied, then there is a stochastic process* $\tilde{X}_T(t, \omega)$ *such that*

$$\int_I E\left|\sum_{n=-N}^{N-1} A_n(T, \omega)e^{2n\pi it/T}-\tilde{X}_T(t, \omega)\right|^2 dt\to 0, \tag{5.20}$$

as $N\to\infty$, *for every finite interval I.*

(ii) *Let* $1<p<2$. *Suppose* (5.7) *and moreover for large u*

$$\int_{2u>|v|>u} |b(v)|\,dv=O(u^{-r}) \tag{5.21}$$

for some $r>p(2-p)^{-1}$. *Then there is a stochastic process* $\tilde{X}_T(t, \omega)$ *such that* (5.20) *holds as* $N\to\infty$, *for every finite interval I.*

Proof. It is sufficient to prove (5.20) for $I=(-T/2, T/2)$, because of the periodicity of the series involved in (5.20). Namely we have only to show

$$Q=\frac{1}{T}\int_{-T/2}^{T/2} E\left|\sum_{n=M}^{N} e^{2n\pi it/T}A_n(T, \omega)\right|^2 dt\to 0, \tag{5.22}$$

as $N\to\infty$, $M\to\infty$ or $M\to-\infty$, $N\to-\infty$.

Using Parseval relation, we have

$$
\begin{aligned}
Q &= E\frac{1}{T}\int_{-T/2}^{T/2}\left|\sum_{n=M}^{N} e^{2n\pi it/T}A_n(T,\omega)\right|^2 dt \\
&= E\sum_{n=M}^{N}|A_n(T,\omega)|^2 = \sum_{n=M}^{N} E|A_n(T,\omega)|^2 \\
&= \frac{1}{2\pi}\sum_{n=M}^{N}\lim_{\alpha\to-\infty,\beta\to\infty}(J_1+J_2),
\end{aligned}
$$

where J_1 and J_2 are those in (5.17) with $a=2n\pi/T$, $b=(2n+1)\pi/T$. From (5.18), we have

$$
\lim_{\alpha\to-\infty,\beta\to\infty}|J_1|\leqq C\int_{-\infty}^{\infty}\frac{|dG(\lambda)|}{1+|\lambda|^p}\left(\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p}
$$

and hence

$$
\sum_{n=M}^{N}\lim_{\alpha\to-\infty,\beta\to\infty}|J_1|\leqq C\sum_{n=M}^{N}\left(\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p}. \tag{5.23}
$$

Consider the case (i). Since $2-p\geqq 1$, the series on the right hand side of (5.23) is, by Jensen inequality,

$$
\leqq\left(\sum_{n=M}^{N}\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p}=\left(\int_{2M\pi/T}^{2(N+1)\pi/T}|b(u)|\,du\right)^{2-p} \tag{5.24}
$$

which converges to zero, as $N, M\to\infty$ or $M, N\to-\infty$.

As we have shown that J_2 converges to zero in (5.19) for each $N>M>0$ or $M<N<0$, we now complete the proof of Q going to zero as $M, N\to-\infty$ or $M, N\to\infty$.

Now we handle the case (ii).

$$
\begin{aligned}
\sum_{n=M}^{N}\left(\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p} &= \sum_{k=K}^{L}\sum_{n=2^k+1}^{2^{k+1}}(\cdot)^{2-p} \\
&= \sum_{k=K}^{L}\left(\sum_{n=2^k+1}^{2^{k+1}}1\right)^{p-1}\left(\sum_{n=2^k+1}^{2^{k+1}}\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p} \\
&= \sum_{k=K}^{L}2^{k(p-1)}\left(\int_{2(2^k+1)\pi/T}^{2(2^{k+1}+1)\pi/T}|b(u)|\,du\right)^{2-p}.
\end{aligned}
$$

By (5.21) the last one is

$$
\leqq CT^{(2-p)r}\sum_{k=K}^{L}2^{k(p-1-r(2-p))}
$$

which converges to zero as $L, K\to\infty$ or $-\infty$, because $p-1-r(2-p)<-1$. The proof is now complete.

6. Approximation by approximate Fourier series

Let $X(t, \omega)$ be a GLP with $EX(t, \omega)=0$ for all t. We here give a result on approximation of $X(t, \omega)$ by $\tilde{X}(t, \omega)$.

Theorem 5. *Assume Condition A.*

(i) *If* (2.3) *holds and*

$$ub'(u)\in L^1(-\infty, \infty)\,, \tag{6.1}$$

then, for every $1<p<2$, *for every finite interval* $I=(A, B)$ *and for* $T>2|A|, 2|B|$,

$$\int_I E|X(t, \omega)-\tilde{X}_T(t, \omega)|^2 dt \leqq CT^{-(2-p)}\,. \tag{6.2}$$

(ii) *Suppose* (6.1) *and* (5.6) *for some constant* v_0 *and for some* $0\leqq p\leqq 1$. *Then for every finite interval* $I=(A, B)$ *and for* $T>2|A|, 2|B|$, (6.2) *holds for this* p.

In both cases, C is a constant independent of T and depends on I in such a way that $C=C_0\int_I(1+t^2)dt$, where C_0 is a constant independent of T and of I as well.

Proof. Denote the integral in (6.2) by $R=R(T)$.

$$\begin{aligned}
R&=\int_I \lim_{N\to\infty} E\left|X(t, \omega)-\sum_{n=-N}^{N-1} A_n(T, \omega)e^{2n\pi it/T}\right|^2 dt\\
&\leqq \liminf_{N\to\infty}\int_I E\left|X(t, \omega)-\sum_{n=-N}^{N-1} A_n(T, \omega)e^{2n\pi it/T}\right|^2 dt\\
&\leqq \liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}\int_I E\left|\int_\alpha^\beta a(t-\lambda)\xi(d\lambda, \omega)-\sum_{n=-N}^{N-1} A_n(T, \omega)e^{2n\pi it/T}\right|^2 dt\\
&\leqq 3\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}\int_I E\left|\int_\alpha^\beta a(t-\lambda)\xi(d\lambda, \omega)\right.\\
&\qquad\left.-\sum_{n=-N}^{N-1} e^{2n\pi it/T}(2\pi)^{-1/2}\int_{2n\pi/T}^{2(n+1)\pi/T} b(u)du\int_\alpha^\beta e^{i\lambda u}\xi(d\lambda, \omega)\right|^2 dt\\
&\qquad+3\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}\int_I\lim_{\beta'\to\infty} E\left|\sum_{n=-N}^{N-1} e^{2n\pi it/T}\right.\\
&\qquad\left.\times(2\pi)^{-1}\int_{2n\pi/T}^{2(n+1)\pi/T} b(u)du\int_\beta^{\beta'} e^{i\lambda u}\xi(d\lambda, \omega)\right|^2 dt\\
&\qquad+3\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}\int_I\lim_{\alpha'\to-\infty} E\left|\sum_{n=-N}^{N-1} e^{2n\pi it/T}\right.\\
&\qquad\left.\times(2\pi)^{-1/2}\int_{2n\pi/T}^{2(n+1)\pi/T} b(u)du\int_{\alpha'}^{\alpha} e^{i\lambda u}\xi(d\lambda, \omega)\right|^2 dt\\
&=3\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}(R_1+R_2+R_3)\,,\qquad\text{say}.
\end{aligned}$$

Let T be so large that

$$R_2 \leqq \frac{C}{T}\int_{-T/2}^{T/2} \lim_{\beta' \to +\infty} E|\cdot|^2 dt$$

for some constant C which is independent of T but depends on the interval I. Obviously this is possible, since the series involved in the last inequality (inside of E) is periodic with period T. Then Parseval relation shows that

$$\begin{aligned} R_2 &\leqq C \liminf_{\beta' \to \infty} \frac{1}{2\pi} \sum_{n=-N}^{N-1} E\left| \int_{2n\pi/T}^{2(n+1)\pi/T} b(u)du \int_{\beta}^{\beta'} e^{i\lambda u}\xi(d\lambda, \omega)\right|^2 \\ &\leqq C \liminf_{\beta' \to \infty} \sum_{n=-N}^{N-1} (J_1 + J_2), \end{aligned}$$

where J_1 and J_2 are those in (5.17) with $a = 2n\pi/T$, $b = 2(n+1)\pi/T$ and β, β' in place of α, β there respectively.

Therefore because of (5.18) with $p = 1$ and (5.19), J_1 and J_2 converge to zero for each n, when $\beta' \to \infty$, $\beta \to \infty$. We thus have

$$\lim_{\beta \to \infty} R_2 = 0 .$$

Similarly we have

$$\lim_{\alpha \to -\infty} R_3 = 0 .$$

Hence we have

$$R \leqq 3 \liminf_{N \to \infty} \lim_{\alpha \to -\infty, \beta \to \infty} R_1 . \tag{6.3}$$

Inserting $a(t-\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} b(u) e^{i(t-\lambda)u} du$ into R_1

$$\begin{aligned} |R| &\leqq C \liminf_{N \to \infty} \lim_{\alpha \to -\infty, \beta \to \infty} \int_I E \left| \int_{-\infty}^{\infty} b(u) e^{itu} du \int_{\alpha}^{\beta} e^{-i\lambda u} \xi(d\lambda, \omega) \right. \\ &\qquad \left. - \sum_{n=-N}^{N-1} e^{2n\pi i t/T} \int_{2n\pi/T}^{2(n+1)\pi/T} b(u)du \int_{\alpha}^{\beta} e^{-i\lambda u}\xi(d\lambda, \omega) \right|^2 dt \\ &\leqq C \liminf_{N \to \infty} \lim_{\alpha \to -\infty, \beta \to \infty} \int_I E \left| \sum_{n=N}^{N-1} \int_{2n\pi/T}^{2(n+1)\pi/T} (e^{itu} - e^{2n\pi i t/T}) b(u)du \int_{\alpha}^{\beta} e^{-i\lambda u}\xi(d\lambda, \omega)\right|^2 dt \\ &\quad + C \liminf_{N \to \infty} \lim_{\alpha \to -\infty, \beta \to \infty} \int_I E \left| \int_{|u| > 2N\pi/T} e^{itu} b(u)du \int_{\alpha}^{\beta} e^{i\lambda u}\xi(d\lambda, \omega)\right|^2 dt \\ &= C \liminf_{N \to \infty} \lim_{\alpha \to -\infty, \beta \to \infty} (U_1 + U_2), \qquad \text{say}. \end{aligned}$$

Write

$$a_N(v) = (2\pi)^{-1/2} \int_{|u| > 2N\pi/T} e^{ivu} b(u)du .$$

Then

$$U_2=\int_I E\left|\int_\alpha^\beta a_N(t-\lambda)\xi(d\lambda,\omega)\right|^2 dt$$
$$=\int_A^B dt\int_\alpha^\beta |a_N(t-\lambda)|^2 dF(\lambda)$$
$$=\int_\alpha^\beta dF(\lambda)\int_{A-\lambda}^{B-\lambda}|a_N(u)|^2 du\,.$$

By the same reasoning as in the calculation of (3.9),

$$U_2=\int_{A-\beta}^{B-\beta}\Big|a_N(u)|^2du\int_{A-u}^{\beta}dF(\lambda)$$
$$+\int_{B-\beta}^{A-\alpha}|a_N(u)|^2du\int_{A-u}^{B-u}dF(\lambda)+\int_{A-\alpha}^{B-\beta}|a_N(u)\Big|^2du\int_\alpha^{B-u}dF(\lambda)\,.$$

Using the condition (2.3), we see, noting $0\leqq\beta-A+u\leqq B-A$, as before, that the first integral is

$$\leqq C\int_{A-\beta}^{B-\beta}|a_N(u)|^2du\rightarrow 0$$

as $\beta\rightarrow\infty$, since $a_N(u)\in L^2(-\infty,\infty)$ because of $b(u)\in L^1\cap L^\infty$. The third integral is shown, in the same way, to be $o(1)$. The second integral is

$$\leqq C\int_{A-\beta}^{B-\alpha}|a_N(u)|^2du\leqq C\int_{-\infty}^{\infty}|a_N(u)|^2du$$
$$=C\int_{|u|\geqq 2N\pi/T}|b(u)|^2du\rightarrow 0\qquad\text{as } N\rightarrow\infty.$$

Hence we have

$$R\leqq C\liminf_{N\rightarrow\infty}\ \lim_{\alpha\rightarrow-\infty,\beta\rightarrow\infty}U_1\,. \tag{6.4}$$

Write

$$g_N(u,t)=g_N(u,t;T)=e^{itu}-e^{2n\pi it/T}\,,\qquad \text{for}\quad 2n\pi/T\leqq u<2(n+1)\pi/T\,,\ n=-N,\cdots,N-1\,,$$
$$=0\,,\qquad\text{for}\quad |u|>2N\pi/T\,.$$

Then

$$U_1=\int_I dtE\left|\int_{-\infty}^{\infty}g_N(u,t)b(u)du\int_\alpha^\beta e^{-i\lambda u}\xi(d\lambda,\omega)\right|^2 \tag{6.5}$$
$$=\int_I dt\int_\alpha^\beta dF(\lambda)\left|\int_{-\infty}^{\infty}g_N(t,u)b(u)e^{-iu\lambda}du\right|^2$$

$$= \int_I dt \int_\alpha^\beta dG(\lambda) P_N(t, \lambda) + v_0 \int_I dt \int_\alpha^\beta P_N(t, \lambda) d\lambda$$
$$= U_{1,1} + U_{1,2}, \qquad \text{say},$$

where

(6.6) $$P_N(t, \lambda) = \left| \int_{-\infty}^{\infty} g_N(t, u) b(u) e^{-iu\lambda} du \right|^2.$$

Noting that

$$|g_N(t, u)| \leqq 2\pi |t|/T,$$

we see

(6.7) $$\left| \int_{-\infty}^{\infty} g_N(t, u) b(u) e^{-iu\lambda} \right| \leqq \frac{2\pi |t|}{T} \int_{-\infty}^{\infty} |b(u)| du$$
$$= C|t|/T.$$

We also see

$$\int_{-\infty}^{\infty} g_N(t, u) b(u) e^{-iu\lambda} du = \sum_{n=-N}^{N-1} \int_{2n\pi/T}^{2(n+1)\pi/T} (e^{itu} - e^{2n\pi it/T}) b(u) e^{-iu\lambda} du$$
$$= \left| \sum_{n=-N}^{N-1} (-i\lambda)^{-1} (e^{itu} - e^{2n\pi it/T}) b(u) \right|_{2n\pi/T}^{2(n+1)\pi/T}$$
$$+ \sum_{n=-N}^{N-1} (t/\lambda)^{-1} \int_{2n\pi/T}^{2(n+1)\pi/T} e^{itu} b(u) e^{-iu\lambda} du$$
$$+ (i\lambda)^{-1} \sum_{n=-N}^{N-1} \int_{2n\pi/T}^{2(n+1)\pi/T} (e^{itu} - e^{2n\pi it/T}) b(u) e^{-iu\lambda} du \Bigg|$$
$$\leqq \frac{C}{|\lambda| T} \sum_{n=-N}^{N-1} |b(2(n+1)\pi/T)|$$
$$+ |t|/|\lambda| \int_{-\infty}^{\infty} |b(u)| du + C|t|/(|\lambda| T) \int_{-\infty}^{\infty} |b'(u)| du$$

in which

$$\frac{1}{T} \sum_{n=0}^{N-1} |b(2(n+1)\pi/T)| \leqq \frac{1}{T} \sum_{n=0}^{N-1} \left| \int_{2(n+1)\pi/T}^{\infty} b'(u) du \right|$$
$$\leqq \frac{1}{T} \sum_{n=0}^{N-1} \int_{2(n+1)\pi/T}^{\infty} |b'(u)| du$$
$$\leqq \sum_{n=0}^{N-1} \frac{1}{2\pi} \int_{2n\pi/T}^{2(n+1)\pi/T} dv \int_v^{\infty} |b'(u)| du = \frac{1}{2\pi} \int_0^{\infty} |ub'(u)| du$$

which is finite because of (6.1).

Handling

$$\frac{1}{T}\sum_{n=-N}^{-1}|b(2(n+1)\pi/T)|$$

in the same way, we have

$$\frac{1}{T}\sum_{n=-N}^{N-1}|b(2(n+1)\pi/T)|\leqq C.$$

Thus we get

$$\left|\int_{-\infty}^{\infty} g_N(t, u)b(u)e^{-iu\lambda}du\right|\leqq C\frac{1+|t|}{1+|\lambda|}, \tag{6.8}$$

because this is trivial for small λ. (6.7) and (6.8) together give us

$$P_N(t, \lambda)\leqq C\left(\frac{1+t^2}{1+|\lambda|}\right)^p T^{p-2}\leqq C\frac{1+|t|^{2p}}{1+|\lambda|^p}T^{p-2}$$

and hence

$$\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}|U_{1,1}|\leqq\frac{C}{T^{2-p}}\int_{-\infty}^{\infty}\frac{|dG(\lambda)|}{1+|\lambda|^p}. \tag{6.9}$$

We mention that C here is written by $C_0\int_I(1+|t|^{2p})dt$ where C_0 is independent of I as well as of T.

Now we shall deal with $U_{1,2}$.

$$\begin{aligned} U_{1,2}&=v_0\int_I dt\int_\alpha^\beta P_N(t, \lambda)d\lambda \\ &=v_0\int_I dt\left|\int_{-\infty}^{\infty} g_N(u)b(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda\right|^2 \end{aligned} \tag{6.10}$$

in which

$$\begin{aligned}\int_{-\infty}^{\infty} g_N(u)b(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda=&\left(\sum_{n=1}^{N-1}+\sum_{n=-N}^{-2}\right)\int_{2n\pi/T}^{2(n+1)\pi/T}(e^{itu}-e^{2n\pi it/T})b(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda \\ &+\int_0^{2\pi/T}(e^{itu}-1)b(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda \\ &+\int_{-2\pi/T}^{0}(e^{itu}-e^{-2\pi it/T})b(u)du\int_\alpha^\beta e^{-iu\lambda}d\lambda.\end{aligned}$$

Each term of the series on the right hand side is easily seen to go to zero as $\alpha\to-\infty$ and $\beta\to\infty$ by integrating out the last integral and using Riemann-Lebesgue lemma since the domain of u does not contain zero. The second term converges to zero by Lemma 3 in which $f(u)$ in (5.11) stands for $(e^{itu}-1)b(u)$. The third term converges as $\alpha\to-\infty$, $\beta\to\infty$, to $(1-e^{2\pi it/T})b(0)$ which is, in absolute value, not greater than $|t|/T\cdot|b(0)|\leqq C|t|/T$.

Thus we have

$$\liminf_{N\to\infty}\lim_{\alpha\to-\infty,\beta\to\infty}|U_{12}|\leqq\frac{C}{T^2}.$$

Therefore

$$R\leqq\frac{C}{T^{2-p}}\int_{-\infty}^{\infty}\frac{|dG(\lambda)|}{1+|\lambda|^p}+\frac{C}{T^2}\leqq C\frac{1}{T^{2-p}},$$

for $T>1$, where C is a constant independent of T. C is as before $C=C_0\int_I(1+|t|^{2p})dt$, C_0 being a constant independent of T as well as of I.

7. Sample continuity of a GLP

Let $X(t,\omega)$, $-\infty<t<\infty$, be a GLP with $EX(t,\omega)=0$, $-\infty<t<\infty$. All conditions in Lemma 2 are supposed to be satisfied. Consider the approximate Fourier series of $X(t,\omega)$

(7.1) $$\sum_{n=-\infty}^{\infty}A_n(T,\omega)e^{2n\pi it/T}, \qquad T>1.$$

First we mention that the inequality

(7.2) $$E|A_n(T,\omega)|^2\leqq C\left(\int_{2n\pi/T}^{2(n+1)\pi/T}|b(u)|\,du\right)^{2-p}$$

for $|n|>2$, holds, (i) if, for some $0\leqq p\leqq 1$ and for some constant v_0, (5.6) holds, or (ii) if, for some $1<p<2$, (5.7) holds.

The proof of this inequality is involved in the proof of Theorem 4 and actually it follows from (5.16), (5.17), (5.18) and Lemma 2. It is noted again that (5.7) is automatically satisfied when (2.3) is satisfied. In (7.2), C is, as we mentioned before, a constant

(7.3) $$C=C_0\int_I(1+|t|^{2p})dt,$$

where C_0 is a constant independent of T and the interval I.

We first deal with the almost sure absolute convergence of $\sum A_n(T,\omega)$.

Theorem 6. *Suppose Condition A. Let $g(x)$ be a positive even function which is nondecreasing for $x\geqq 0$ and is such that, for some $0\leqq p<2$,*

(7.4) $$\sum_{n=0}^{\infty}[g(n)]^{-1/(2-p)}<\infty.$$

(i) *If $0\leqq p\leqq 1$ and (5.6) holds and moreover*

(7.5) $$\int_{-\infty}^{\infty} |b(u)|\, g(u)du < \infty\,,$$

then

(7.6) $$\sum_{n=-\infty}^{\infty} |A_n(T,\omega)| < \infty\,,$$

almost surely for each T.

(ii) *If* $1<p<2$ *and* (5.7) *holds and moreover for* $r>p(2-p)^{-1}$,

(7.7) $$\int_{2u>|v|>u} |b(v)|\, g(v)dv = O(u^{-r})$$

for large u, then (7.6) *holds almost surely for each T.*

Proof. It is sufficient to prove

$$E\sum_{n=-\infty}^{\infty} |A_n(T,\omega)| < \infty\,.$$

Now

$$E\sum_{n=1}^{\infty} |A_n(T,\omega)| = E\sum_{n=1}^{\infty} g^{-1/2(2-p)}(2n\pi/T)g^{1/2(2-p)}(2n\pi/T)\,|A_n(T,\omega)|$$
$$\leqq \left[\sum_{n=1}^{\infty} g^{-1/(2-p)}(2n\pi/T)\right]^{1/2}\left[\sum_{n=1}^{\infty} g^{1/(2-p)}(2n\pi/T)E\,|A_n(T,\omega)|^2\right]^{1/2},$$

which is, by (7.2) and (7.4)

(7.8) $$\leqq C\left[\sum_{n=1}^{\infty} g^{-1/(2-p)}(2n\pi/T)\left(\int_{2n\pi/T}^{2(n+1)\pi/T} |b(u)|\,du\right)^{2-p}\right]^{1/2}$$
$$\leqq C\left[\sum_{n=1}^{\infty}\left(\int_{2n\pi/T}^{2(n+1)\pi/T} g(u)\,|b(u)|\,du\right)^{2-p}\right]^{1/2}.$$

The same arguments as in deriving (5.23) and (5.24) lead to the convergence of the last series in the both cases (i) and (ii). The same thing is true for

$$\sum_{n=-\infty}^{-1} |A_n(T,\omega)|\,.$$

Theorem 6 is thus proved.

Under the conditions in Theorem 6, in the both cases (i) and (ii), the approximate series (7.1) is absolutely and hence uniformly convergent for every $T>1$ almost surely. The sum of (7.1) is denoted by $X_T(t,\omega)$. It is continuous on every finite interval of t almost surely.

Now we shall prove

Theorem 7. *Assume Condition A. Let $g(x)$ be a function in Theorem 6. Under the* *conditions* (5.6) *and* (7.5) *when* $0 \leqq p \leqq 1$ *and under the conditions* (5.7) *and* (7.7) *when* $1 < p < 2$, *there is an* $\hat{X}(t, \omega)$ *which is continuous almost surely such that*

$$X(t, \omega) = \hat{X}(t, \omega)$$

almost surely for almost all t on every finite interval.

Proof. It is sufficient to prove that $X_{2^k}(t, \omega)$ converges uniformly on every finite interval I. This is readily seen. First note that the condition (7.7) implies (5.21). We then have

$$\begin{aligned}\hat{X}_{2^{k+1}}(t, \omega) - \hat{X}_{2^k}(t, \omega) = &\sum_{n=-\infty}^{\infty} e^{2n\pi i t/2^{k+1}} A_n(2^{k+1}, \omega) \\ &- \sum_{m=-\infty}^{\infty} [e^{2\pi i(2m)t/2^{k+1}} A_{2m}(2^{k+1}, \omega) \\ &+ e^{2\pi i(2m+1)t/2^{k+1}} A_{2m+1}(2^{k+1}, \omega) - e^{2m\pi i t/2^k} A_m(2^k, \omega)] .\end{aligned}$$

As we easily see, the identity

$$A_m(2^k, \omega) = A_{2m}(2^{k+1}, \omega) + A_{2m+1}(2^{k+1}, \omega)$$

holds and hence

$$\begin{aligned}|\hat{X}_{2^{k+1}}(t, \omega) - \hat{X}_{2^k}(t, \omega)| &= \left| \sum_{m=-\infty}^{\infty} (e^{2\pi i(2m+1)t/2^{k+1}} - e^{2\pi i(2m)/2^{k+1}}) A_{2m+1}(2^{k+1}, \omega) \right| \\ &\leqq \frac{2\pi |t|}{2^{k+1}} \sum_{m=-\infty}^{\infty} |A_{2m+1}(2^{k+1}, \omega)| .\end{aligned}$$

Let ε_k be a decreasing sequence of positive numbers converging to zero. Then

$$\begin{aligned}Q_k &= P\left(\max_{|t| \leqq A} |\hat{X}_{2^{k+1}}(t, \omega) - \hat{X}_{2^k}(t, \omega)| > \varepsilon_k \right) \\ &\leqq P\left(\frac{\pi A}{2^k} \sum_{p=-\infty}^{\infty} |A_p(2^{k+1}, \omega)| > \varepsilon_k \right) \\ &\leqq \frac{1}{\varepsilon_k{}^2} E \frac{\pi^2 A^2}{2^{2k}} \left[\sum_{p=-\infty}^{\infty} |A_p(2^{k+1}, \omega)| \right]^2 \\ &= \frac{\pi^2 A^2}{\varepsilon_k{}^2 2^{2k}} \sum_{p=-\infty}^{\infty} [g(p\pi/2^k)]^{-1/(2-p)} \left\{ \sum_{p=-\infty}^{\infty} [g(p\pi/2^k)]^{1/(2-p)} E |A_p(2^{k+1}, \omega)|^2 \right\}^{1/2} .\end{aligned}$$

In view of

$$\sum_{n=1}^{\infty} [g(an)]^{-1/(2-p)} \leqq a^{-1} \sum_{n=0}^{\infty} [g(n)]^{-1/(2-p)}$$

(which we have used in deriving (7.9)), we see that in the last expression, the factor outside of $\{\cdot\}^{1/2}$ is not greater than $CA^2/(\varepsilon_k{}^2 2^k)$, where C is a constant depending only on $g(x)$, while the last factor $\{\cdot\}$ is seen to be bounded in k as in the proof of Theorem 6, that is

$$Q_k \leqq C\varepsilon_k^{-2}2^{-k},$$

C being a constant independent of k.

Choose ε_k in such a way that

$$\sum \varepsilon_k < \infty, \qquad \sum \varepsilon_k^{-2}2^k < \infty.$$

We then have

$$\sum_{k=1}^{\infty} Q_k < \infty$$

which shows by Borel-Cantelli lemma that

$$\sum |\hat{X}_{2^{k+1}}(t, \omega) - \hat{X}_{2^k}(t, \omega)|$$

converges uniformly in $|t| \leqq A$. This completes the proof.

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