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PROJECTIONS, SYMMETRIES AND NILPOTENT OPERATORS ON A HILBERT SPACE

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1. Introduction

Let $B(X)$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space X, the conventional definition of a projection $T\in B(X)$ says that T is self-adjoint and idempotent. This has been extended so that we can replace "self-adjoint" by "dominant" ([5], Corollary 4). Note that the class of dominant operators [10] contains hyponormal operators (T is hyponormal iff $T^*T\geqslant TT^*$). Aside from this point of view of normality it seems natural to ask the question: "What kind of idempotent operator T is a projection?" The following are well-known results in terms of the restriction on the norm of T. The idempotent operator T is a projection if

(a) T is a contraction ($\|T\|\leqslant 1$). This has been known for quite sometime (see a remark below), or

(b) $w(T)\leqslant 1$ ([1], Lemma), where $w(T)=\sup\{|(Tx, x)|: x\in X, \|x\|=1\}$, the numerical radius of T, or

(c) $T\in\mathscr{C}_\rho$ ([2], Theorem 1); or equivalently, $w_\rho(T)\leqslant 1$.

Recall that $T\in\mathscr{C}_\rho$ iff for some $\rho>0$, T satisfies the condition:

$$(\rho h, h)-2\,\mathrm{Re}\,(z(\rho-1)Th, h)+|z|^2((\rho-2)Th, Th)\geqslant 0$$

for all $h\in X$, $|z|\leqslant 1$ ([7], p. 45). It turns out that $T\in\mathscr{C}_\rho$, iff $T^*\in\mathscr{C}_\rho$, iff $w_\rho(T)\leqslant 1$, where $w_\rho(T)=\inf\{u: u>0, T/u\in\mathscr{C}_\rho\}$. $w_\rho(\cdot)$ is a norm on $B(X)$ whenever $0<\rho\leqslant 2$ (but it is not, otherwise), $w_1(\cdot)=\|\cdot\|$ and $w_2(\cdot)=w(\cdot)$ ([4], Theorem 3.1). Thus, (a) and (b) are nothing but special cases of (c). In fact, if $T\in\mathscr{C}_\rho$, $|z|=1$ and $m\geqslant 1$, then zT is a projection iff $(I-zT)^mT=0$, and zT is a symmetry (T is a symmetry iff $T=T^*=T^{-1}$) iff $(I-zT)^m(I+zT)=0$ ([6], Corollary 2).

The purpose of this note is to answer the previous question, more precisely, to look for various types of operators $T\in B(X)$ satisfying the condition: $N(T^*)\supseteq N(T)$, where $N(T)$ means the null space of T.

Lemma 1. *Let $T\in B(X)$, $m\geqslant 1$ and $k\geqslant 1$. If $N(T^*)\supseteq N(T)$, then*

(1) $N(T)=N(T^m)$.

(2) T^k *is a projection iff* $T^m(T^k-I)=0$.

(3) T^k+I *is a projection iff* $T^m(T^k+I)=0$.

(4) *$T+I$ is a symmetry iff $T^m(T+2I)=0$.*
(5) *T is a nilpotent operator iff $T=0$.*

Proof. The proof is elementary and can be found in ([6], Lemma and Corollary 1). Note that the proofs of (2) and (3) are similar to that of (1) in ([6], Corollary 1) and (5) follows from (1).

In 1943 Riesz and Sz.-Nagy [8] showed that if $T\in B(X)$ is a contraction, then $N(T-I)=N(T^*-I)$. In 1966 Hildebrandt [3] weakened the hypothesis on T so that $W(T)$, the numerical range of T, is contained in the closed unit disc. Therefore, by applying Lemma 1, if $\|T\|\leqslant 1$, or if $w(T)\leqslant 1$ and $m\geqslant 1$, then T is a projection iff $(T-I)^mT=0$, T is a symmetry iff $(T-I)^m(T+I)=0$, and $T=I$ iff $T-I$ is a nilpotent operator.

Incidentally, Riesz, Sz.-Nagy and Hildebrandt's result can be generalized as follows: $N(zT-I)^*=N(zT-I)=N(zT-I)^n$ for $n>1$ and any complex number z with $|z|=1$ (cf. Corollary 2, [6]).

2. The class S_z

Let us generalize the condition on $T\in\mathscr{C}_\rho$.

Definition. An operator $T\in B(X)$ is said to be in the class S_z if the inequality

$$(Sh, h)-2\operatorname{Re}(z(S-I)Th, h)+|z|^2((S-2I)Th, Th)\geqslant 0$$

holds for all $h\in X$, some self-adjoint operator $S\in B(X)$ and some complex number $z\neq 0$.

The following are immediate results: (a) $T\in\mathscr{C}_\rho$ iff $T\in\rho_z$ (in our sense) for all z, $|z|\leqslant 1$. (b) $\|T\|\leqslant 1$ iff $T\in 1_z$ for any z with $|z|=1$. (c) If $w(T)\leqslant 1$, then $T\in 2_z$ for any z with $|z|\leqslant 1$. (d) If $w(T)\leqslant 1$ and $T\in 0_z$ for any z with $|z|\geqslant 1$, then $\|T\|\leqslant 1$. (e) If T is a projection, then $T\in T_z$ for any z with $|z|\leqslant 1$. (f) $1/z\in S_z$.

Theorem 1. *If $T\in B(X)$ is in S_z and $m\geqslant 1$, then*
(1) *zT is a projection iff $(I-zT)^mT=0$.*
(2) *zT is a symmetry iff $(I-zT)^m(I+zT)=0$.*
(3) *$T=1/z$ iff $I-zT$ is a nilpotent operator.*

Proof. By Lemma 1 it suffices to show that $N(I-\bar{z}T^*)\supseteq N(I-zT)$. Let $x\in N(I-zT)$, i.e., $zTx=x$ and suppose that $(I-zT)^*x=y$, then $(x, y)=(x, (I-zT)^*x)=((I-zT)x, x)=0$. Let $h=x+ty$ for $t<0$, then

$$(S(x+ty), x+ty)-2\operatorname{Re}(z(S-I)T(x+ty), x+ty)$$
$$+|z|^2((S-2I)T(x+ty), T(x+ty))\geqslant 0\,.$$

To simplify this inequality is cumbersome. Let us write some of the simplified forms only.

$$\operatorname{Re}(Sx+tSy-2Sx-2ztSTy+2x+2ztTy,\ x+ty)$$
$$+(Sx+ztSTy-2x-2ztTy,\ x+ztTy)\geqslant 0\,,$$
$$\operatorname{Re}[t^2(Sy,y)-2zt^2(STy,y)-2zt(Ty,x)+2zt^2(Ty,y)$$
$$+|z|^2t^2(STy,Ty)-2|z|^2t^2(Ty,Ty)]\geqslant 0\,,$$
$$(Sy,y)-2\operatorname{Re} z(STy-Ty,y)-|z|^2(2Ty-STy,Ty)\geqslant -2t^{-1}(y,y)\,.$$

In order that the last inequality hold for any negative number t it is necessary that $y=0$ and the theorem is verified.

Note that a stronger version of the class S_z is that if S is replaced by $A\in B(X)$, a positive and invertible operator, and the inequality holds for all z, $|z|\leqslant 1$, then such a T is said to be in the class $\mathscr{C}_A$, and it turns out that $\mathscr{C}_A\subseteq\mathscr{C}_\rho$ if $\rho\geqslant\|A\|$ ([7], p. 55).

An alternative definition of the class S_z is that $T\in S_z$ iff

$$(I-zT)^*(S-2I)(I-zT)+(I-zT)+(I-zT)^*\geqslant 0$$

holds. Because the inequality in Definition can be rewritten as follows:

$$\operatorname{Re}(S-2z(S-I)T+|z|^2T^*(S-2I)T)\geqslant 0\,,$$
$$\operatorname{Re}((S-\bar{z}T^*S+2\bar{z}T^*)-(S-\bar{z}T^*S+2\bar{z}T^*)zT)\geqslant 0\,,$$
$$\operatorname{Re}(((I-zT)^*(S-2I)+2I)(I-zT))\geqslant 0\,.$$

The desired relation follows since $\operatorname{Re}E=\frac{1}{2}(E+E^*)$ for $E\in B(X)$. Immediately, if $T\in S_z$ and $S\leqslant S'\in B(X)$, another self-adjoint operator, then $T\in S_z'$. In particular, $T\in k_z$ for $k\geqslant\|S\|$.

Now, let us give an alternative proof of Theorem 1. By the definition in [6] $T\in C_S$ iff the relation $T^*ST+T+T^*\geqslant 0$ holds for some self-adjoint operator S. Hence, $T\in S_z$ iff $I-zT\in C_{S-2I}$, and if $(I-zT)x_n\to 0$ for some bounded sequence $\{x_n\}$ in X, then $(I-zT)^*x_n\to 0$ ([6], Lemma). The proof follows by applying Lemma 1.

3. Generalized sequentially G_1 operators and proper boundary points

Let $d(v)$ denote the distance between v and $\sigma(T)$, and $\partial\sigma(T)$ the boundary of $\sigma(T)$. Let us recall the following well-known conditions on T: (a) T is called a sequentially G_1 operator [11] if for every $v\in\partial\sigma(T)$ there exists a sequence $v_n\notin\sigma(T)$ such that $v_n\to v$ and $\|(v_n-T)^{-1}\|=1/d(v_n)$ for all n. This is, of course, an extension of the G_1 growth condition: $\|(v-T)^{-1}\|=1/d(v)$ for all $v\notin\sigma(T)$. For example, a hyponormal operator satisfies this. (b) A point $v\in\partial\sigma(T)$ will be called proper [9] if there exists a bounded sequence $v_n\notin\sigma(T)$ such that $\|(v_n-v)(v_n-T)^{-1}\|\to 1$. Two examples of such v are: (1) If $\|T\|=|v|$, and (2) If $v\in\sigma(T)$ is a boundary point of $W(T)$ [9].

In virtue of $w_\rho(\cdot)$ indicated previously (also note that $w_\rho(zT)=|z|\,w_\rho(T)$) the above two relations on T can be naturally extended as follows:

(a′) $w_\rho((v_n-T)^{-1})\leqslant k/d(v_n)$, or equivalently, $k^{-1}d(v_n)(v_n-T)^{-1}\in\mathscr{C}_\rho$ for some constant k and all n.

(b′) $w_\rho((v_n-v)(v_n-T)^{-1})\to 1$. Hence, either $(v_m-v)(v_m-T)^{-1}\in\mathscr{C}_\rho$ for some integer $m>0$, or, given $\varepsilon>0$, there exists an integer $m>0$ such that $(1+\varepsilon)^{-1}(v_m-v)(v_m-T)^{-1}\in\mathscr{C}_\rho$.

Here, we add another one: $w_\rho((v-T)^{-1})\leqslant 1/|v|$, or equivalently, $v(v-T)^{-1}\in\mathscr{C}_\rho$ for some $v\notin\sigma(T)$ and $v\neq 0$.

We want to consider above situations in terms of the class S_z.

Theorem 2. (1) *If there exists a sequence $v_n\notin\sigma(T)$ such that $v_n\to 0\in\partial\sigma(T)$ and $k^{-1}d(v_n)(v_n-T)^{-1}\in S_z$ for some constant k and all n, or*

(2) *If $v(v-T)^{-1}\in S_1$ for some $v\notin\sigma(T)$ and $v\neq 0$,*

then, the statements (2), (3), (4) *and* (5) *in Lemma* 1 *hold.*

Proof. In both cases we need only show that $N(T^*)\supseteq N(T)$.

(1) For every n we have

$$\begin{aligned}N(v_n-T-zk^{-1}d(v_n))^* &= N(I-zk^{-1}d(v_n)(v_n-T)^{-1})^*\\ &\supseteq N(I-zk^{-1}d(v_n)(v_n-T)^{-1})\\ &= N(v_n-T-zk^{-1}d(v_n))\,.\end{aligned}$$

Hence, $N(T^*)\supseteq N(T)$ as $n\to\infty$.

(2) $N(T^*)=N(I-v(v-T)^{-1})^*\supseteq N(I-v(v-T)^{-1})=N(T)$.

Theorem 3. *For $0\neq v\in\partial\sigma(T)$, if there exists a bounded sequence $v_n\notin\sigma(T)$ which satisfies one of the following conditions:*

(i) $(v_m-v)(v_m-T)^{-1}\in S_1$ *for some integer $m>0$,*

(ii) *for any $\varepsilon>0$, there exists an integer $m>0$ such that*

$$(1+\varepsilon)^{-1}(v_m-v)(v_m-T)^{-1}\in S_1\,,$$

then, for $k\geqslant 1$,

(1) *$v^{-1}T$ is a projection iff $(T-v)^kT=0$.*

(2) *$v^{-1}T$ is a symmetry iff $(T-v)^k(T+v)=0$.*

(3) *$T=v$ iff $T-v$ is a nilpotent operator.*

Proof. The proof of the first case is the same as (2) in Theorem 2. Suppose that

$$(1+\varepsilon)^{-1}(v_m-v)(v_m-T)^{-1}\in S_1\,,$$

then

$$\begin{aligned}N(v_m-T-(1+\varepsilon)^{-1}(v_m-v))^* &= N(I-(1+\varepsilon)^{-1}(v_m-v)(v_m-T)^{-1})^*\\ &\supseteq N(I-(1+\varepsilon)^{-1}(v_m-v)(v_m-T)^{-1})\\ &= N(v_m-T-(1+\varepsilon)^{-1}(v_m-v))\,.\end{aligned}$$

Since $\{v_n\}$ is bounded and ε was arbitrary we see that $N(v^{-1}T-I)^*\supseteq N(v^{-1}T-I)$. Now, Lemma 1 is applicable.

Corollary 1. (a) *If* $0 \in \partial\sigma(T)$ *is replaced by* $v \neq 0$ *in* (1) *of Theorem* 2, *then the statements* (1), (2) *and* (3) *in Theorem* 3 *hold.*

(b) *If* $v \in \partial\sigma(T)$ *is replaced by* 0 *in Theorem* 3, *then the statements* (2), (3), (4) *and* (5) *in Lemma* 1 *hold.*

Corollary 2. (a) *Let* $T \in B(X)$ *be a sequentially* G_1 *operator. If* $0 \in \partial\sigma(T)$, *then the statements* (2), (3), (4) *and* (5) *in Lemma* 1 *hold. If* $0 \neq v \in \partial\sigma(T)$, *then the statements* (1), (2) *and* (3) *in Theorem* 3 *hold.*

(b) *If* $T \in B(X)$ *has a proper boundary point* $0 \in \partial\sigma(T)$, *then the statements* (2), (3), (4) *and* (5) *in Lemma* 1 *hold. If it has a proper boundary point* $v \in \partial\sigma(T)$, $v \neq 0$, *then the statements* (1), (2) *and* (3) *in Theorem* 3 *hold.*

Remark that if T is nilpotent, then $\sigma(T) = \{0\}$. Hence, above statements about a nilpotent operator can be stated in a more simpler forms. For example, if T is a nilpotent and sequentially G_1 operator, then $T = 0$ ([11], Proposition 1.2).

Theorem 4. *For* $0 \neq T \in B(X)$ *it is either*

(a) *there exists a complex number* $v \neq 0$ *such that* (1), (2) *and* (3) *in Theorem* 3 *hold, or*

(b) (2), (3), (4) *and* (5) *in Lemma* 1 *hold.*

Proof. We note that for any $T \in B(X)$, there exists a complex number v such that $N(T-v)^* = N(T-v)$. To see this, take $z \notin$ the closure of $W(T)$ and let $c =$ the distance between z and $W(T)$, then $\|c(T-z)^{-1}\| \leqslant 1$, i.e., $w_1(c(T-z)^{-1}) \leqslant 1$. Let $v = z + c$ and we have the desired relation (cf. [6], (1) in Corollary 2). Of course, for $T \neq 0$, $N(T - \|T\|)^* = N(T - \|T\|)$ since $\| \|T\|^{-1} T \| = 1$.

4. The direct sum of the zero and a unitary operator

It is well known that an idempotent operator in the class $\mathscr{C}_\rho$ is not only a projection, but also the direct sum of the zero and a unitary operator ([2], Theorem 2). Instead of an idempotent operator it has been generalized to a polynomial equation of that operator ([11], Theorem 2.5). The next result shows that the same holds for an operator in S_z. But first let us prove

Lemma 2. *Suppose that* $T \in B(X)$ *is in* S_z, *or in* S_{z_0} *and* $0 \neq z \neq z_0 \neq 0$. *If* z^{-1} *and* $z_0{}^{-1} \in \sigma(T)$, *and if* $(I - zT)x_n \to 0$ *and* $(I - z_0 T)y_n \to 0$ *for some bounded sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X, *then* $(x_n, y_n) \to 0$. *In particular,* $N(I - zT) \perp N(I - z_0 T)$.

Proof. Suppose that $T \in S_{z_0}$.

$$\begin{aligned} |(z^{-1} - z_0{}^{-1})(x_n, y_n)| &= |z^{-1}(x_n, y_n) - z_0{}^{-1}(x_n, y_n)| \\ &= |(z^{-1}x_n - Tx_n, y_n) + (x_n, T^* y_n - \bar{z}_0{}^{-1} y_n)| \\ &\leqslant \|z^{-1}(I - zT)x_n\| \, \|y_n\| + \|x_n\| \, \|\bar{z}_0{}^{-1}(I - z_0 T)^* y_n\| \to 0 \end{aligned}$$

by the alternative proof of Theorem 1. Hence $(x_n, y_n) \to 0$.

Theorem 5. *Let $T\in B(X)$ and suppose that $p(z)$ is a polynomial so that $p(T)=0$. If $T\in S_{z^{-1}}$ for all reciprocals of the roots z of $p(z)$ except for, perhaps, a root z_0 of multiplicity one, then*

$$T=\sum_{0}^{m}\oplus z_iP_i ,$$

where z_i $(i=0, 1, \cdots, m)$ is a root of $p(z)$, $z_i\in\sigma(T)$ and P_i $(i=0, 1, \cdots, m)$ is a projection. Moreover, if $|z_i|=1$ $(i=1, 2, \cdots, m)$, then

$$T=U\oplus z_0P_0 ,$$

where U is a unitary operator.

Proof. Let z_i be the distinct roots of $p(z)$ of multiplicity n_i $(i=1, 2, \cdots, m)$, then $p(z)=(z-z_0)(z_1-z)^{n_1}\cdots(z_n-z)^{n_n}$ and hence

$$X=\sum_{1}^{n}N(z_i-T)^{n_i}+N(T-z_0I) .$$

Note that $N(z_i-T)^{n_i}=N(z_i-T)$, and this is equal to $\{0\}$ if $z_i\notin\sigma(T)$. $N(z_i-T)\perp N(z_j-T)$ for $i\neq j$ by Lemma 2. $N(z_i-T)\subseteq N(z_i-T)^*$ and $N(T-z_0I)\perp N(z_i-T)$ hold for every i. Therefore, we conclude from these remarks that

$$T=\sum_{0}^{m}\oplus z_iP_i , \quad m\leq n ,$$

where P_i and P_0 are projections of X onto $N(z_i-T)$ $(i=1, 2, \cdots, m)$ and $N(T-z_0I)$, respectively. Moreover, if $|z_i|=1$ $(i=1, 2, \cdots, m)$, then $\sum_1^m\oplus z_iP_i$ is a unitary operator and hence $T=U\oplus z_0P_0$.

Corollary 3. *Suppose that $T\in B(X)$ is in S_z. If $T=zT^2$, 0 and $z^{-1}\in\sigma(T)$ and $|z|=1$, then $T=U\oplus 0$, where $U\in B(X)$ is a unitary operator.*

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