YOKOHAMA MATHEMATICAL
JOURNAL VOL. 29, 1981

RIEMANNIAN MANIFOLDS WITH COMPACT BOUNDARY

By

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(Received August 4, 1981)

§ 0. Introduction

The main purpose of this paper is to prove the following theorem.

Theorem. *Let* $\bar{M}=M\cup\partial M$ *be an* $n+1$*-dimensional* ($n\geqq 1$) *connected* C^{∞} *Riemannian manifold with compact* C^{∞} *boundary* ∂M. *Suppose that* M *is of non-negative Ricci curvature and the mean curvature (with respect to the inner normal direction) of* ∂M *is non-negative. Then* ∂M *has at most two connectecd components. Moreover, if* ∂M *has just two connected components, then* $\bar{M}$ *is isometric to a Riemannian product manifold* $\Gamma\times[0, a]$ ($a>0$) *where* Γ *is an n-dimensional compact connected* C^{∞} *Riemannian manifold of non-negative Ricci curvature without boundary.*

In order to prove this theorem, in § 1, we study properties of solutions of some non-linear differential equation. We show that the minimum principle holds for solutions of a non-linear differential equation which is closely related to geometrical problems. In § 2, we give an application (Theorem 2.1) of the minimum principle showed in § 1. Theorem 2.1 will be used to prove the theorem stated above. The proof of the main theorem is given in § 3.

§ 1. Minimum principle

Let D be a bounded domain in n-dimensional ($n\geqq 1$) Euclidean space R^{n}. We denote the set of real-valued functions of class C^{k} on D by $C^{k}(D)$ where k is a non-negative integer. In the following, for a $u\in C^{2}(D)$ we use the following notations:

$$u_{i}=\partial u/\partial x_{i}\ , \qquad \nabla u=(u_{1}, \cdots, u_{n}) \qquad \text{and} \qquad u_{ij}=\partial^{2}u/\partial x_{i}\partial x_{j}$$

where $x_{1}, \cdots, x_{n}$ are the canonical coordinate functions in R^{n}. Let $\|\ \ \|$ be the standard Euclidean norm of R^{n}.

Let us consider a non-linear differential equation of second order on D:

$$L(u)=\sum_{i,j=1}^{n}A_{ij}(x, u, \nabla u)u_{ij}-B(x, u, \nabla u)\equiv 0 \tag{1.1}$$

where A_{ij} and B are real-valued continuous functions on $D\times[a, b]\times R^{n}$, $a<b$, and

$A_{ij}=A_{ji}$ $(1\leqq i, j\leqq n)$. In the case $n=1$, of course, (1.1) is an ordinary differential equation of second order. We denote by (x, t, p) a point of $D\times[a, b]\times R^n$.

We assume that the equation (1.1) is elliptic, that is, for every non vanishing real vector $X=(X_1, \cdots, X_n)\in R^n$

$$\sum_{i,j=1}^{n} A_{ij}(x, t, p)X_iX_j>0 \tag{1.2}$$

holds on $D\times[a, b]\times R^n$.

It is called that $u\in C^2(D)$, $a\leqq u\leqq b$, is a supersolution of equation (1.1) if $L(u)\leqq 0$ holds.

Theorem 1.1. *For equation* (1.1), *assume that B is of class C^1 with respect to the variables p_i* $(1\leqq i\leqq n)$ *and that*

$$B(x, t, 0)\leqq 0 \quad on \quad D\times[a, b]\times\{0\}\ . \tag{1.3}$$

If $u\in C^2(D)$, $a\leqq u\leqq b$, is a supersolution of equation (1.1), *then u can not take the minimum value in D unless u is constant.*

Proof. Suppose for contradiction that u takes the minimum value m in D and that u is not constant. We set $E=\{x\in D;\ u(x)=m\}$. E can not be open in D. Then we can choose a $x_0\in D-E$ and the closed metric ball $\bar{D}_0$ of radius r_0 and center x_0 in R^n such that

$$\bar{D}_0\cap E=\{y_0\}\ ,\quad \bar{D}_0\subset D\ . \tag{1.4}$$

Let $\bar{D}_1$ be the closed metric ball in R^n of radius r_1 and center y_0 such that $0<r_1<r_0$ and that $\bar{D}_1\subset D$. Then we have

$$r_2\leqq\|x-x_0\|\leqq r_3\ ,\quad x\in\bar{D}_1 \tag{1.5}$$

where $r_2=r_0-r_1$, $r_3=r_0+r_1$. There exists a constant δ $(0<\delta<1)$ satisfying the condition

$$u>m+\delta \quad \text{on} \quad \bar{D}_0\cap\partial\bar{D}_1 \tag{1.6}$$

where $\partial\bar{D}_1=\{x\in R^n;\ \|x-y_0\|=r_1\}$. Since equation (1.1) satisfies the condition (1.2), there exists positive constants λ_1 and λ_2 such that

$$\lambda_1\|X\|^2\leqq\sum_{i,j=1}^{n} A_{ij}(x, u(x), p(x))X_iX_j\leqq\lambda_2\|X\|^2\ ,\quad x\in\bar{D}_1 \tag{1.7}$$

where $X=(X_1, \cdots, X_n)\in R^n$ and we put $\nabla u(x)=p(x)=(p, \cdots, p_n)$. (In the following we use this notation.) Since B is of class C^1 with respect to the variables p_i $(1\leqq i\leqq n)$, on $\bar{D}_1$ we have

$$|B(x, u(x), p(x))-B(x, u(x), 0)|\leqq c\|p(x)\|$$

where

$$c=\sup_{\bar{D}}\sum_{i=1}^{n}\int_{0}^{1}\left|\frac{\partial B}{\partial p_i}(x, u(x), tp(x))\right|dt . \tag{1.8}$$

Then by (1.3) we have

$$B(x, u(x), p(x))\leqq c\|p(x)\| , \quad x\in \bar{D}_1 . \tag{1.9}$$

Let us consider the auxiliary function w on $\bar{D}_1$ defined by

$$w(x)=u(x)-h(x) , \quad x\in \bar{D}_1 \tag{1.10}$$

where

$$h(x)=\exp(-\alpha\|x-x_0\|^2)-\exp(-\alpha r_0^2) , \tag{1.11}$$

α being a positive constant such that

$$\alpha>\max\{\log(1/\delta)/r_2^2, (n\lambda_2+cr_3)/2\lambda_1 r_2^2\} . \tag{1.12}$$

Since $\|x-x_0\|>r_0$ on $\partial\bar{D}_1-\bar{D}_0$, $h<0$ on $\partial\bar{D}_1-\bar{D}_0$. Hence, we have

$$w>m \quad \text{on} \quad \partial\bar{D}_1-\bar{D}_0 . \tag{1.13}$$

From (1.5), (1.6) and (1.12), on $\partial\bar{D}_1\cap\bar{D}_0$ we have

$$w>m+\delta-\exp(-\alpha r_2^2)>m . \tag{1.14}$$

On the other hand, at y_0 we have

$$w(y_0)=u(y_0)=m . \tag{1.15}$$

Thus it follows from (1.13), (1.14) and (1.15) that w takes the minimum value at an interior point y of $\bar{D}_1$. From (1.9) and (1.10), at y we have

$$\sum_{i,j=1}^{n}A_{ij}(y, u(y), p(y))(w_{ij}(y)+h_{ij}(y))\leqq c\|p(y)\| . \tag{1.16}$$

In the following we shall estimate the inequality (1.16). By (1.11) we have

$$h_i(y)=-2\alpha z_i\xi , \qquad h_{ij}(y)=-2\alpha(\delta_{ij}-2\alpha z_i z_j)\xi \tag{1.17}$$

where $z=(z_1, \cdots, z_n)=y-x_0$ and $\xi=\exp(-\alpha\|y-x_0\|^2)$. Since w takes the minimum value at y, we have

$$u_i(y)=h_i(y) , \quad 1\leqq i\leqq n , \tag{1.18}$$

and

$$\sum_{i,j=1}^{n} A_{ij}(y, u(y), p(y))w_{ij}(y) \geqq 0 . \tag{1.19}$$

From (1.5), (1.7), (1.17) and (1.19), at y we have

$$\begin{aligned} \text{the left-hand side of } (1.16) &\geqq 2\alpha\xi(2\alpha\lambda_1\|z\|^2 - n\lambda_2) \\ &\geqq 2\alpha\xi(2\alpha\lambda_1 r_2^2 - n\lambda_2) . \end{aligned} \tag{1.20}$$

By (1.17) and (1.18), we have $\|p(y)\| = 2\alpha\xi\|z\| \neq 0$. Hence, from (1.5), (1.16) and (1.20), we get

$$2\alpha\lambda_1 r_2^2 - n\lambda_2 \leqq c\|z\| \leqq c r_3 .$$

This contradicts (1.12). We complete the proof.

§ 2. A geometrical application of Theorem 1.1

Let N be a C^∞ Riemannian manifold without boundary and let $\langle\ ,\ \rangle$ be the inner product defined by the Riemannian metric of N. Let $e_1, \cdots, e_n$ be an orthonormal basis of the tangent vector space T_pN at a point p of N, $n = \dim N$, and let X be a unit vector at p. The quantity $\operatorname{Ric}(X) = \sum_{i=1}^{n} \langle R(e_i, X)X, e_i \rangle$ is called the Ricci curvature of N with respect to X direction where R denotes the Riemannian curvature tensor of N. We say that N is of non-negative (resp. positive) Ricci curvature if $\operatorname{Ric}(X) \geqq 0$ (resp. $\operatorname{Ric}(X) > 0$) for every unit vector X at every point of N. Let M be an imbedded hypersurface in N. It is called that M is totally geodesic if the second fundamental form of M vanishes everywhere.

Now, let D be an open metric ball in n-dimensional ($n \geqq 1$) Euclidean space R^n. Let $(x_1, \cdots, x_n)$ be the canonical coordinate system in R^n. For a $\varepsilon > 0$, let us consider a Riemannian manifold $N = (D \times (-\varepsilon, \varepsilon), ds^2)$ whose line element is given by $ds^2 = \sum_{i,j=1}^{n} g_{ij}(x, t) dx_i dx_j + dt^2$ where $g_{ij} \in C^\infty(D \times (-\varepsilon, \varepsilon))$ and the matrix (g_{ij}) is symmetric and positive definite everywhere. Let ∇ be the Riemannian connection of N. For a t, $|t| < \varepsilon$, we denote the mean curvature (with respect to $\partial/\partial t$) of the level hypersurface $S_t = \{(x, t); x \in D\}$ in N by H_t. In the case $n = 1$, by the mean curvature we mean the geodesic curvature.

Lemma 2.1. *Suppose that N is of non-negative Ricci curvature. Then $H_t \leqq H_{t'}$ holds for any $t < t'$. If $H_t = H_{t'}$ for $t < t'$, then for each r ($t \leqq r \leqq t'$) S_r is totally geodesic.*

Proof. Let $\{e_1, \cdots, e_n, \partial/\partial t\}$ be an orthonormal frame on N such that $\nabla_{\partial/\partial t} e_i = 0$. We put $h_{ij} = \langle \nabla_{e_i} e_j, \partial/\partial t \rangle$. Since $H_t = \frac{1}{n} \sum_{i=1}^{n} \langle \nabla_{e_i} e_i, \partial/\partial t \rangle$, we have

$$\partial H_t/\partial t=\frac{1}{n}\{\mathrm{Ric}\,(\partial/\partial t)+\sum_{i,j=1}^{n}(h_{ij})^2\}\ .$$

The lemma follows directly from the above formula.

Now, for a $u\in C^2(D)$, $|u|<\varepsilon$, we consider the hypersurface $S(u)=\{(x, u(x)); x\in D\}$ in N. We put $X_i=\partial/\partial x_i+u_i\partial/\partial t$, $\tilde{g}_{ij}=g_{ij}+u_iu_j$, $1\leqq i, j\leqq n$. We can give a unit normal vector field $\xi=\sum_{i=1}^{n}\xi^i\partial/\partial x_i+\xi^{n+1}\partial/\partial t$ on $S(u)$ as follows:

$$\xi^i=-u^i/(1+\|\nabla u\|^2)^{1/2}\quad(1\leqq i\leqq n)\qquad\text{and}\qquad\xi^{n+1}=1/(1+\|\nabla u\|^2)^{1/2}$$

where $\|\nabla u\|^2=\sum_{i,j=1}^{n}g^{ij}(x, u(x))u_iu_j$, $u^i=\sum_{j=1}^{n}g^{ij}(x, u(x))u_j$ and here g^{ij} is the (i, j)-component of the inverse matrix of (g_{ij}). Let Λ be the mean curvature of $S(u)$ with respect to ξ. Λ is given by $\Lambda=\frac{1}{n}\sum_{i,j=1}^{n}\tilde{g}^{ij}\langle\nabla_{X_i}X_j, \xi\rangle$ where $\tilde{g}^{ij}=g^{ij}(x, u(x))-u^iu^j/(1+\|\nabla u\|^2)$. Rewriting it we get

$$\begin{aligned}(2.1)\qquad &\sum_{i,j=1}^{n}\{(1+\|\nabla u\|^2)g^{ij}(x, u(x))-u^iu^j\}u_{ij}\\ &=n\Lambda(x)(1+\|\nabla u\|^2)^{3/2}-nH(x, u(x))(1+\|\nabla u\|^2)+\frac{1}{2}\sum_{i,j=1}^{n}\partial g_{ij}/\partial t(x, u(x))u^iu^j\\ &\quad+\sum_{i,j,k=1}^{n}\{(1+\|\nabla u\|^2)g^{ij}(x, u(x))-u^iu^j\}\Gamma_{ij}^k(x, u(x))u_k\end{aligned}$$

where $H(x, u(x))=-\frac{1}{2n}\sum_{i,j=1}^{n}g^{ij}(x, u(x))\partial g_{ij}/\partial t(x, u(x))$ and Γ_{ij}^k denotes the Christoffel's symbol.

In (2.1), if we regard Λ as a given real-valued continuous function on D, then (2.1) is a non-linear differential equation of second order on D. We see that the equation (2.1) satisfies the condition (1.2). We put

$$\begin{aligned}B(x, t, p)=&n\Lambda(x)(1+\|p\|^2)^{3/2}-nH(x, t)(1+\|p\|^2)\\ &+\frac{1}{2}\sum_{i,j=1}^{n}\partial g_{ij}/\partial t(x, t)p^ip^j+\sum_{i,j,k=1}^{n}\{(1+\|p\|^2)g^{ij}(x, t)-p^ip^j\}\Gamma_{ij}^kp_k\end{aligned}$$

where $p=(p_1, \cdots, p_n)\in R^n$, $\|p\|^2=\sum_{i,j=1}^{n}g^{ij}(x, t)p_ip_j$ and $p^i=\sum_{j=1}^{n}g^{ij}(x, t)p_j$. Then it is clear that B is of class C^1 on $D\times(-\varepsilon, \varepsilon)\times R^n$. If N is of non-negative Ricci curvature and $\Lambda\leqq H_0$ holds on D, then by Lemma 2.1 we have $B(x, t, 0)=n\Lambda(x)-nH_t(x)\leqq 0$ on $D\times[0, \varepsilon)\times\{0\}$. Applying Theorem 1.1 to the equation (2.1), we have the following.

Theorem 2.1. *Suppose that N is of non-negative Ricci curvature. Let Λ be a given real-valued continuous function on D such that $\Lambda\leqq H_0$ on D. Then any solution u of the equation* (2.1) *such that $0\leqq u<\varepsilon$ can not take the minimum value in D unless u is constant.*

§ 3. Proof of the main theorem

Let $\bar{M}=M\cup\partial M$ be an $n+1$-dimensional ($n\geqq 1$) connected C^∞ Riemannian manifold with compact C^∞ boundary ∂M. Let ρ be the distance function on $\bar{M}$ which is defined by the Riemannian metric of $\bar{M}$. Compactness of ∂M implies that $\bar{M}$ is complete as a metric space. For each point p of M there is a geodesic from p to ∂M whose length is equal to $\rho(p, \partial M)$. A geodesic c: $[a, b]\to\bar{M}$ is called minimal if the length of c is equal to $\rho(c(a), c(b))$. Let p be a point of M and let r be a positive such that $r<\rho(p, \partial M)$. We put $B_r(p)=\{q\in M;\ \rho(p, q)<r\}$, $\bar{B}_r(p)=\{q\in\bar{M};\ \rho(p, q)\leqq r\}$ and $\partial B_r(p)=\bar{B}_r(p)-B_r(p)$. We can choose a positive r ($2r<\rho(p, \partial M)$) such that for any distinct points q and q' of $\bar{B}_r(p)$ there is a unique minimal geodesic from q to q' whose interior is contained in $B_r(p)$ ([2], pp. 103–105). Such an open metric ball is called strongly convex.

Now, since ∂M is compact, it can be expressed by $\partial M=\bigcup_{i=1}^{k}\Gamma_i$ where each Γ_i is an n-dimensional compact connected Riemannian manifold without boundary. For a $\delta>0$ we put $\perp_\delta^+(\Gamma_i)=\{\xi\in T\bar{M};\ \xi$ is an inner normal vector to Γ_i and $\|\xi\|<\delta\}$, $\perp_\delta^+(U_i)=\{\xi\in\perp_\delta^+(\Gamma_i);\ \xi$ is an inner normal vector to $U_i\}$ where $T\bar{M}$ denotes the tangent vector bundle of $\bar{M}$ and $\|\ \ \|$ stands for the norm defined by the Riemannian metric of $\bar{M}$ and U_i is an open subset of Γ_i, $1\leqq i\leqq k$. When exp: $\perp_\delta^+(\Gamma_i)\to\bar{M}$ (resp. exp: $\perp_\delta^+(U_i)\to\bar{M}$) is an imbedding of C^∞ for a $\delta>0$, we put $\Gamma_i(\delta)=\{\exp_p \delta\eta_i(p);\ p\in\Gamma_i\}$ (resp. $U_i(\delta)=\{\exp_p \delta\eta_i(p);\ p\in U_i\}$ where exp stands for the exponential map and $\eta_i(p)$ is the inner normal vector to Γ_i at $p\in\Gamma_i$.

Under the situation described above, we shall prove the following.

Lemma 3.1. *Let $\bar{M}=M\cup\partial M$ be as above. Suppose $\rho(\Gamma_1, \Gamma_2)=\min\{\rho(\Gamma_i, \Gamma_j);\ 1\leqq i<j\leqq k\}$. Let p_1 and p_2 be points of Γ_1 and Γ_2 such that $\rho(p_1, p_2)=\rho(\Gamma_1, \Gamma_2)$, respectively. Then there is a unique minimal geodesic c: $[0, a]\to\bar{M}$, $a=\rho(\Gamma_1, \Gamma_2)$, with unit speed such that $c(0)=p_1$, $c(a)=p_2$, $c((0, a))\subset M$ and $\dot{c}(0)$ (resp. $\dot{c}(a)$) is orthogonal to Γ_1 (resp. Γ_2) respectively, where $\dot{c}(t)$ denotes the velocity vector of c.*

Proof. Since Γ_1 is compact, we can take $a\varepsilon$ ($0<2\varepsilon<a$) such that exp: $\perp_{2\varepsilon}^+(\Gamma_1)\to\bar{M}$ is an imbedding of C^∞ and $\Gamma_1(\varepsilon)=\{p\in M;\ \rho(p, \Gamma_1)=\varepsilon\}$. There is a point p_3 of $\Gamma_1(\varepsilon)$ such that $\rho(p_1, p_2)=\rho(p_1, p_3)+\rho(p_3, p_2)$. Then, using Gauss' lemma, we see $\exp_{p_1}\varepsilon\xi=p_3$ where ξ is the inner unit normal vector to Γ_1 at p_1. Let $t_0=\sup\{t\in[0, a];\ \exp_{p_1}s\xi\in M\ (0<s\leqq t)$ and $\rho(\exp_{p_1}t\xi, p_2)=a-t\}$. It is clear $\varepsilon\leqq t_0$. We shall show $t_0=a$. Suppose $t_0<a$. By completeness of $\bar{M}$ the geodesic $\exp_{p_1}t\xi$ ($0\leqq t<t_0$) can be extended to the geodesic $c_1(t)=\exp_{p_1}t\xi$ ($0\leqq t\leqq t_0$). Since $a=\rho(\Gamma_1, \Gamma_2)\leqq\rho(\Gamma_i, \Gamma_j)$ ($1\leqq i<j\leqq k$), $c_1(t_0)\in M$. We take a δ ($0<2\delta<\rho(c_1(t_0), \partial M)$) such that $B_\delta(c_1(t_0))$ is strongly convex. Let p be a point of $\partial B_\delta(c_1(t_0))$ such that $\rho(c_1(t_0), p_2)=\rho(c_1(t_0), p)+\rho(p, p_2)$, and let $c_2(t)=\exp_{c_1(t_0)} tX$ $(0\leqq t\leqq\delta)$ be a unique minimal geodesic from $c_1(t_0)$ to p where X is a unit tangent vector at $c_1(t_0)$. Since $\rho(c_1(t_0), p_2)=a-t_0$, we have $\rho(p_1, p)\geqq t_0+\delta$. Hence, $\rho(p_1, p)=t_0+\delta$ because t_0 is equal to the length of the geodesic c_1 and δ is equal to the length of the geodesic c_2. This implies $\dot{c}_1(t_0)=X$. Thus we can extend the geodesic c_1 to the geodesic $\exp_{p_1} t\xi$ $(0\leqq t\leqq t_0+\delta)$ which satisfies $\exp_{p_1} t\xi\in M$ $(0<t\leqq t_0+\delta)$ and $\rho(\exp_{p_1}(t_0+\delta)\xi, p_2)=a-(t_0+\delta)$. This contradicts the definition of t_0. Hence we have $t_0=a$. Thus there is a minimal geodesic c: $[0, a]\to\bar{M}$ with unit speed such that $c(0)=p_1$, $c(a)=p_2$ and $c((0, a))\subset M$. Since Γ_1 and Γ_2 are hypersurfaces in $\bar{M}$ and c is a shortest geodesic from Γ_1 to Γ_2, $\dot{c}(0)$ (resp. $\dot{c}(a)$) is orthogonal to Γ_1 (resp. Γ_2), respectively. The uniqueness is then clear. We complete the proof.

Proof of the main theorem. Since ∂M is compact, it can be expressed by $\partial M=\bigcup_{i=1}^{k}\Gamma_i$ where each Γ_i is an n-dimensional compact connected Riemannian manifold without boundary. Suppose $k\geqq 2$, and let $a=\min\{\rho(\Gamma_i, \Gamma_j);\ 1\leqq i<j\leqq k\}$. By exchanging the indecies, we assume $a=\rho(\Gamma_1, \Gamma_2)$. Then we shall prove that $\partial M=\Gamma_1\cup\Gamma_2$ and Γ_1, Γ_2 are totally geodesic hypersurfaces in $\bar{M}$ and that $\bar{M}$ is isometric to the Riemannian product manifold $\Gamma_1\times[0, a]$. We put $C=\{q\in\Gamma_2;\ \rho(q, \Gamma_1)=a\}$. It is clear that C is a non-empty closed subset of Γ_2. We shall show that C is open in Γ_2. Let p_2 be an arbitrary point of C. We choose a $p_1\in\Gamma_1$ such that $\rho(p_1, p_2)=a$. By Lemma 3.1, there is a unique minimal geodesic c: $[0, a]\to\bar{M}$ with unit speed such that $c(0)=p_1$, $c(a)=p_2$, $c((0, a))\subset M$ and $\dot{c}(0)$ (resp. $\dot{c}(a)$) is orthogonal to Γ_1 (resp. Γ_2), respectively. Since Γ_2 is compact, we can choose a δ $(0<2\delta<a)$ so that exp: $\perp^+_{2\delta}(\Gamma_2)\to\bar{M}$ is an imbedding of C^∞ and $\Gamma_2(\delta)=\{q\in M;\ \rho(q, \Gamma_2)=\delta\}$. Then $\Gamma_2(\delta)$ is a compact connected hypersurface of C^∞ in M and $\dot{c}(a-\delta)$ is the outer unit normal vector to $\Gamma_2(\delta)$ at $c(a-\delta)$. Since c is a shortest geodesic from Γ_1 to Γ_2, $c(t)$, $0<t<a$, is not focal point of Γ_1 along c. Therefore we can take a local coordinate system $(U_1, (x_1, \cdots, x_n))$ about p_1 in Γ_1 and $a\varepsilon$ $(a-\delta<\varepsilon<a)$ such that exp: $\perp^+_\varepsilon(U_1)\to\bar{M}$ is an imbedding of C^∞ and $\exp(\perp^+_\varepsilon(U_1))\cap\Gamma_i=\phi$, $2\leqq i\leqq k$. By using Gauss' lemma the line element on $\exp(\perp^+_\varepsilon(U_1))$ can be expressed by $ds^2=\sum_{i,j=1}^{n} g_{ij}(x, t)dx_i dx_j+dt^2$, $(x, t)\in U_1\times[0, \varepsilon]$. Since $c(a-\delta)\in\Gamma_2(\delta)$ and $\rho(\Gamma_2(\delta), \Gamma_1)\geqq a-\delta$, by the implicit function theorem, there exists an open neighborhood V_1 of p_1 in Γ_1, $V_1\subset U_1$, and a $u\in C^\infty(V_1)$ satisfying the conditions: (1) V_1 is diffeomorphic to an open metric ball in R^n; (2) $a-\delta\leqq u<\varepsilon$ on V_1 and $u(p_1)=a-\delta$; (3) $\Gamma_2(\delta)$ can be locally expressed by a hypersurface $W=\{\exp_x u(x)\eta_1(x);\ x\in V_1\}$ about $c(a-\delta)$ where $\eta_1(x)$ is the inner unit normal vector to

Γ_1 at $x \in V_1$. Let W_2 be the open neighborhood of p_2 in Γ_2 such that $W_2(\delta)=W$. Now, let Λ be the mean curvature of $\Gamma_2(\delta)$ with respect to the outer normal direction. From the argument in § 2 we see that u is a solution of the equation (2.1) having the minimum value $a-\delta$ at an interior point p_1 of V_1. We denote the mean curvature of $V_1(t)$ with respect to $\partial/\partial t$ by H_t where $V_1(0)=V_1$ and $0\leqq t\leqq\varepsilon$. Since M is of non-negative Ricci curvature and the mean curvature (with respect to the inner normal direction) of ∂M is non-negative, by Lemma 2.1 $H_t\geqq 0$ $(0\leqq t\leqq\varepsilon)$ and $\Lambda\leqq 0$. Applying Theorem 2.1 to the present situation, we have $u\equiv a-\delta$ on V_1. This implies $W_2(\delta)=W=V_1(a-\delta)$. Since Λ is the mean curvature of $\Gamma_2(\delta)$ with respect to the outer normal direction, then we have $\Lambda=H_{a-\delta}$ on $W_2(\delta)=V_1(a-\delta)$. This yields $\Lambda=H_{a-\delta}\equiv 0$ because they have an opposite sign each other. Then, by Lemma 2.1, $V_1(t)$ $(0\leqq t\leqq a-\delta)$ and $W_2(r)$ $(0\leqq r\leqq\delta)$ are all totally geodesic hypersurfaces in $\bar{M}$ where $W_2(0)=W_2$. Thus we see that $\exp\colon \perp_a^+(V_1)\to\bar{M}$ is an imbedding of C^∞, $W_2=V_1(a)$ and $\exp(\perp_a^+(V_1))-(V_1\cup V_1(a))\subset M$. Hence we have $\rho(W_2,\Gamma_1)=a$, which implies that C is open in Γ_2. By connectedness of Γ_2 we have $C=\Gamma_2$. Thus we have proved that $\Gamma_2\subset\{q\in\bar{M};\ \rho(q,\Gamma_1)=a\}$. By a similar argument as above, we have $\Gamma_1\subset\{p\in\bar{M};\ \rho(p,\Gamma_2)=a\}$. From the above argument, we have the following.

Lemma. *For each point p_1 (resp. p_2) of Γ_1 (resp. Γ_2) there exists an open neighborhood V_1 of p_1 (resp. V_2 of p_2) in Γ_1 (resp. Γ_2) such that* (1) $\exp\colon\perp_a^+(V_i)\to\bar{M}$ *is an imbedding of C^∞, $i=1,2$;* (2) $V_1(a)\subset\Gamma_2$, $V_2(a)\subset\Gamma_1$; (3) $\exp(\perp_a^+(V_i))-(V_i\cup V_i(a))\subset M$, $i=1,2$; (4) *for each $p\in V_i$ $\exp_p t\eta_i(p)$ $(0\leqq t\leqq a)$ is a minimal geodesic where $\eta_i(p)$ is the inner unit normal vector to Γ_i at $p\in V_i$, $i=1,2$;* (5) $V_i(t)$ $(0\leqq t\leqq a)$ *are all totally geodesic hypersurfaces in $\bar{M}$ where $V_i(0)=V_i$, $i=1,2$.*

By virtue of this lemma and Lemma 3.1, we see that $\exp\colon\perp_a^+(\Gamma_i)\to\bar{M}$ is an imbedding of C^∞ $(i=1,2)$, $\Gamma_1(a)=\Gamma_2$, $\Gamma_2(a)=\Gamma_1$ and $\exp(\perp_a^+(\Gamma_i))-(\Gamma_1\cup\Gamma_2)\subset M$, $i=1,2$. By the connectedness of $\bar{M}$, $\exp(\perp_a^+(\Gamma_i))=\bar{M}$, $i=1,2$. Hence we have $\partial M=\Gamma_1\cup\Gamma_2$. From the above lemma, for each t $(0\leqq t\leqq a)$ the level hypersurface $\Gamma_1(t)$ is totally geodesic where $\Gamma_1(0)=\Gamma_1$. Now let $\Phi\colon\Gamma_1\times[0,a]\to\bar{M}$ be a map defined by $\Phi(p,t)=\exp_p t\eta_1(p)$ where $\eta_1(p)$ is the inner unit normal vector to Γ_1 at p. Then Φ is an isometry from the Riemannian product manifold $\Gamma_1\times[0,a]$ onto $\bar{M}$. We complete the proof.

As a corollary of the main theorem we have the following.

Corollary. *Let $\bar{M}=M\cup\partial M$ be an $n+1$-dimensional $(n\geqq 1)$ connected C^∞ Riemannian manifold with compact C^∞ boundary ∂M. Suppose that M is of non-*

negative (resp. positive) Ricci curvature and the mean curvature (with respect to the inner normal direction) of ∂M is positive (resp. non-negative). Then ∂M is connected.

Theorem 2.1 has interesting geometrical applications. For example, let M be an n-dimensional ($n \geqq 2$) connected complete real analytic Riemannian manifold of non-negative Ricci curvature without boundary. Suppose that M contains isometrically imbedded compact connected real analytic minimal hypersurfaces which are disjoint. Then M is isometric to one of some four types of Riemannian manifolds, in the case $n=2$ such Riemannian manifolds are flat torus, cylider, Klein bottle and Möbius band. The proof of this result will be given in the author's paper [4].

Remark. Recently Mr. Atsushi Kasue informed the author that he had independently proved our main theorem by a different method ([5]).

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