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ON INCREASING CONVEX FUNCTION OF $\log\sigma$

By

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1. Introduction

Consider a Dirichlet series $f(s)=\sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s=\sigma+it,\ \lambda_1\geqq 0,\ \lambda_n<\lambda_{n+1}\to\infty$ with $n)$, which we shall assume to be absolutely convergent everywhere in the complex plane C and is bounded in any left strip and hence it defines an entire function. The logarithmic mean of $f(s)$ is defined as

$$L(\sigma)=\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T}\log|f(\sigma+it)|dt\right\} .$$

For any $\delta>0$, we define [2, p. 231] the generalized logarithmic mean of $f(s)$ as

$$L_\delta^*(\sigma)=\lim_{T\to\infty}\left\{\frac{\sigma^{-\delta-1}}{2T}\int_0^{\sigma}\int_{-T}^{T} x^\delta \log|f(x+it)|\,dxdt\right\} . \tag{1.1}$$

Since $\log L_\delta^*(\sigma)$ is an increasing convex function of $\log\sigma$ [2, p. 232], we may represent it in terms of an integral given by

$$\log L_\delta^*(\sigma)=\log L_\delta^*(\sigma_0)+\int_{\sigma_0}^{\sigma}\frac{U(x)}{x}dx\ ,\quad \sigma\geqq\sigma_0\ , \tag{1.2}$$

where $U(x)$ is a positive real valued indefinitely increasing function of x.

In this paper we are mainly interested in studying certain growth relations of $U(\sigma)$ and the generalized logarithmic mean function $L_\delta^*(\sigma)$ relative to each other.

2. Main Results

Theorem 1. *For $m>0$, let*

$$I_1=\int_{\sigma_0}^{\infty}\frac{\log L_\delta^*(\sigma)}{\sigma^{m+1}}d\sigma\ ,\qquad I_2=\int_{\sigma_0}^{\infty}\frac{U(\sigma)}{\sigma^{m+1}}d\sigma\ .$$

Then I_1 and I_2 converge or diverge together.

Proof. From (1.2), we have

$$\int_{\sigma_0}^{\sigma} \frac{U(x)}{x} dx = \log L_\delta^*(\sigma) - \log L_\delta^*(\sigma_0) .$$

Hence, for any $u \geqq \sigma_0$,

$$(2.1) \qquad \int_{\sigma_0}^{u} \frac{d\sigma}{\sigma^{m+1}} \int_{\sigma_0}^{\sigma} \frac{U(x)}{x} dx = \int_{\sigma_0}^{u} \{\log L_\delta^*(\sigma) - \log L_\delta^*(\sigma_0)\} \frac{d\sigma}{\sigma^{m+1}}$$
$$= \frac{\log L_\delta^*(u) - \log L_\delta^*(\sigma_0)}{-mu^m} + \frac{1}{m} \int_{\sigma_0}^{u} \frac{U(\sigma)}{\sigma^{m+1}} d\sigma .$$

Also,

$$(2.2) \qquad \int_{\sigma_0}^{u} \frac{d\sigma}{\sigma^{m+1}} \int_{\sigma_0}^{\sigma} \frac{U(x)}{x} dx = \int_{\sigma_0}^{u} \frac{\log L_\delta^*(\sigma)}{\sigma^{m+1}} d\sigma + \frac{\log L_\delta^*(\sigma_0)}{m} \left(\frac{1}{u^m} - \frac{1}{\sigma_0^{\,m}} \right) .$$

(2.1) and (2.2) give,

$$\int_{\sigma_0}^{u} \frac{\log L_\delta^*(\sigma)}{\sigma^{m+1}} d\sigma + \frac{\log L_\delta^*(\sigma_0)}{m} \left(\frac{1}{u^m} - \frac{1}{\sigma_0^{\,m}} \right) = \frac{\log L_\delta^*(u) - \log L_\delta^*(\sigma_0)}{-mu^m} + \frac{1}{m} \int_{\sigma_0}^{u} \frac{U(\sigma)}{\sigma^{m+1}} d\sigma ,$$

or,

$$(2.3) \qquad m \int_{\sigma_0}^{\sigma} \frac{\log L_\delta^*(x)}{x^{m+1}} dx - \frac{\log L_\delta^*(\sigma_0)}{\sigma_0^{\,m}} + \frac{\log L_\delta^*(\sigma)}{\sigma^m} = \int_{\sigma_0}^{\sigma} \frac{U(x)}{x^{m+1}} dx .$$

Let us now suppose that I_1 is convergent, then, for any $\varepsilon > 0$ and for sufficiently large σ,

$$\varepsilon > \int_{\sigma}^{2\sigma} \frac{\log L_\delta^*(x)}{x^{m+1}} dx > \frac{\log L_\delta^*(\sigma)}{m\sigma^m} (1 - 2^{-m}) .$$

So,

$$\frac{\log L_\delta^*(\sigma)}{\sigma^m} \to 0 , \qquad \text{as } \sigma \to \infty .$$

Hence, from (2.3), we find that I_2 is also convergent.

Now, if I_2 is convergent, then, from (2.3), we get

$$(2.4) \qquad m \int_{\sigma_0}^{\sigma} \frac{\log L_\delta^*(x)}{x^{m+1}} dx + \frac{\log L_\delta^*(\sigma)}{\sigma^m} < k$$

for some $k > 0$. But

$$\int_{\sigma_0}^{\sigma} \frac{\log L_\delta^*(x)}{x^{m+1}} dx > \frac{\log L_\delta^*(\sigma_0)}{m} \left(\frac{1}{\sigma_0^{\,m}} - \frac{1}{\sigma^m} \right) > 0 ,$$

so, both terms on the left hand side of (2.4) are positive. Hence I_1 is also convergent. Thus I_1 converges if, and only if, I_2 converges. Appealing to Modus

Tollendo Tollen's [1, p. 32] the divergence part of this theorem follows from its convergence part.

Theorem 2. *Let*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log U(\sigma)}{F(\sigma)} = \frac{P}{Q}, \quad 0 \leqq Q \leqq P \leqq \infty ,$$

where $F(\sigma)$ is a logarithmico exponential function of σ, such that, as $\sigma\to\infty$, $F(k\sigma)\approx F(\sigma)$ (k is a constant >0) and $\log\log\sigma = o(F(\sigma))$. Then

$$\liminf_{\sigma\to\infty} \frac{\sigma F'(\sigma)\log L_\delta^*(\sigma)}{U(\sigma)} \leqq \frac{1}{P} \leqq \frac{1}{Q} \leqq \limsup_{\sigma\to\infty} \frac{\sigma F'(\sigma)\log L_\delta^*(\sigma)}{U(\sigma)} . \tag{2.5}$$

In order to prove this theorem we need the following lemma:

Lemma 1. *Let*

$$\phi(x) = A + \int_{x_0}^{x} \frac{g(t)}{t} dt , \tag{2.6}$$

where $g(x)$ is a positive non-decreasing function of x for $x \geqq x_0$ and A is a constant >0. If

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log g(x)}{F(x)} = \frac{M}{N}, \quad 0 \leqq N \leqq M \leqq \infty .$$

Then

$$\liminf_{x\to\infty} \frac{g(x)}{x\phi(x)F'(x)} \leqq N \leqq M \leqq \limsup_{x\to\infty} \frac{g(x)}{x\phi(x)F'(x)} . \tag{2.7}$$

Proof. We have

$$\phi(x) = A + \int_{x_0}^{x} \frac{g(t)}{t} dt \leqq g(x)\log x + \text{const.}$$

So,

$$\limsup_{x\to\infty} \frac{\log\phi(x)}{F(x)} \leqq \limsup_{x\to\infty} \left\{ \frac{\log g(x)}{F(x)} + \frac{\log\log x + \text{cost.}}{F(x)} \right\} = M .$$

Now,

$$\phi(2x) \geqq \int_{x}^{2x} \frac{g(t)}{t} dt \geqq g(x)\log 2 .$$

Therefore,

$$\limsup_{x\to\infty} \frac{\log\phi(2x)}{F(2x)} \geqq \limsup_{x\to\infty} \left\{ \frac{\log g(x)}{F(x)} \cdot \frac{F(x)}{F(2x)} + \frac{\log\log 2}{F(2x)} \right\} = M .$$

Hence

$$\limsup_{x\to\infty} \frac{\log \phi(x)}{F(x)} = M .$$

Similarly,

$$\liminf_{x\to\infty} \frac{\log \phi(x)}{F(x)} = N .$$

Now, from (2.6), we get, for $x \geqq x_0$,

$$\frac{\phi'(x)}{\phi(x)} = \frac{g(x)}{x\phi(x)} .$$

Integrating in the Lebesgue sense between x_0 and x, we find

$$\log \phi(x) = \int_{x_0}^{x} \frac{g(t)}{t\phi(t)} dt + \text{const.} \tag{2.8}$$

Let,

$$\lim_{x\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{g(x)}{x\phi(x)F'(x)} = \frac{C}{D}, \quad 0 \leqq D \leqq C \leqq \infty .$$

We first suppose that $0 < D$, $C < \infty$. Then, for any $\varepsilon > 0$ and sufficiently large x,

$$(D-\varepsilon)F'(x) < \frac{g(x)}{x\phi(x)} < (C+\varepsilon)F'(x) .$$

Integrating in the Lebesgue sense, we get

$$(D-\varepsilon)(1-o(1)) < \frac{\log \phi(x)}{F(x)} < (C+\varepsilon)(1-o(1)) ,$$

or,

$$D \leqq N \leqq M \leqq C , \tag{2.9}$$

which also holds, when $D=0$ or $C=\infty$. If $D=\infty$, then so is C and $\lim_{x\to\infty} (g(x)/x\phi(x)F'(x)) = \infty$. So, taking an arbitrary large real number in place of $D-\varepsilon$ and proceeding as above, we obtain $M=N=\infty$. Similarly, if $C=0$, it can be shown that $M=N=0$. Hence, for $0 \leqq D \leqq C \leqq \infty$, (2.9) implies (2.7).

Proof of theorem 2. Replacing $\phi(\sigma)$ by $\log L_\delta{}^*(\sigma)$ and $g(\sigma)$ by $U(\sigma)$ in (2.7), we get Theorem 2.

Theorem 3. *Let*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{l_{p+1}L_\delta{}^*(\sigma)}{l_q\sigma} = \frac{H}{h}, \quad 0 \leqq h \leqq H \leqq \infty .$$

Then

$$\liminf_{\sigma\to\infty} \frac{(l_1L_\delta{}^*(\sigma))(l_2L_\delta{}^*(\sigma))\cdots(l_pL_\delta{}^*(\sigma))}{U(\sigma)(l_1\sigma)(l_2\sigma)\cdots(l_{q-1}\sigma)} \tag{2.10}$$
$$\leqq\frac{1}{H}\leqq\frac{1}{h}\leqq \limsup_{\sigma\to\infty}\frac{(l_1L_\delta{}^*(\sigma))(l_2L_\delta{}^*(\sigma))\cdots(l_pL_\delta{}^*(\sigma))}{U(\sigma)(l_1\sigma)(l_2\sigma)\cdots(l_{q-1}\sigma)},$$

where $l_k\sigma$ denotes k-th iterate of $\log \sigma$.

The proof of this theorem is based on the following lemma:

Lemma 2. *Let*

$$G(x)=A+\int_{x_0}^{x}\frac{\Psi(t)}{t}dt, \tag{2.11}$$

where $\Psi(x)$ is a positive and non-decreasing function of x for $x\geqq x_0$. If

$$\lim_{x\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{l_pG(x)}{l_qx}=\frac{T}{S},\quad 0\leqq S\leqq T\leqq\infty .$$

Then

$$\liminf_{x\to\infty}\frac{G(x)(l_1G(x))(l_2G(x))\cdots(l_{p-1}G(x))}{\Psi(x)(l_1x)(l_2x)\cdots(l_{q-1}x)} \tag{2.12}$$
$$\leqq\frac{1}{T}\leqq\frac{1}{S}\leqq\limsup_{x\to\infty}\frac{G(x)(l_1G(x))(l_2G(x))\cdots(l_{p-1}G(x))}{\Psi(x)(l_1x)(l_2x)\cdots(1_{q-1}x)}.$$

Proof. Let

$$\lim_{x\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{G(x)(l_1G(x))(l_2G(x))\cdots(l_{p-1}G(x))}{\Psi(x)(l_1x)(l_2x)\cdots(l_{q-1}x)}=\frac{c}{d},\quad 0\leqq d\leqq c\leqq\infty ,$$

and suppose that $d>0$. Then, for any $\varepsilon>0$ and $x\geqq x_0$, we have

$$G(x)(l_1G(x))(l_2G(x))\cdots(l_{p-1}G(x))>(d-\varepsilon)\Psi(x)(l_1x)(l_2x)\cdots(l_{q-1}x).$$

Differentiating (2.11), we get

$$G'(x)=\frac{\Psi(x)}{x}.$$

Therefore,

$$\frac{G'(x)}{G(x)(l_1G(x))(l_2G(x))\cdots(l_{p-1}G(x))}<\frac{\Psi(x)}{(d-\varepsilon)\Psi(x)x(l_1x)(l_2x)\cdots(l_{q-1}x)}. \tag{2.13}$$

Integrating (2.13) in the Lebesgue sense, between x_0 and x, we obtain

$$l_pG(x)=\int_{x_0}^{x}\frac{G'(t)}{G(t)(l_1G(t))(l_2G(t))\cdots(l_{p-1}G(t))}dt<\frac{l_qx}{d-\varepsilon},$$

or,

$$\frac{l_p G(x)}{l_q x} < \frac{1}{d-\varepsilon} .$$

So,

$$d \leqq \frac{1}{T} , \tag{2.14}$$

which also holds when $d=0$. If $d=\infty$, the above argument with an arbitrary large real number instead of $d-\varepsilon$ gives $T=0$. Hence, for $0 \leqq d \leqq \infty$, (2.14) gives the left hand side of (2.12). Similarly, the right hand side follows.

Proof of theorem 3. Replacing $G(\sigma)$ and $\Psi(\sigma)$ by $\log L_\delta{}^*(\sigma)$ and $U(\sigma)$, respectively, we get the required result.

Theorem 4. *If $F(\sigma)$ is a logarithmico exponential function of σ, such that, $F(k\sigma) \approx F(\sigma)$ and $\log L_\delta{}^*(\sigma) \approx F(\sigma)$. Then,*

$$\lim_{\sigma\to\infty} \frac{\log L_\delta{}^*(\sigma)}{U(\sigma)} = \infty .$$

Proof. For any $\varepsilon > 0$ and $\sigma \geqq \sigma_0$,

$$\frac{F(\sigma)}{(1-\varepsilon)^2} > \frac{F(2\sigma)}{1-\varepsilon} > \log L_\delta{}^*(2\sigma) = \log L_\delta{}^*(\sigma) + \int_\sigma^{2\sigma} \frac{U(x)}{x} dx \geqq \log L_\delta{}^*(\sigma) + U(\sigma) \log 2 ,$$

and

$$\frac{F(\sigma)}{(1+\varepsilon)^2} < \frac{F(\sigma)}{1+\varepsilon} < \log L_\delta{}^*(\sigma) .$$

So,

$$\{(1-\varepsilon)^{-2} - (1+\varepsilon)^{-2}\} F(\sigma) > U(\sigma) \log 2.$$

Thus,

$$\lim_{\sigma\to\infty} \frac{U(\sigma)}{F(\sigma)} = 0 .$$

Since, $F(\sigma) \approx \log L_\delta{}^*(\sigma)$. Hence

$$\lim_{\sigma\to\infty} \frac{\log L_\delta{}^*(\sigma)}{U(\sigma)} = \infty .$$

Thus the proof of Theorem 4 follows.

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References

[1] Suppes, P.: *Introduction to logic*, Van Nostrand (East-West edition), 1969.
[2] Vaish, S. K.: *Generalized logarithmic means of an entire Dirichlet series* (*I*), Arch. Math., No. 4, **16** (1980), 231-238.

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