YOKOHAMA MATHEMATICAL
JOURNAL VOL. 29, 1981

FACTOR STATE EXTENSION ON NUCLEAR C^*ALGEBRAS

By

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(Received March 19, 1981. Revised July 9, 1981)

ABSTRACT. Let $\mathscr{B}$ be a C^*-subalgebra of a C^*-algebra $\mathscr{A}$. Every factor state φ on $\mathscr{B}$ with $\overline{\pi_\varphi(\mathscr{B})}^w$, the weak-operator closure of the GNS representation $\pi_\varphi(\mathscr{B})$ induced by φ, injective extends to a factor state on $\mathscr{A}$.

Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{B}$ a C^*-subalgebra of $\mathscr{A}$ with the identity element. In this note we show that every factor state on $\mathscr{B}$ extends to a factor state on $\mathscr{A}$ if $\mathscr{B}$ is nuclear. The theorem is based on a theorem (Theorem 1) proposed by S. Sakai in Wabash conference 1973.

Several attempts have been undertaken on the factor state extension problem (see [5], [6] for example). The author would like to thank Professor John Bunce for kindly bringing Sakai's lecture in Wabash conference to his attention. Furthermore, using a result in [1] we show every factor state φ on $\mathscr{B}$ with $\overline{\pi_\varphi(\mathscr{B})}^w$, the weak-operator closure of $\pi_\varphi(\mathscr{B})$, injective extends to a factor state on $\mathscr{A}$, where π_φ is the GNS representation induced by φ. If there exists a projection of norm one from $\mathscr{A}$ onto $\mathscr{B}$, then it follows from Theorem 1 that every factor state on $\mathscr{B}$ extends to a factor state on $\mathscr{A}$.

The author would also like to thank Professor S. Sakai for letting the author include his result in this paper.

Theorem 1. *Let φ be a factor state on $\mathscr{B}$ and $\{\pi_\varphi \mathscr{H}_\varphi\}$ be the GNS representation induced by φ with $\varphi(b)=\langle\pi_\varphi(b)f_\varphi, f_\varphi\rangle$ for all b in $\mathscr{B}$. Suppose that there exists a positive linear map Q of $\mathscr{A}$ into $\overline{\pi_\varphi(\mathscr{B})}^w$ $(\equiv\mathscr{R})$, the weak-operator closure of $\pi_\varphi(\mathscr{B})$, satisfying the following*:

$$(1)\qquad \begin{aligned} &Q(b)=\pi_\varphi(b) \quad (b\in\mathscr{B}) \\ &Q(b_1xb_2)=Q(b_1)Q(x)Q(b_2) \quad (b_1, b_2\in\mathscr{B}) \quad (x\in\mathscr{A}) . \end{aligned}$$

Then φ extends to a factor state on $\mathscr{A}$.

Proof. Let Q be a positive linear map satisfying (1) and the linear functional, $x\mapsto\langle Q(x)f_\varphi, f_\varphi\rangle$ denoted by φ_Q. Consider $\mathscr{E}=\{Q' \mid Q'$: positive linear map of $\mathscr{A}$ into

AMS (MOS) subject classifications (1980): 46L30, 46L35.

$\mathscr{R}$ satisfying (1)} and $\mathscr{S}=\{\varphi_{Q'} \mid Q' \in \mathscr{E}\}$. $\mathscr{E}$ is nonempty by the assumption of the theorem, and is σ-compact, convex in $\mathscr{B}(\mathscr{A}, \mathscr{R})$ in the topology of simple weak* convergence, where $\mathscr{B}(\mathscr{A}, \mathscr{R})$ is the set of all bounded linear maps from $\mathscr{A}$ into $\mathscr{R}$. $\mathscr{S}$ is weak* compact and convex in the dual space of $\mathscr{A}$, and has extreme points. Let ψ be an extreme point of $\mathscr{S}$. We show $\psi(\equiv\varphi_{Q_0})$ is a factor state on $\mathscr{A}$.

Let $\{\pi_{Q_0}, \mathscr{H}_{Q_0}\}$ be the GNS representation due to φ_{Q_0} with $\varphi_{Q_0}(x)=\langle\pi_{Q_0}(x)f_{Q_0}, f_{Q_0}\rangle$ for all x in $\mathscr{A}$. We show that the center of the weak-operator closure of $\pi_{Q_0}(\mathscr{A})$ consists of scalars only.

Since $\varphi_{Q_0}|_{\mathscr{B}}=\varphi|_{\mathscr{B}}$, $\mathscr{H}_\varphi$ can be embedded into $\mathscr{H}_{Q_0}$. Suppressing the embedding relationship, we consider $\mathscr{H}_\varphi$ as a closed subspace of $\mathscr{H}_{Q_0}$ and denote the orthogonal projection of $\mathscr{H}_{Q_0}$ onto $\mathscr{H}_\varphi$ by E.

Consider a linear map Q^0 of $\mathscr{A}$ into $\mathscr{R}$ defined by $Q^0(x)=E\pi_{Q_0}(x)E$ for x in $\mathscr{A}$. We note that $E\pi_{Q_0}(b)E=\pi_{Q_0}(b)E=\pi_\varphi(b)$ for all b in $\mathscr{B}$, and Q^0 is a positive linear map of $\overline{\pi_{Q_0}(\mathscr{A})}^w$ onto $\mathscr{R}$. In fact, $Q^0=Q_0$. For x in $\mathscr{A}$ and y, z in $\mathscr{B}$ we have

$$\begin{aligned}\langle(E\pi_{Q_0}(x)E)\pi_{Q_0}(y)f_{Q_0}, \pi_{Q_0}(z)f_{Q_0}\rangle &= \varphi_{Q_0}(z^*xy)\\ &=\langle Q_0(z^*xy)f_\varphi, f_\varphi\rangle\\ &=\langle Q_0(x)Q_0(y)f_\varphi, Q_0(z)f_\varphi\rangle\\ &=\langle Q_0(x)\pi_\varphi(y)f_\varphi, \pi_\varphi(y)f_\varphi\rangle\\ &=\langle Q_0(x)\pi_{Q_0}(y)f_{Q_0}, \pi_{Q_0}(y)f_{Q_0}\rangle .\end{aligned}$$

Hence, $E\pi_{Q_0}(x)E=Q_0(x)$ for all x in $\mathscr{A}$. Suppose P is a central projection in $\overline{\pi_{Q_0}(\mathscr{A})}^w$, the weak-operator closure of $\pi_{Q_0}(\mathscr{A})$. Since, for b in $\mathscr{B}$,

$$(EPE)(E\pi_{Q_0}(b)E)=EP\pi_{Q_0}(b)E=E\pi_{Q_0}(b)PE=(E\pi_{Q_0}(b)E)(EPE) ,$$

EPE is in the center of $\mathscr{R}$. Thus $EPE=\lambda I_{\mathscr{R}}$ for $0\leqq\lambda\leqq1$.

The positive linear functional η defined on $\mathscr{A}$ by $\eta(x)=\langle P\pi_{Q_0}(x)f_{Q_0}, f_{Q_0}\rangle$ satisfies $\eta\leqq\varphi_{Q_0}$, and $\eta(1)=\langle Pf_{Q_0}, f_{Q_0}\rangle=\langle EPEf_{Q_0}, f_{Q_0}\rangle=\lambda$. Let $\theta=\varphi_{Q_0}-\eta$. Then $\theta(x)=\langle(I-P)\pi_{Q_0}(x)f_{Q_0}, f_{Q_0}\rangle$ for all x in $\mathscr{A}$. If $\lambda=0$ or $\lambda=1$, then $\varphi_{Q_0}=\theta$ or $\varphi_{Q_0}=\eta$. Thus $P=0$ or $P=I$.

Suppose λ is neither 0 nor 1. Define $Q_1(x)=(1/\lambda)EP\pi_{Q_0}(x)E$, for all x in $\mathscr{A}$. Q_1 is a positive linear map of $\mathscr{A}$ into $\mathscr{R}$, and for b in $\mathscr{B}$, we have

$$Q_1(b)=\frac{1}{\lambda}EP\pi_{Q_0}(b)E=E\pi_{Q_0}(b)E=Q_0(b) .$$

For x in $\mathscr{A}$ and b in $\mathscr{B}$, we have

$$\begin{aligned}Q_1(xb)&=\frac{1}{\lambda}EP\pi_{Q_0}(x)\pi_{Q_0}(b)E\\&=\frac{1}{\lambda}(EP\pi_{Q_0}(x)E)(E\pi_{Q_0}(b)E)\\&=Q_1(x)\frac{1}{\lambda}(EPE\pi_{Q_0}(b)E)\\&=Q_1(x)Q_1(b)\,.\end{aligned}$$

Hence, Q_1 satisfies condition (1), and $\varphi_{Q_1}=(1/\lambda)\eta$. Similarly, $(1/1-\lambda)\theta=\varphi_{Q_2}$, where Q_2 is in $\mathscr{E}$. Since φ_{Q_0} is extreme in $\mathscr{S}$ and $\varphi_{Q_0}=\lambda\varphi_{Q_1}+(1-\lambda)\varphi_{Q_2}$, it follows that $\varphi_{Q_0}=\varphi_{Q_1}=\varphi_{Q_2}=(1/\lambda)\eta$. Therefore, $P=\lambda I$ and $\lambda=1$ or $\lambda=0$, which is a contradiction to the assumption that λ is neither 0 nor 1. Q.E.D.

Theorem 2. *Let $\mathscr{B}$ be a nuclear C^*-subalgebra of $\mathscr{A}$. Then every factor state of $\mathscr{B}$ extends to a factor state of $\mathscr{A}$.*

Proof. Since the second dual $\mathscr{B}^{**}$ of $\mathscr{B}$ is injective in the second dual $\mathscr{A}^{**}$ of $\mathscr{A}$, when $\mathscr{B}^{**}$ is considered as a subalgebra of $\mathscr{A}^{**}$. Then there exists a norm one projection P of $\mathscr{A}^{**}$ onto $\mathscr{B}^{**}$. Let φ be a factor state of $\mathscr{B}$ and $\{\pi_\varphi, \mathscr{H}_\varphi\}$ be the GNS construction induced by φ. π_φ extends uniquely to $\pi_\varphi{}^{**}$ on $\mathscr{B}^{**}$ with range in $\overline{\pi_\varphi(\mathscr{B})}^w=\mathscr{R}$.

Let $Q=\pi_\varphi{}^{**}P|_{\mathscr{A}}$. We have

(i) Q is a positive linear map of $\mathscr{A}$ into $\mathscr{R}$;

(ii) $Q(b)=\pi_\varphi{}^{**}P(b)=\pi_\varphi{}^{**}(b)=\pi_\varphi(b)$ $(b\in\mathscr{B})$

(iii) $Q(axb)=\pi_\varphi{}^{**}(P(a)P(x)P(b))$
$=Q(a)Q(x)Q(b)$
$(x\in\mathscr{A},\ a, b\in\mathscr{B})$

Therefore, by the previous theorem φ can extend to a factor state on $\mathscr{A}$. Q.ED.

Actually it would be sufficient to prove the above theorem with the assumption that $\overline{\pi_\varphi(\mathscr{B})}^w$ is injective. We show this in the following:

Theorem 3. *In the same notation as in Theorem 2, if $\overline{\pi_\varphi(\mathscr{B})}^w$ $(=\mathscr{R})$ is an injective factor, then φ extends to a factor state on $\mathscr{A}$.*

Note. Due to the work of Connes [2], $\mathscr{R}$ is simply a hyperfinite factor, if the Hilbert space it acts on is separable.

Proof. Let ψ be a state on $\mathscr{A}$ extending φ, and $\{\pi_\psi, \mathscr{H}_\psi\}$ the GNS construction induced by ψ. Let V be an embedding of $\mathscr{H}_\varphi$ into $\mathscr{H}_\psi$, since ψ extends φ. By suppressing the embedding we consider $\mathscr{H}_\psi$, as a closed subspace of $\mathscr{H}_\psi$, and

denote the orthogonal projection of $\mathscr{H}_\phi$ onto $\mathscr{H}_\varphi$ by E. We note that $\pi_\phi(b)E= E\pi_\phi(b)E=\pi_\varphi(b)$ for b in $\mathscr{B}$. Since the compression map $x\to ExE$ of $\overline{\pi_\phi(\mathscr{A})}^w$ is completely positive, $\overline{E\pi_\phi(\mathscr{A})E}^w$ becomes a C^*-algebra when provided with the Banach space operations and *-operation, and the new product $(ExE, EyE)\to ExyE$ [1, Theorem 3.1]. However, $ExEyE=ExyE$ for x, y in $\overline{\pi_\phi(\mathscr{B})}^w$. It follows that there exists a projection P of norm one of $\overline{E\pi_\phi(\mathscr{A})}^wE$ onto $\overline{\pi_\varphi(\mathscr{B})}^w$ with $P(E\pi_\phi(b)E)= \pi_\varphi(b)$ for all b in $\mathscr{B}$. Therefore, by Theorem 1 φ extends to a factor state on $\mathscr{A}$.
Q.E.D.

Remark 4. Suppose that there exists a projection of norm one from a unital C^*-algebra $\mathscr{A}$ onto its C^*-subalgebra $\mathscr{B}$. Then it is an easy consequence of Theorem 1 that every factor state on $\mathscr{B}$ extends to a factor state of $\mathscr{A}$. It would be interesting to investigate the factor state extension problem from a C^*-subalgebra $\mathscr{B}$ to the full C^*-algebra $\mathscr{A}$ which is UHF in Glimm's sense [3].

Addendum

Four months after the completion of this paper, in an American Mathematical Society's Summer Institute on Operator Algebras and Applications at Kingston, Ontario, Canada, John Bunce announced a result similar to Theorem 3, and it is slightly more general than Theorem 3 and proved by a somewhat different method. (For details please see the forthcoming Conference Proceedings of AMS Summer Institute of Operator Algebras and Applications, 1980).

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