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ON SPIRAL-LIKE FUNCTIONS OF ORDER α AND TYPE β

By

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1. Introduction

A function f analytic in the open disk $E=\{z \mid |z|<1\}$ is said to be λ-spiral-like if $f(0)=0$, $f'(0)=1$, and $\operatorname{Re}\{e^{i\lambda}zf'(z)/f(z)\}>0$ for some real $\lambda(|\lambda|<\pi/2)$. Let S^{λ} be the class of all such functions. It was shown by Špaček [10] that spiral-like functions are univalent in E. In recent years, such functions have been the source of useful and important counter-examples in geometric function theory [1, 4].

In [6], Libera introduced the notion of order α $(0\leqq\alpha<1)$ for λ-spiral-like functions in E. Motivated by [3], we, in the present paper, introduce the concept of 'type' for the class of λ-spiral-like functions of order α.

A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, is a λ-spiral-like function of order α and type β, $f\in S^{\lambda}(\alpha,\beta)$, if and only if for all $z\in E$, the inequality

$$\left|\frac{zf'(z)/f(z)-1}{2\beta(zf'(z)/f(z)-1+(1-\alpha)e^{-i\lambda}\cos\lambda)-(zf'(z)/f(z)-1)}\right|<1\,, \tag{1.1}$$

holds for some $\alpha\in[0,1)$, $\beta\in(0,1]$, and $\lambda\in(-\pi/2,\pi/2)$. Since $S^{\lambda}(\alpha,\beta)\subset S^{\lambda}$, it follows that the functions in $S^{\lambda}(\alpha,\beta)$ are univalent.

By specializing α, β, and λ we obtain several subclasses of univalent functions. For instance, $S^{\lambda}(\alpha,1)\equiv S^{\lambda}(\alpha)$; $S^{\lambda}(0,(2-\cos\lambda)/2)\equiv H(\lambda)$, $|\lambda|<\pi/2$; $S^{\lambda}(0,(2M-1)/2M)\equiv F_{\lambda,M}$, $M>1/2$; and $S^{0}(\alpha,\beta)\equiv S^{*}(\alpha,\beta)$ are the classes introduced and studied, respectively, by Libera [6], Goel [2], Kulshrestha [5], and Juneja and Mogra [3]. Further, replacement of α by $(1-\beta+2\alpha\beta)/(1+\beta)$ and β by $(1+\beta)/2$ in the class $S^{\lambda}(\alpha,\beta)$, gives the class, $S^{\lambda}_{\alpha,\beta}$, defined by Makówka [7].

Since our class $S^{\lambda}(\alpha,\beta)$ includes various subclasses as noticed above, a study of its various properties will lead to a unified study of these subclasses. In the present paper, we shall give at first a representation formula for the class $S^{\lambda}(\alpha,\beta)$. We deal next with the distortion properties and coefficient bounds for a λ-spiral function of order α and type β. Finally, γ-spiral radius is obtained for $S^{\lambda}(\alpha,\beta)$.

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In particular, we deduce some results for the classes $S^\lambda(\alpha)$, $H(\lambda)$, $F_{\lambda,M}$ and $S^\lambda_{\alpha,\beta}$; the analogues of which have not been obtained by earlier authors.

2. A Representation for $S^\lambda(\alpha, \beta)$

Let A denote the class of functions ϕ which are analytic in E and which satisfy $|\phi(z)|\leqq 1$ for all z in E. We first give the following lemma.

Lemma. *If a function* $H(z)=1+\sum_{n=1}^{\infty} c_n z^n$, *analytic in* E, *satisfies the condition*

$$|(H(z)-1)/\{2\beta(H(z)-1+(1-\alpha)e^{-i\lambda}\cos\lambda)-(H(z)-1)\}|<1\,, \tag{2.1}$$

for some $\alpha\in[0, 1)$, $\beta\in(0, 1]$, $\lambda\in(-\pi/2, \pi/2)$, *and for all* $z\in E$, *then*

$$H(z)=\frac{1+((2\beta-1)-2\beta(1-\alpha)e^{i\lambda}\cos\lambda)z\phi(z)}{1+(2\beta-1)z\phi(z)} \tag{2.2}$$

for some $\phi\in A$. *Conversely, a function* H *given by* (2.2) *for some* $\phi\in A$ *is analytic in* E *and satisfies* (2.1) *for all* z *in* E.

Proof. The first half of the Lemma is obtained immediately by an application of Schwarz's Lemma; and the converse part follows from the observation that the function

$$w=\frac{1+((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)z}{1+(2\beta-1)z}$$

maps $|z|<1$ onto the disk

$$|(1-w)/\{2\beta((w-1)+(1-\alpha)e^{-i\lambda}\cos\lambda)-(w-1)\}|<1$$

in the w-plane.

Theorem 1. *A function* $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, *analytic in* E, *is in the class* $S^\lambda(\alpha, \beta)$ *if and only if*

$$f(z)=z\exp\left\{-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\int_0^z\frac{\phi(t)dt}{1+(2\beta-1)t\phi(t)}\right\}, \tag{2.3}$$

for some $\phi\in A$.

Proof. First suppose $f(z)\in S^\lambda(\alpha, \beta)$. Noting that $zf'(z)/f(z)$ satisfies the hypothesis of the first part of the Lemma, we see that

$$\frac{zf'(z)}{f(z)}=\frac{1+((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)z\phi(z)}{1+(2\beta-1)z\phi(z)}$$

for some $\phi \in A$. Thus we have

$$\frac{f'(z)}{f(z)}-\frac{1}{z}=\frac{-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot\phi(z)}{1+(2\beta-1)z\phi(z)}. \tag{2.4}$$

An integration from 0 to z in (2.4) followed by an exponentiation leads to (2.3). Conversely, if (2.3) holds, then

$$\frac{zf'(z)}{f(z)}=\frac{1+((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)z\phi(z)}{1+(2\beta-1)z\phi(z)}.$$

Now the theorem follows by the converse part of the Lemma.

Remarks. (i) Putting $\beta=1$, we obtain a representation formula determined by Libera [6].

(ii) The representation formula determined by Juneja and Mogra [3] for starlike functions of order α and type β can be obtained by putting $\lambda=0$ in Theorem 1.

(iii) $\alpha=0$ and $\beta=(2-\cos\lambda)/2$ in Theorem 1 gives the result obtained by Goel [2].

(iv) Replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$ and β by $(1+\beta)/2$ in Theorem 1, we get the corresponding result obtained by Makówka [7].

(v) By taking appropriate values of α, β, λ in Theorem 1, we obtain the corresponding representation formulae for the functions of the classes introduced by Spacék [10], Kulshrestha [5], Padmanabhan [8], and many others.

3. A sufficient condition.

We now establish a sufficient condition for a function to be in $S^{\lambda}(\alpha, \beta)$.

Theorem 2. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f(z)\in S^{\lambda}(\alpha,\beta)$, if for some $\alpha\in[0,1)$ and $\lambda\in(-\pi/2,\pi/2)$,*

$$\sum_{n=2}^{\infty}\{2n(1-\beta)-1+|(1-2\beta)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|\leqq 2\beta(1-\alpha)\cos\lambda, \quad \text{whenever}\quad \beta\in(0,1/2], \tag{3.1}$$

$$\sum_{n=2}^{\infty}\{(n-1)+|(2\beta-1)(n-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|\leqq 2\beta(1-\alpha)\cos\lambda, \quad \text{whenever}\quad \beta\in[1/2,1], \tag{3.2}$$

holds.

Proof. Let $|z|=r<1$. Noting that

$$|zf'(z)-f(z)|<\sum_{n=2}^{\infty}(n-1)|a_n|r, \tag{3.3}$$

and

(3.4) $$|2\beta((zf'(z)-f(z))+(1-\alpha)e^{-i\lambda}\cos\lambda\cdot f(z))-(zf'(z)-f(z))|$$
$$\geqq\{2\beta(1-\alpha)\cos\lambda-\sum_{n=2}^{\infty}(1-2\beta)n|a_n|-\sum_{n=2}^{\infty}|(1-2\beta)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\,|a_n|\},$$

we see that

$$|zf'(z)-f(z)|-|2\beta((zf'(z)-f(z))+(1-\alpha)e^{-i\lambda}\cos\lambda\cdot f(z))-(zf'(z)-f(z))|$$
$$\leqq\sum_{n=2}^{\infty}\{2n(1-\beta)-1+|(1-2\beta)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|-2\beta(1-\alpha)\cos\lambda,$$

provided $0<\beta\leqq 1/2$. The last quantity is $\leqq 0$ by (3.1), so that $f(z)\in S^{\lambda}(\alpha,\beta)$. For the second part, we assume that (3.2) holds for $\beta\in[1/2,1]$. In this case,

(3.5) $$|2\beta((zf'(z)-f(z))+(1-\alpha)e^{-i\lambda}\cos\lambda\cdot f(z))-(zf'(z)-f(z))|$$
$$\geqq\{2\beta(1-\alpha)\cos\lambda-\sum_{n=2}^{\infty}|(2\beta-1)(n-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|\,|a_n|\}r.$$

Now the theorem follows, as before, from (3.3), (3.5) and (3.2).

Corollary 1. *A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, is a λ-spiral-like function of order α if*

$$\sum_{n=2}^{\infty}\{(n-1)+|(n-1)+2(1-\alpha)e^{-i\lambda}\cos\lambda|\}|a_n|\leqq 2(1-\alpha)\cos\lambda,$$

for some $\alpha\in[0,1)$, $\lambda\in(-\pi/2,\pi/2)$.

Corollary 2. *A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, is in the class $H(\lambda)$ if*

$$\sum_{n=2}^{\infty}\{(n-1)+|(1-\cos\lambda)(n-1)+(2-\cos\lambda)e^{-i\lambda}\cos\lambda|\}|a_n|\leqq(2-\cos\lambda)\cos\lambda,$$

for some $\lambda\in(-\pi/2,\pi/2)$.

Corollary 3. *A function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, analytic in E, is in the class $F_{\lambda,M}$ if for some $\lambda\in(-\pi/2,\pi/2)$,*

$$\sum_{n=2}^{\infty}\{(n-M)+\sqrt{(1-M)^2+(2M-1)\cos^2\lambda}\}|a_n|\leqq(2M-1)\cos\lambda,$$
whenever $1/2<M\leqq 1$,

$$\sum_{n=2}^{\infty}\{(n-1)M+|(M-1)(n-1)+(2M-1)e^{-i\lambda}\cos\lambda|\}|a_n|\leqq(2M-1)\cos\lambda,$$
whenever $M\geqq 1$,

holds.

Remarks. (i) For $\lambda=0$ in Theorem 2, we obtain the sufficient condition determined by Juneja and Mogra [3].

(ii) Replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$, β by $(1+\beta)/2$ in Theorem 2 we obtain the corresponding sufficient condition for a function $f(z)$ to be in the class introduced by Maköwka [7].

(iii) By fixing the parameters α, β, and λ in Theorem 2, we can obtain sufficient conditions for a function to be in the classes introduced by Padmanabhan [8], Wright [11], and others.

4. Distortion Theorems

Theorem 3. *Let $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be analytic in E. If $f(z)\in S^\lambda(\alpha,\beta)$, then for $|z|=r$, $0<r<1$, and for all $\alpha\in[0,1)$, $\beta\in(0,1/2)\cup(1/2,1]$, $\lambda\in(-\pi/2,\pi/2)$,*

$$|f(z)|\leqq r\left[\frac{(1+(2\beta-1)r)^{(1-\cos\lambda)}}{(1-(2\beta-1)r)^{(1+\cos\lambda)}}\right]^{\beta(1-\alpha)\cos\lambda/(2\beta-1)}, \tag{4.1}$$

and

$$|f(z)|\geqq r\left[\frac{(1-(2\beta-1)r)^{(1-\cos\lambda)}}{(1+(2\beta-1)r)^{(1+\cos\lambda)}}\right]^{\beta(1-\alpha)\cos\lambda/(2\beta-1)}; \tag{4.2}$$

whereas for $\alpha\in[0,1)$, $\beta=1/2$, $\lambda\in(-\pi/2,\pi/2)$,

$$|f(z)|\leqq r\exp((1-\alpha)\cos\lambda\cdot r), \tag{4.3}$$

and

$$|f(z)|\geqq r\exp(-(1-\alpha)\cos\lambda\cdot r). \tag{4.4}$$

All these estimates are sharp for all admissible values of α, β, λ.

Proof. Since $f(z)\in S^\lambda(\alpha,\beta)$, the condition (1.1) coupled with an application of Schwarz's Lemma, implies $|zf'(z)/f(z)-\xi|<R$, where

$$\xi=\frac{1-(2\beta-1)((2\beta-1)-2\beta(1-\alpha)\cos^2\lambda)r^2-i\beta(2\beta-1)(1-\alpha)\sin 2\lambda\cdot r^2}{1-(2\beta-1)^2r^2},$$

and

$$R=\frac{2\beta(1-\alpha)\cos\lambda\cdot r}{1-(2\beta-1)^2r^2},\quad (|z|=r).$$

Hence we have

$$\begin{aligned}&\frac{1-2\beta(1-\alpha)\cos\lambda\cdot r+(2\beta-1)(2\beta(1-\alpha)\cos^2\lambda-(2\beta-1))r^2}{1-(2\beta-1)^2r^2}\\&\leqq\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right)\leqq\frac{1+2\beta(1-\alpha)\cos\lambda\cdot r+(2\beta-1)(2\beta(1-\alpha)\cos^2\lambda-(2\beta-1))r}{1-(2\beta-1)^2r^2}.\end{aligned} \tag{4.5}$$

Noting

$$\log\left(\left|\frac{f(z)}{z}\right|\right)=\mathrm{Re}\int_0^z\left(\frac{f'(s)}{f(s)}-\frac{1}{s}\right)ds=\int_0^r\frac{1}{t}\,\mathrm{Re}\left(te^{i\theta}\frac{f'(te^{i\theta})}{f(te^{i\theta})}-1\right)dt\,,$$

and using (4.5), we see that

$$\log\left(\left|\frac{f(z)}{z}\right|\right)\leqq 2\beta(1-\alpha)\cos\lambda\cdot\int_0^r\frac{1+(2\beta-1)\cos\lambda\cdot t}{1-(2\beta-1)^2t^2}dt\,. \tag{4.6}$$

Now suppose $\alpha\in[0,1)$, $\beta\in(0,1/2)\cup(1/2,1]$ and $\lambda\in(-\pi/2,\pi/2)$. Then from (4.6), we get

$$\log\left(\left|\frac{f(z)}{z}\right|\right)\leqq\frac{\beta(1-\alpha)\cos\lambda}{2\beta-1}\log\left\{\frac{(1+(2\beta-1)r)^{(1-\cos\lambda)}}{(1-(2\beta-1)r)^{(1+\cos\lambda)}}\right\}\,,$$

which gives (4.1). For the case when $\alpha\in[0,1)$, $\beta=1/2$, and $\lambda\in(-\pi/2,\pi/2)$, (4.6) immediately proves (4.3). In view of

$$\log\left(\left|\frac{f(z)}{z}\right|\right)=\mathrm{Re}\left(\log\frac{f(z)}{z}\right)=\int_0^r\mathrm{Re}\left(\frac{\partial}{\partial t}\left(\log\frac{f(t)}{t}\right)\right)dt$$
$$=\int_0^r\frac{1}{t}\,\mathrm{Re}\left(t\frac{f'(t)}{f(t)}-1\right)dt\,,$$

and with the aid of (4.5) we may write

$$\log\left(\left|\frac{f(z)}{z}\right|\right)\geqq -2\beta(1-\alpha)\cos\lambda\cdot\int_0^r\frac{1-(2\beta-1)\cos\lambda\cdot t}{1-(2\beta-1)^2t^2}dt\,. \tag{4.7}$$

If $\beta\neq 1/2$, then carrying out the integration in (4.7), we obtain (4.2). Further, when $\beta=1/2$, then we immediately get (4.4) from (4.7). The extremal function for all the inequalities is given by

$$f(z)=\begin{cases} z/\{(1-(2\beta-1)e^{i\theta}z\}^{2\beta(1-\alpha)\cos\lambda\cdot e^{-i\lambda}/(2\beta-1)}\,, & \beta\neq 1/2\\ z\exp\{(1-\alpha)\cos\lambda\cdot e^{i(\theta-\lambda)}z\}\,, & \beta=1/2\end{cases} \tag{4.8}$$

where $0\leqq\alpha<1$, $-\pi/2<\lambda<\pi/2$; and θ $(0\leqq\theta\leqq 2\pi)$ is determined by

$$\tan(\theta/2)=((1-(2\beta-1)r)/(1+(2\beta-1)r))\cot(\pi/2-\lambda/2)$$

for the equality in (4.1) and (4.3), and by the equation

$$\tan(\theta/2)=((1-(2\beta-1)r)/(1+(2\beta-1)r)\cot(-\lambda/2)$$

for the equality in (4.2) and (4.4).

Corollary 4. *If* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, *analytic in* E, *is a* λ*-spiral-like function of order* α, *then for* $|z|=r<1$,

$$r\left[\frac{(1-r)^{(1-\cos\lambda)}}{(1+r)^{(1+\cos\lambda)}}\right]^{(1-\alpha)\cos\lambda}\leqq|f(z)|\leqq r\left[\frac{(1+r)^{(1-\cos\lambda)}}{(1-r)^{(1+\cos\lambda)}}\right]^{(1-\alpha)\cos\lambda}.$$

By choosing $\alpha=0$ and $\beta=(2M-1)/2M$ in Theorem 3, we get the following result.

Corollary 5. *If* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, *analytic in* E, *is in* $F_{\lambda,M}$, *then for* $|z|=r<1$, $M>1/2$, $(M\neq 1)$

$$r\left[\frac{(1-((M-1)/M)r)^{(1-\cos\lambda)}}{(1+((M-1)/M)r)^{(1+\cos\lambda)}}\right]^{(2M-1)\cos\lambda/2(M-1)}$$
$$\leqq|f(z)|\leqq r\left[\frac{(1+((M-1)/M)r)^{(1-\cos\lambda)}}{(1-((M-1)/M)r)^{(1+\cos\lambda)}}\right]^{(2M-1)\cos\lambda/2(M-1)};$$

whereas for $M=1$,

$$r\exp(-\cos\lambda\cdot r)\leqq|f(z)|\leqq r\exp(\cos\lambda\cdot r).$$

Remarks. (i) Taking $\lambda=0$, Theorem 3 coincides with the result of Juneja and Mogra [3].

(ii) Putting $\alpha=0$, $\beta=(2-\cos\lambda)/2$; and replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$, β by $(1+\beta)/2$ in Theorem 3 we get, respectively, the distortion theorems obtained by Goel [2], and Makówka [7].

(iii) Different values of α, β, and λ in Theorem 3 lead to the corresponding distortion theorems for the respective classes defined by Padmanabhan [8], Wright [11] and others.

5. Coefficient Bounds

Theorem 4. *Let* $f\in S^{\lambda}(\alpha,\beta)$, *and* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, $z\in E$.

(a) *If* $\beta(1-\alpha)(2-\alpha)\cos^2\lambda>(1-\beta)(1+(1-\alpha)\cos^2\lambda)$, *let*

$$N=\left[\frac{\beta(1-\alpha)(2-\alpha)\cos^2\lambda}{(1-\beta)(1+(1-\alpha)\cos^2\lambda)}\right].$$

Then

$$|a_n|\leqq\frac{1}{(n-1)!}\prod_{k=2}^{n}|(2\beta-1)(k-2)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|, \tag{5.1}$$

for $n=2,3,\cdots,N+2$; *and*

$$|a_n|\leqq\frac{1}{(N+1)!\,(n-1)}\prod_{k=2}^{N+3}|(2\beta-1)(k-2)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|,\quad n>N+2. \tag{5.2}$$

(b) *If* $\beta(1-\alpha)(2-\alpha)\cos^2\lambda\leqq(1-\beta)(1+(1-\alpha)\cos^2\lambda)$, *then*

(5.3) $$|a_n| \leqq \frac{2\beta(1-\alpha)\cos\lambda}{n-1}, \qquad \textit{for} \quad n \geqq 2.$$

The bounds in (5.1) *and* (5.3) *are sharp for all admissible* α, β, λ, *and for each* n.

Proof. Since $f \in S^{\lambda}(\alpha, \beta)$, (2.3) gives

(5.4) $$\frac{zf'(z)}{f(z)} = \frac{1+((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)w(z)}{1+(2\beta-1)w(z)},$$

where $w(z)=z\phi(z)$ is analytic in E and satisfies the conditions $w(0)=0$, and $|w(z)|<1$ for $z \in E$. Now (5.4) may be written as

$$\{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot z + \sum_{k=2}^{\infty}((2\beta-1)(k-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)a_k z^k\}w(z) = \sum_{k=2}^{\infty}(1-k)a_k z^k,$$

which is equivalent to

$$\{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot z + \sum_{k=2}^{n-1}((2\beta-1)(k-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)a_k z^k\}w(z)$$
$$= \sum_{k=2}^{n}(1-k)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k,$$

where $\sum_{k=n+1}^{\infty} b_k z^k$ converges in E. Then, since $|w(z)|<1$,

(5.5) $$|2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot z + \sum_{k=2}^{n-1}((2\beta-1)(k-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)a_k z^k|$$
$$\geqq |\sum_{k=2}^{n}(1-k)a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k|.$$

Writing $z=re^{i\theta}$, $r<1$, squaring both sides of (5.5), and then integrating, we get

$$4\beta^2(1-\alpha)^2\cos^2\lambda\cdot r^2 + \sum_{k=2}^{n-1}|(2\beta-1)(k-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda|^2|a_k|^2 r^{2k}$$
$$\geqq \sum_{k=2}^{n}(k-1)^2|a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty}|b_k|^2 r^{2k}.$$

Let $r \to 1$, then on some simplification we obtain

(5.6) $$(n-1)^2|a_n|^2 \leqq 4\beta^2(1-\alpha)^2\cos^2\lambda$$
$$+\sum_{k=2}^{n-1}\{|(2\beta-1)(k-1)+2\beta(1-\alpha)e^{-i\lambda}\cos\lambda| - (k-1)^2\}|a_k|^2, \quad n \geqq 2.$$

Now there may be following two cases:

Let $\beta(1-\alpha)(2-\alpha)\cos^2\lambda \geqq (1-\beta)(1+(1-\alpha)\cos^2\lambda)$. Suppose that $n \leqq N+2$; then for $n=2$, (5.6) gives

$$|a_2| \leqq 2\beta(1-\alpha)\cos\lambda,$$

which proves (5.1) for $n=2$. We establish (5.1) for $n\leqq N+2$, from (5.6), by mathematical induction.

Suppose (5.1) is valid for $k=2,3,\cdots,n-1$. Then it follows from (5.6),

$$
\begin{aligned}
(n-1)^2|a_n|^2 &\leqq 4\beta^2(1-\alpha)^2\cos^2\lambda \\
&\quad+\sum_{k=2}^{n-1}\left\{\{|(2\beta-1)(k-1)+M|^2-(k-1)^2\}\frac{1}{((k-1)!)^2}\prod_{p=2}^{k}|(2\beta-1)(p-2)+M|^2\right\} \\
&=\frac{1}{((n-2)!)^2}\prod_{k=2}^{n}|(2\beta-1)(k-2)+M|^2\,,
\end{aligned}
$$

where $M=2\beta(1-\alpha)e^{-i\lambda}\cos\lambda$. Thus we get

$$|a_n|\leqq\frac{1}{(n-1)!}\prod_{k=2}^{n}|(2\beta-1)(k-2)+M|\,,$$

which completes the proof of (5.1).

Next, we suppose $n>N+2$. Then (5.6) gives

$$
\begin{aligned}
(n-1)^2|a_n|^2 &\leqq 4\beta^2(1-\alpha)^2\cos^2\lambda \\
&\quad+\sum_{k=2}^{N+2}\{|(2\beta-1)(k-1)+M|^2-(k-1)^2\}|a_k|^2 \\
&\quad+\sum_{k=N+3}^{n-1}\{|(2\beta-1)(k-1)+M|^2-(k-1)^2\}|a_k|^2 \\
&\leqq 4\beta^2(1-\alpha)^2\cos^2\lambda+\sum_{k=2}^{N+2}\{|(2\beta-1)(k-1)+M|^2-(k-1)^2\}|a_k|^2\,.
\end{aligned}
$$

On substituting upper estimates for $a_2, a_3, \cdots, a_{N+2}$ obtained above, and simplifying, we obtain (5.2).

(b) Let $\beta(1-\alpha)(2-\alpha)\cos^2\lambda\leqq(1-\beta)(1+(1-\alpha)\cos^2\lambda)$, then it follows from (5.6)

$$(n-1)^2|a_n|^2\leqq 4\beta^2(1-\alpha)^2\cos^2\lambda\,, \quad (n\geqq 2)$$

which proves (5.3).

The bound in (5.1) is sharp for the function given by

$$f(z)=z/(1-(2\beta-1)z)^{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot(2\beta-1)^{-1}}\,. \tag{5.7}$$

The bounds in (5.3) are sharp for the functions given by

$$f_n(z)=z/(1-(2\beta-1)z^{n-1})^{2\beta(1-\alpha)e^{-i\lambda}\cos\lambda\cdot(2\beta-1)^{-1}(n-1)^{-1}}\,,$$

for $\beta\neq 1/2$; whereas for $\beta=1/2$,

$$f_n(z)=z\exp\{((1-\alpha)e^{-i\lambda}\cos\lambda\cdot(n-1)^{-1})z^{n-1}\}\,, \quad (n\geqq 2)\,.$$

Remarks. (i) Taking $\beta=1$, we obtain a theorem of Libera [6].

(ii) By choosing $\alpha=0$, $\beta=(2-\cos\lambda)/2$ in Theorem 4, we get the result of Goel [2].

(iii) $\lambda=0$ leads to the result obtained by Juneja and Mogra [3].

(iv) Replacing α by $(1-\beta+2\alpha\beta)/(1+\beta)$ and β by $(1+\beta)/2$ in Theorem 4, we obtain the corresponding results of Makówka [7].

(v) The coefficient estimates determined by Kulshrestha [5], Zamorski [12], Wright [11], and many others can be obtained from Theorem 4 by taking different values of α, β, and λ.

6. The γ-spiral radius

Let S be the family of all normalized functions which are analytic and univalent in E. Following Libera [6], if $f\in S$ and $\gamma\in(-\pi/2,\pi/2)$, then γ-spiral radius of f is

$$\gamma\text{—s.r.}\,\{f\}=\sup\left\{r\colon \operatorname{Re}\left(e^{i\gamma}\frac{zf'(z)}{f(z)}\right)>0,\ |z|<r\right\}\ ; \tag{6.1}$$

and if $U\subset S$, then the γ-spiral radius of U is

$$\gamma\text{—s.r.}\ U=\operatorname*{Inf}_{f\in U}\{\gamma\text{—s.r.}\,\{f\}\}\ . \tag{6.2}$$

We now determine the γ-spiral radius of the class $S^{\lambda}(\alpha,\beta)$.

Theorem 5. *γ—s.r. $S^{\lambda}(\alpha,\beta)$ is the smallest positive root r of the equation*

$$(2\beta-1)(2\beta(1-\alpha)\cos(\gamma-\lambda)\cdot\cos\lambda-(2\beta-1)\cos\gamma)r^2-2\beta(1-\alpha)\cos\lambda\cdot r+\cos\gamma=0\ . \tag{6.3}$$

The result is sharp for the extremal function given in (5.7).

Proof. Let $f\in S^{\lambda}(\alpha,\beta)$. Then by Lemma,

$$\frac{zf'(z)}{f(z)}=\frac{1+((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)w(z)}{1+(2\beta-1)w(z)}\ , \tag{6.4}$$

where $w(z)$ satisfies the conditions $w(0)=0$ and $|w(z)|<1$. If $B(z)=e^{i\gamma}zf'(z)/f(z)$, then (6.4) may be written

$$w(z)=\frac{e^{i\gamma}-B(z)}{(2\beta-1)B(z)-e^{i\gamma}((2\beta-1)-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)}\ ,\quad (z\in E)\ .$$

Now by applying Schwarz's Lemma, it follows that $B(z)$ maps the disk $|z|\leqq r$ onto a disk $|B(z)-\eta|<R$, where

$$\eta=\frac{e^{i\gamma}(1-(2\beta-1)(2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)r^2)}{1-(2\beta-1)^2r^2}\ ,$$

and

$$R=\frac{2\beta(1-\alpha)\cos\lambda\cdot r}{1-(2\beta-1)^2r^2}.$$

Hence $\mathrm{Re}\,(e^{i\gamma}zf'(z)/f(z))\geqq 0$ if and only if

$$\mathrm{Re}\left\{\frac{e^{i\gamma}(1-(2\beta-1)(2\beta-1-2\beta(1-\alpha)e^{-i\lambda}\cos\lambda)r^2)}{1-(2\beta-1)^2r^2}\right\}\geqq\frac{2\beta(1-\alpha)\cos\lambda\cdot r}{1-(2\beta-1)^2r^2},$$

which, on simplification, and with the aid of (6.2) concludes the proof of the theorem.

By choosing appropriate values of α, β, λ and γ in Theorem 5 we obtain the corresponding results for several subclasses of S.

Corollary 6. *γ—s.r. $F_{\lambda,M}$ is the smallest positive root r of the equation*

$$(M-1)((2M-1)\cos(\gamma-\lambda)\cdot\cos\lambda-(M-1)\cos\gamma)r^2-(2M-1)M\cos\lambda\cdot r+M^2\cos\gamma=0,$$

for $M>1/2$.

Taking $\alpha=0$ and $\beta=(2-\cos\lambda)/2$ in Theorem 5, we get the following result.

Corollary 7. *γ—s.r. $H(\lambda)$ is the smallest positive root of the equation*

$$(1-\cos\lambda)((2-\cos\lambda)\cos(\gamma-\lambda)\cdot\cos\lambda-(1-\cos\lambda)\cos\gamma)r^2-(2-\cos\lambda)\cos\lambda\cdot r+\cos\gamma=0.$$

When $\gamma=0$ in Corollary 7, we get the radius of starlikeness of $H(\lambda)$ as obtained by Goel [2].

Remarks. Different values of the parameters α, β, λ, and γ lead to γ-spiral radius and radius of starlikeness of the classes studied by Robertson [9], Libera [6], Maköwka [7], and others.

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