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IRREDUCIBLE 3-MANIFOLDS WITH NON-TRIVIAL π_2

By

SEIYA NEGAMI

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§ 1. Introduction

Let Π be the set of compact connected irreducible 3-manifolds with non-trivial π_2. Using the sphere theorem, one can prove that under the Poincaré conjecture, an orientable 3-manifold M^3 is irreducible if and only if $\pi_2(M^3)$ is trivial. So any 3-manifold in Π is non-orientable. From the projective plane theorem, it follows that any 3-manifold M^3 in Π admits a 2-sided embedding of a projective plane, and the converse is also true; Feustel [2] showed that if there is an embedding $h: P^2 \to M^3$ such that $h_*: \pi_2(P^2) \to \pi_2(M^3)$ is trivial, then M^3 is homeomorphic to $P^3 \# \Sigma^3$, where P^3 and Σ^3 are a 3-dimensional projective space and a homotopy 3-sphere, respectively.

Let Π_c and Π_b be the subsets of Π consisting of closed 3-manifolds and boundary-irreducible 3-manifolds with non-empty boundary, respectively. A few 3-manifolds in Π_c or Π_b have been known. For example, $P^2 \times S^1$ belongs to Π_c, and $P^2 \times I$ to Π_b. It is however easy to construct infinitely many elements of the complement Π_a of $\Pi_b \cup \Pi_c$; attach 1-handles to $P^2 \times I$.

In [8], Tao proved that under the Poincaré conjecture, the orientable double covering space of any prime closed 3-manifold is also prime, with respect to connected sums $\#$, if and only if $P^2 \times S^1$ is the only one connected closed irreducible 3-manifold with non-trivial π_2. Recently Ochiai [6] has shown that if a closed 3-manifold M^3 which has a Heegaard splitting of genus two admits a 2-sided embedding of a projective plane, then M^3 is homeomorphic to $P^2 \times S^1$. Naturally one may expect that Π_c would be the singleton $\{P^2 \times S^1\}$. But we shall show that Π_c includes infinitely many elements which can be distinguished by several well-defined invariants:

Main Theorem. *There are infinitely many closed connected irreducible 3-manifolds with non-trivial* π_2.

The existence is due to knot and link theory and especially the fact that the boundary of a knot space of a non-trivial knot is an incompressible torus. To prove the infiniteness, we shall define a connected graph $G(M^3)$, for an element M^3 of Π, which describes a structure of M^3 induced by a specific system of projective planes in M^3.

We shall work in the piecewise linear category through this paper, and use mostly the terminology and the notation in [5].

I would like to express my hearty thanks Prof. T. Homma for helpful conversations.

§ 2. Complete systems of projective planes

Let M^3 be a 3-manifold and F_0, F_1 closed 2-manifolds or surfaces embedded in Int (M^3). Then F_0 and F_1 are said to be *parallel* if there is an embedding h: $F\times I\to M^3$ such that $h(F\times\{0\})=F_0$ and $h(F\times\{1\})=F_1$. A *complete system of projective planes* in M^3 is defined to be a system $\{P_1^2, \cdots, P_q^2\}$ of mutually disjoint 2-sided projective planes in Int (M^3) satisfying the following conditions 1 and 2:

1. Every P_i^2 is not parallel to each other.
2. If P_{q+1}^2 is a 2-sided projective plane in M^3 disjoint from $P_1^2\cup\cdots\cup P_q^2$, then P_{q+1}^2 is parallel to some P_i^2 $(i=1, \cdots, q)$.

For any compact irreducible 3-manifold M^3, we define $q(M^3)$ by the number of projective planes contained in a complete system $\{P_1^2, \cdots, P_q^2\}$ in M^3, and $p(M^3)$ by the number of components of $M^3-P_1^2\cup\cdots\cup P_q^2$. Haken's finiteness theorem [3] states that for any compact 3-manifold M^3, there is an integer $n(M^3)$ such that M^3 can not admit more than $n(M^3)$ pairwise disjoint 2-sided incompressible closed surfaces which are not parallel to each other. So $p(M^3)$ and $q(M^3)$ are finite integers.

In this section, we shall show uniqueness of a complete system of projective planes. From this it follows that the integers $p(M^3)$ and $q(M^3)$ do not depend on the choice of a complete system of projective planes, that is, $p(M^3)$ and $q(M^3)$ are well-defined invariants. So we shall say that a compact connected irreducible 3-manifold M^3 is of *type* (p, q), when $p=p(M^3)$ and $q=q(M^3)$. Then M^3 is not an element of Π if and only if M^3 is of type $(1, 0)$.

Using the following lemma, one can prove that $P^2\times S^1$ is of type $(1, 1)$ and that $P^2\times I$ is of type $(2, 1)$:

Lemma 1.1. *Any projective plane in* Int $(P^2\times I)$ *is parallel to* $P^2\times\{0\}$.

Proof. Let Q^2 be a projective plane in Int $(P^2\times I)$, necessarily 2-sided, and g: $S^2\times I\to P^2\times I$ a natural double covering with a covering translation ρ: $S^2\times I\to$

$S^2\times I$ such that $\rho(S^2\times\{t\})=S^2\times\{t\}$ for $t\in I$. Let A be a 1-sided annulus properly embedded in $P^2\times I$ which splits $P^2\times I$ into a 3-ball; for example, take $l\times I$ as A, where l is a non-trivial simple loop in P^2.

By the annulus theorem, 2-spheres $g^{-1}(Q^2)$ and $S^2\times\{0\}$ bound a submanifold E of $S^2\times I$ homeomorphic to $S^2\times I$. We wish to define an embedding h: $S^2\times I\to S^2\times I$ so that $h(S^2\times I)=E$ and $h\cdot\rho=\rho\cdot h$ in order to show that $g(E)$ is homeomorphic to $P^2\times I$. It is not so difficult to do so in the case that $g^{-1}(Q^2\cap A)$ has precisely one component, as illustrated in Figure 1. Then we shall observe that there is an ambient isotopy of $P^2\times I$ which carries Q^2 into a projective plane Q_1^2 such that $g^{-1}(Q_1^2\cap A)$ is a single circle.

Let Q^2 be in general position with respect to A. Then we have two type of components of $A\cap Q^2$; one is a circle parallel to each component of ∂A and the other bounds a 2-disk in A. If there is a component of $A\cap Q^2$ of the second type, choose an innermost one l_2 in A which bounds a 2-disk D_2 in A. Since Q^2 is incompressible in $P^2\times I$, l_2 bounds a 2-disk D_1 in Q^2 so that $D_1\cup D_2$ is a 2-sphere. Since $P^2\times I$ is irreducible, $D_1\cup D_2$ bounds a 3-ball B^3 in $P^2\times I$. Therefore there is an ambient isotopy of $P^2\times I$ which first carries D_1 into D_2 through B^3 and which next pushes D_2 so that l_2 and possibly some other components of $A\cap Q^2$ vanish. So we have a composition of such ambient isotopies H_t: $P^2\times I\to P^2\times I$ $(t\in I)$ such that $A\cap H_1(Q^2)$ contains no component of the second type.

Note that $A\cap H_1(Q^2)\neq\phi$; otherwise, a 3-ball in $P^2\times I-A$ could admit the 2-

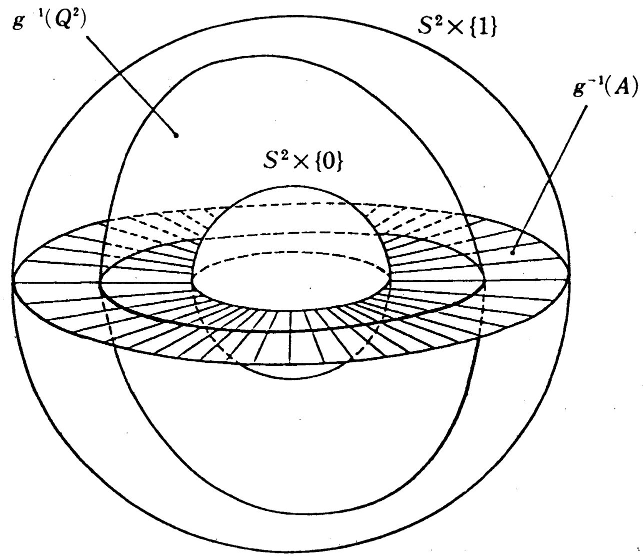

Figure 1.

sided embedding of a projective plane $H_1(Q^2)$. And note that each component of $A\cap H_1(Q^2)$ is an orientation-reversing loop in $P^2\times I$ and hence it bounds no 2-disk in $H_1(Q^2)$. Because two non-trivial simple loops in a projective plane always intersect each other, $A\cap H_1(Q^2)$ and $g^{-1}(A\cap H_1(Q_2))$ must be single circles. So H_t is the required ambient isotopy with $Q_1^2=H_1(Q^2)$. This completes the proof.

Lemma 1.2. *Let $\{S_1, \cdots, S_n\}$ be a system of mutually disjoint 2-spheres or 2-sided projective planes and $\{F_1, \cdots, F_m\}$ a system of mutually disjoint 2-sided incompressible surfaces properly embedded in a 3-manifold M^3. Suppose that every component of $M^3-F_1\cup\cdots\cup F_m$ is irreducible. Then there is an ambient isotopy of M^3 which carries $\{S_1, \cdots, S_n\}$ into a system disjoint from $\{F_1, \cdots, F_m\}$.*

Proof. We observe that a Möbius band and an annulus (or another Möbius band) can not be embedded properly and 2-sidedly in a solid Klein bottle so that they intersect each other transversely along their center lines. So any component l of $S_i\cap F_j$ bounds a 2-disk D_1 in S_i. If l is innermost in S_i, then l bounds a 2-disk D_2 in F_j and $D_1\cup D_2$ is a 2-sphere; for F_j is incompressible in M^3. Since $M^3-F_1\cup\cdots\cup F_m$ is irreducible, we can push $D_1\cup D_2$ slightly into $M^3-F_1\cup\cdots\cup F_m$ so that $D_1\cup D_2$ bounds a 3-ball in M^3. Therefore we have an ambient isotopy of M^3, like H_t in the proof of Lemma 1.1, which decreases the number of components of $(S_1\cup\cdots\cup S_n)\cap(F_1\cup\cdots\cup F_m)$. This completes the proof.

Theorem 1. *For any compact irreducible 3-manifold, there is a unique finite, possibly empty, complete system of projective planes, up to ambient isotopy.*

Proof. Finiteness of a complete system of projective planes follows from Haken's finiteness theorem, as above-mentioned. So we shall show only uniqueness.

Let $\{P_1^2, \cdots, P_n^2\}$ and $\{Q_1^2, \cdots, Q_m^2\}$ $(n\leqq m)$ be two complete systems of projective planes in a compact irreducible 3-manifold M^3. Using Lemma 1.2, we may assume that $(P_1^2\cup\cdots\cup P_n^2)\cap(Q_1^2\cup\cdots\cup Q_m^2)=\phi$. By completeness of $\{P_1^2, \cdots, P_n^2\}$, each Q_i^2 is parallel to some $P^2_{\tau(i)}$, that is, $Q_i^2\cup P^2_{\tau(i)}$ bounds a submanifold E_i of M^3 homeomorphic to $P^2\times I$. Using Lemma 1.1, we observe that if $E_i\cap E_j\neq\phi$ $(i\neq j)$, then Q_i^2 and Q_j^2 are parallel, which is contrary to completeness of $\{Q_1^2, \cdots, Q_m^2\}$. Thus $E_i\cap E_j=\phi$ $(i\neq j)$, so there is an ambient isotopy which carries Q_i^2 into $P^2_{\tau(i)}$ through E_i. Necessarily $n=m$ and τ is a bijection. This completes the proof.

Corollary 1.1. *We have two well-defined invariants $p(M^3)$ and $q(M^3)$ for a compact irreducible 3-manifold M^3.*

§ 3. Construction of elements of Π

In this section, we shall construct infinitely many elements of Π from several $P^2\times S^1$'s. Let τ, ρ: $\boldsymbol{R}^3\to\boldsymbol{R}^3$ be homeomorphisms defined by $\tau(x)=(1/2)x$ and $\rho(x)=-x$, respectively, and g: $\boldsymbol{R}^3-\{0\}\to P^2\times S^1$ the universal covering the group of whose covering translations is generated by τ and ρ. We can take $E_+=\{x=(x_1, x_2, x_3)\in\boldsymbol{R}^3: 1\leqq\|x\|\leqq 2,\ x_3\geqq 0\}$ as a fundamental domain homeomorphic to a 3-ball.

Setting $E_-=\rho(E_+)$ and $S_t^2=\{x\in\boldsymbol{R}^3: \|x\|=t\}$, consider a non-splittable link $k=k_1\cup\cdots\cup k_n$ (or a non-trivial knot if $n=1$) within Int $(E_+\cup E_-)$ such that for each component k_i $\rho(k_i)=k_i$. For example, the square knot with $n-1$ trivial knots linking, as shown in Figure 2, is such a link or knot; see the proof of Lemma 2.2. Since $g(k_i)$ is an orientation-reversing loop in $P^2\times S^1$, a regular neighborhood $N(g(k_i), P^2\times S^1)$ of $g(k_i)$ in $P^2\times S^1$ is a solid Klein bottle. So each component K_i^2 of the boundary of the 3-manifold $P(k)=P^2\times S^1-\text{Int}\,(N(g(k), P^2\times S^1))$ is a Klein bottle.

Lemma 2.1. *$P(k)$ above is an irreducible 3-manifold of type (1, 1). Moreover it is boundary-irreducible, that is, K_i^2 $(i=1, \cdots, n)$ is incompressible in $P(k)$.*

Proof. Let S^2 be a 2-sphere in Int $(P(k))$. Since $P^2\times S^1$ is irreducible, S^3 bounds a 3-ball B^3 in $P^2\times S^1$. Since B^3 can not contain orientation-reversing loops $g(k_1), \cdots, g(k_n)$, $B^3\subset P(k)$, so $P(k)$ is irreducible.

Let Q^2 be a projective plane in Int $(P(k))$ disjoint from $g(S_1^2)$. By Lemma 1.1, Q^2 is parallel to the projective plane $g(S_1^2)$ in $P^2\times S^1$ and there are two submanifold

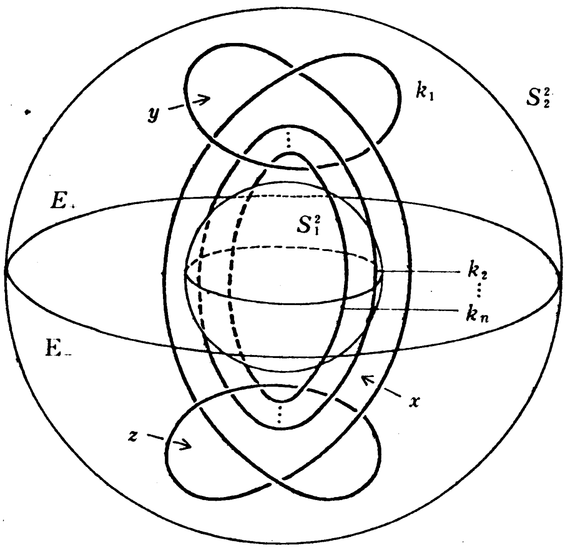

Figure 2.

E_1 and E_2 of $P^2\times S^1$ homeomorphic to $P^2\times I$ such that $P^2\times S^1=E_1\cup E_2$ and $E_1\cap E_2=\partial E_1=\partial E_2=Q^2\cup g(S_1^2)$. Suppose that $E_1\supset g(k_1\cup\cdots\cup k_l)$ and $E_2\supset g(k_{l+1}\cup\cdots\cup k_n)$. Then the 2-sphere $g^{-1}(Q^2)\cap(E_+\cup E_-)$ splits k into $k_1\cup\cdots\cup k_l$ and $k_{l+1}\cup\cdots\cup k_n$, contrary to non-splittability of k. Thus one of E_1 and E_2 includes $g(k)$ and the other is contained in $P(k)$, so Q^2 is parallel to $g(S_1^2)$ in $P(k)$. This implies that $\{g(S_1^2)\}$ is a complete system. By Theorem 1, $q(P(k))=1$ and $p(P(k))=1$ since $P(k)-g(S_1^2)$ is connected.

Boundary-irreducibility of $P(k)$ depends on non-triviality or non-splittability of k. If K_i^2 is compressible in $P(k)$, then there is a 2-disk D^2 in $P(k)$ such that $D^2\cap K_i^2=\partial D^2$. Since D^2 is contractible, there is a lifting $g^{-1}(D^2)\cap(E_+\cup E_-)$ of D^2. This implies that k_i is a trivial knot splittable from the other components of k, contrary to the assumption of k. Therefore K_i^2 is incompressible in $P(k)$.

The proof is complete.

Lemma 2.2. *There are infinitely many closed connected irreducible* 3*-manifolds of type* $(1, q)$ *different from* $P^2\times S^1$, *for any positive integer* q.

Proof. Let $k=k_1\cup\cdots\cup k_n$ be the link or knot in Figure 2. In particular, k_1 is the square knot, and $\pi_1(\boldsymbol{R}^3-k_1)$ has the presentation $(x, y, z;\ xyx=yxy,\ xzx=zxz)$. Each k_i $(i=2, \cdots, n)$ represents yz^{-1} in $\pi_1(\boldsymbol{R}^3-k_1)$. We have the homomorphism Φ from $\pi_1(\boldsymbol{R}^3-k_1)$ onto the group of permutations of the symbols $\{1, 2, 3\}$ defined by $\Phi(x)=\Phi(y)=(12)$ and $\Phi(z)=(23)$. Since $\Phi(yz^{-1})=(321)$, yz^{-1} is not the identity element of $\pi_1(\boldsymbol{R}^3-k_1)$, and hence k is not splittable. Thus $P(k)$ exists.

Let M_1 be a copy of $P(k_1\cup\cdots\cup k_{q+2r-1})$ and $M_2, \cdots, M_q$ copies of $P(k_1)$. M_1 has boundary components $K_1^2, \cdots, K_{q+2r-1}^2$. Attach $M_2, \cdots, M_q$ to M_1 by homeomorphisms from Klein bottles $\partial M_2, \cdots, \partial M_q$ to $K_1^2, \cdots, K_{q-1}^2$, respectively, and identify K_{q+2i-2}^2 with K_{q+2i-1}^2 $(i=1, \cdots, r)$ so that the resulting 3-manifold $M(q, r)$ is closed. The lemma follows from the following claims 1 to 4:

Claim 1. $M(q, r)$ is not homeomorphic to $P^2\times S^1$: $M(q, r)$ admits 2-sided Klein bottles $K_1^2, \cdots, K_{q-1}^2;\ K_q^2=K_{q+1}^2, \cdots, K_{q+2r-2}^2=K_{q+2r+1}^2$. By Lemma 2.1, these Klein bottles are incompressible in $M(q, r)$, so $\pi_1(M(q, r))$ has a non-abelian subgroup isomorphic to $\pi_1(K_i^2)$. Since $\pi_1(P^2\times S^1)\cong Z+Z_2$ is abelian, $M(q, r)$ can not be homeomorphic to $P^2\times S^1$.

Claim 2. $M(q, r)$ is irreducible: By Lemma 1.2, any 2-sphere S^2 can be moved by an ambient isotopy so that afterward $S^2\cap(K_1^2\cup\cdots\cup K_{q+2r-1}^2)=\phi$. Since $P(k)$ is irreducible, S^2 bounds a 3-ball in $M(q, r)$, so $M(q, r)$ is irreducible.

Claim 3. $M(q, r)$ is of type $(1, q)$: Let $P_1^2, \cdots, P_q^2$ be copies of $g(S_1^2)$ lying in $M_1, \cdots, M_q$ respectively. Since $M(q, r)-P_1^2\cup\cdots\cup P_q^2$ is connected, P_i^2 is not parallel to each other. Let P_{q+1}^2 be another projective plane in $M(q, r)$ disjoint from $P_1^2\cup\cdots\cup P_q^2$. By Lemma 1.2, a certain ambient isotopy carries $\{P_1^2, \cdots, P_q^2, P_{q+1}^2\}$ into a system disjoint from $\{K_1^2, \cdots, K_{q-1}^2; K_q^2=K_{q+1}^2, \cdots, K_{q+2r-2}^2=K_{q+2r-1}^2\}$. Since $q(P(k))=1$, P_{q+1}^2 becomes parallel to some P_i^2 $(i=1, \cdots, q)$. This means that $\{P_1^2, \cdots, P_q^2\}$ is a complete system of projective planes. By uniqueness of such a system, we determine $q(M(q, r))=q$ and $p(M(q, r))=1$.

Claim 4. If r_2-r_1 is sufficiently large, then $M(q, r_1)$ and $M(q, r_2)$ are not homeomorphic: $\{K_q^2=K_{q+1}^2, \cdots, K_{q+2r-2}^2=K_{q+2r-1}^2\}$ is a system of r pairwise disjoint 2-sided incompressible closed surfaces in $M(q, r)$ which are not parallel to each other, so Haken's integer $n(M(q, r))$ is greater than or equal to r. Thus if $r_2> n(M(q, r_1))$, then $n(M(q, r_2))>n(M(q, r_1))$, and hence $M(q, r_2)$ is not homeomorphic to $M(q, r_1)$.

The proof is complete.

For an element M^3 of Π, we define, as follows, a connected graph $G(M^3)$ with p vertices $v_1, \cdots, v_p$ and q edges $e_1, \cdots, e_q$, where $p=p(M^3)$ and $q=q(M^3)$: Let $\{P_1^2, \cdots, P_q^2\}$ be a complete system of projective planes in M^3, and identify $P_i^2\times I$ with a small regular neighborhood $N(P_i^2, M^3)$ of P_i^2 in M^3 via a natural homeomorphism. Then $M^3-\mathrm{Int}\,(P_1^2\times I\cup\cdots\cup P_q^2\times I)$ has p components $V_1, \cdots, V_p$ whose boundaries consist of $P_i^2\times\{0\}$ and $P_i^2\times\{1\}$ $(i=1, \cdots, q)$. Join v_i and v_j with e_k if $P_k^2\times\{0\}\subset V_i$ and $P_k^2\times\{1\}\subset V_j$. For example, $G(P^2\times S^1)$ is a single vertex v_1 with a self-loop e_1, and $G(P^2\times I)$ has two vertices v_1, v_2 and one edge e_1 joining them. If one regards $G(M^3)$ as a topological space, then there is an embedding h: $G(M^3)\to M^3$ such that $h(v_i)$ is a point in $\mathrm{Int}\,(V_i)$ and $h(e_k)$ is an arc or loop in M^3 intersecting P_k^2 transversely in one point. By Theorem 1, it is clear that if M_1^3 and M_2^3 are homeomorphic, then $G(M_1^3)$ and $G(M_2^3)$ are isomorphic as graphs.

The *degree* $\deg(v)$ of a vertex v in a graph G is the number of edges of G incident with v, each self-loop counting as two edges. A connected graph G is said to be *eulerian* if each vertex of G has even degree.

Lemma 2.3. *If M^3 is an element of Π_c, then $G(M^3)$ is eulerian.*

Proof. Now $\deg(v_i)$ in $G(M^3)$ is equal to the number of components of ∂V_i. Furthermore $\deg(v_i)=\chi(\partial V_i)$, where $\chi(A)$ is the Euler characteristic of A. For V_i has only boundary components homeomorphic to P^2 and $\chi(P^2)=1$. Since $\chi(M^3)=(1/2)\chi(\partial M^3)$ for a compact 3-manifold M^3, in general, $\deg(v_i)$ must be an even integer.

Theorem 2. *For any eulerian graph G, there are infintely many elements M^3 of Π_e such that $G(M^3)$ is isomorphic to G.*

Proof. Let $v_1, \cdots, v_p$ be vertices and $e_1, \cdots, e_q$ edges of G. Choose a closed connected irreducible 3-manifold M_i of type $(1, q_i)$ corresponding to v_i, where $2q_i = \deg(v_i)$ $(i=1, \cdots, p)$. Let $\{P_{i1}^2, \cdots, P_{iq_i}^2\}$ be a complete system of projective planes in M_i. Then a 3-manifold $V_i = M_i - \mathrm{Int}(N(P_{i1}^2 \cup \cdots \cup P_{iq_i}^2, M_i))$ has $2q_i$ boundary components which are all homeomorphic to a projective plane and $q_1 + \cdots + q_p = q$. Let $Q_1^2, \cdots, Q_{2q}^2$ be the components of $\partial(V_1 \cup \cdots \cup V_p)$ numbered so that if e_k joins v_i and v_j, then $Q_{2k-1}^2 \subset V_i$ and $Q_{2k}^2 \subset V_j$. Let M^3 be the closed 3-manifold obtained from $V_1 \cup \cdots \cup V_p$ by identifying Q_{2k-1}^2 with Q_{2k}^2 for $k=1, \cdots, q$. Using Lemma 1.2 and Theorem 1, we can prove that M^3 is an irreducible 3-manifold of type (p, q) and that $G(M^3)$ is isomorphic to G. By Lemma 2.2, various choice of $M_1, \cdots, M_p$ gives infinitely many required 3-manifolds. This completes the proof.

Corollary 2.1. *Let p and q be positive integers. There is an element of type (p, q) in Π_e if and only if $p \leqq q$.*

Proof. Note that if a connected graph G has p vertices and q edges, then $p \leqq q+1$ and that $p = q+1$ if and only if G is a tree; see [4]. Since a tree is not eulerian, necessity is clear. If $p \leqq q$, then we have an eulerian graph $G(p, q)$ with vertices $v_1, \cdots, v_p$ and edges $e_1, \cdots, e_q$ such that e_i joins v_i and v_{i+1} $(i=1, \cdots, p-1)$, e_p joins v_p and v_1, and e_j is a self-loop attaching to one of $v_1, \cdots, v_p$ $(j=p+1, \cdots, q)$,

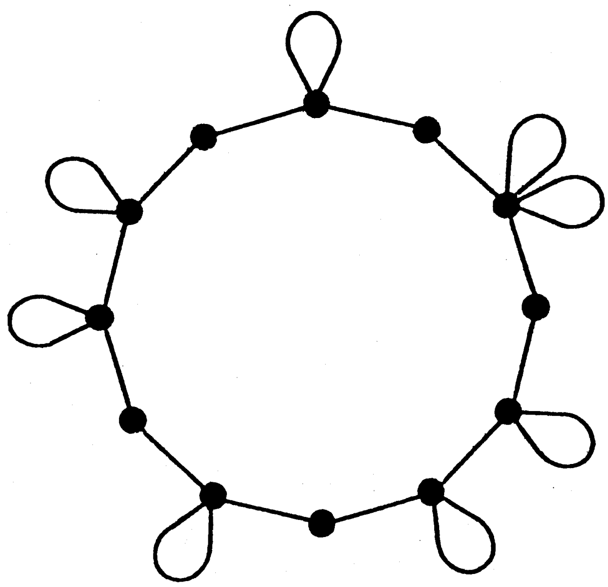

Figure 3.

as illustrated in Figure 3. By Theorem 2, there is a closed connected irreducible 3-manifold M^3 of type (p, q) associated with $G(p, q)$, and sufficiency follows.

Corollary 2.2. *Let p and q be positive integers. There is an element of type (p, q) in Π_b if and only if $p \leqq q+1$.*

Proof. Necessity is clear. When $p \leqq q$, construct a closed irreducible 3-manifold M^3 of type (p, q) associated with $G(p, q)$ above in the way of the proof of Theorem 2. If we use $M(q_i, r)$ $(r>0)$ as M_i, then M^3 contains a 2-sided incompressible Klein bottle K^2 within V_i which does not separate V_i and which is disjoint from a complete system $\{Q_2^2, Q_4^2, \cdots, Q_{2q}^2\}$ of projective planes in M^3. Cut M^3 along K^2. Then the resulting 3-manifold M_b^3 belongs to Π_b and $\{Q_2^2, \cdots, Q_{2q}^2\}$ can be regarded as a complete system in M_b^3, so M_b^3 is of type (p, q).

When $2<p=q+1$, construct a closed irreducible 3-manifold M^3 of type $(p-2, p-2)$ not homeomorphic to $P^2 \times S^1$, associated with the cycle $G(p-2, p-2)$ of length $p-2$, and cut M^3 along a 2-sided projective plane in M^3. The resulting 3-manifold belongs to Π_b and is of type $(p, p-1)$. For $(p, q)=(2, 1)$, we have $P^2 \times I$.

The proof is complete.

Corollary 2.3. *For any connected graph G, there is an element M^3 in Π_a such that $G(M^3)$ is isomorphic to G. Therefore there is an element of type (p, q) in Π_a if and only if $p \leqq q+1$, for positive integers p and q.*

Proof. It is sufficient to construct a bounded irreducible 3-manifold $V(d)$ such that $\partial V(d)$ contains precisely d projective planes and a compressible closed surface and that each projective plane in $\operatorname{Int}(V(d))$ is parallel to a boundary component, for any integer $d \geqq 2$. Copy $V(d)$ for each vertex of G which has degree d, and paste them.

Let $\{P_1^2, \cdots, P_q^2\}$ be a complete system of projective planes in $M(q, r)$ $(q, r>0)$ and K^2 a 2-sided incompressible Klein bottle in $M(q, r)$ disjoint from the complete system such that $M(q, r)-P_1^2 \cup \cdots \cup P_q^2 \cup K^2$ is connected. The 3-manifold $W(q)= M(q, r)-\operatorname{Int}(N(P_1^2 \cup \cdots \cup P_q^2 \cup K^2, M(q, r)))$ has $2q$ projective planes and two incompressible Klein bottles K_+^2, K_-^2 as boundary components. To make a compressible boundary component, attach a 1-handle $D^2 \times I$ to $W(q)$ so that $D^2 \times \partial I$ lies in $K_+^2 \cup K_-^2$. For an even integer $d=2q$, we can take the resulting 3-manifold as $V(d)$. When $d=2q+1$, we have to make not only a compressible boundary component but also one more projective plane. Let D_1^2 and D_2^2 be 2-disks in K_+^2 and $P^2 \times \{0\}$, respectively. Attach $P^2 \times I$ to $W(q)$ via a homeomorphism from D_2^2

to D_1^2, then $V(d)$ will be obtained. This completes the proof.

The question whether there is a bounded 3-manifold in Π_b associated with an arbitrary connected graph or not naturally arises. Unfortunately I don't have the answer. If one would like to give the affirmative solution, he ought to construct a boundary-irreducible 3-manifold whose boundary contains an odd number of projective planes. Construction of such 3-manifolds is easy if we take no account of their boundary-irreducibility, but it is so difficult to check whether they are boundary-irreducible or not.

§ 4. Invariants related to $G(M^3)$

Our theory admits all of invariants in graph theory, since the graph $G(M^3)$ is a stronger invariant. For example, $\delta(M^3)$ is defined to be the minimum degree among the vertices of $G(M^3)$ for $M^3 \in \Pi$. Then $\delta(M^3)=1$ if and only if ∂M^3 contains a projective plane.

We have an invariant which is connected with the homology group of M^3. We define $r(M^3)$ by the 1-dimensional Betti number $\beta_1(G(M^3))=\text{rank } H_1(G(M^3))$ of the topological space $G(M^3)$. If M^3 is of type (p, q), then $r(M^3)=q-p+1$, concretely, and $r(M^3)$ is the number of edges contained in the *cotree* of a spanning tree of the graph $G(M^3)$, graph-theoretically. From a geometric point of view, $r(M^3)$ can be said to be the maximum number of pairwise disjoint projective planes in Int (M^3) whose removal can not disconnect M^3.

Denote the submodule of $H_2(M^3, \partial M^3; Z_2)$ generated by all of embeddings of of a projective plane, by $H_2{}^p(M^3, \partial M^3; Z_2)$.

Theorem 3.1. *For an element* M^3 *of* Π, $\dim H_2{}^p(M^3, \partial M^3; Z_2)=r(M^3)\leqq\beta_1(M^3)$.

Proof. Let $v_1, \cdots, v_p$ be vertices and $e_1, \cdots, e_q$ edges being arcs or loops of the topological space $G(M^3)$. Let $P_k{}^2\times I$ $(k=1, \cdots, q)$ be a regular neighberhood of $P_k{}^2$ in M^3 which belongs to a complete system of projective planes in M^3, corresponding to e_k, and V_i $(i=1, \cdots, p)$ a component of $M^3-\text{Int}\,(P_1{}^2\times I\cup\cdots\cup P_q{}^2\times I)$ corresponding to v_i. Then there is a continuous map f: $M^3\to G(M^3)$ such that $f(V_i)=v_i$ and $f(P_k{}^2\times I)=e_k$. Since f_*: $H_1(M^3)\to H_1(G(M^3))$ is surjective, $\beta_1(M^3)\geqq \beta_1(G(M^3))=r(M^3)$, so the right-hand inequality holds.

Let $\{\alpha_1, \cdots, \alpha_r\}$ be a basis of $H_2{}^p(M^3, \partial M^3; Z_2)$, where $r=\dim H_2{}^p(M^3, \partial M^3; Z_2)$. Since any 2-sphere is homologically zero in M^3, there are r pairwise disjoint 2-sided projective planes $Q_1{}^2, \cdots, Q_r{}^2$ such that $[Q_i{}^2]=\alpha_i$. Necessarily $Q_1{}^2, \cdots, Q_r{}^2$ are not parallel to each other, and $M^3-Q_1{}^2\cup\cdots\cup Q_r{}^2$ is connected. By Theorem

1, an ambient isotopy carries $Q_1^2, \cdots, Q_r^2$ into $P_1^2, \cdots, P_r^2$, after modification of the subscripts.

If the graph obtained from $G(M^3)$ by deleting $e_1, \cdots, e_r$ is not a tree, then there is another projective plane, say P_{r+1}^2, such that $M^3 - P_1^2 \cup \cdots \cup P_r^2 \cup P_{r+1}^2$ is connected, so $P_1^2, \cdots, P_r^2, P_{r+1}^2$ are Z_2-homologically independent. This contradicts the choice of r. Therefore $e_1, \cdots, e_r$ make up a cotree of $G(M^3)$ and $r=q-p+1=r(M^3)$, so the left-hand equality holds.

Most of non-orientable compact 3-manifolds M^3 have infinite $H_1(M^3)$ and exceptions contain projective planes as boundary components. Hempel constructed such exceptions whose boundaries contain $2n$ projective planes and one closed orientable surface (Example 6.10 in [5]). His examples blong to Π_b, which he did not mention. Even if one wish to construct a 3-manifold M^3 such that $H_1(M^3)$ is infinite and each component of ∂M^3 is a projective plane, we can meet his requirement. Cut an element M_1^3 with $r(M_1^3)=r+n$ of Π_c along n mutually disjoint projective planes corresponding to some edges of a cotree in $G(M_1^3)$. Then the resulting 3-manifold M^3 belongs to Π_b, and $\beta_1(M^3) \geqq r(M^3)=r$.

Since we have irreducible closed 3-manifolds with non-trivial π_2 different from $P^2 \times S^1$, it follows from Tao's result stated in Introduction that there are those prime closed 3-manifolds whose orientable double covering spaces are not prime. In fact, each element M^3 of Π_c except $P^2 \times S^1$ is such a 3-manifold. Let $g: \tilde{M} \to M^3$ be the orientable double covering. For any projective plane Q^2 in M^3, $g^{-1}(Q^2)$ is a non-separating 2-sphere in $\tilde{M}$. Thus $\tilde{M}$ has $S^2 \times S^1$ as a prime facter. In particular, the orientable double covering spaces of our examples are homeomorphic to connected sums of several $S^2 \times S^1$'s and unions of knot or link spaces with their boundaries identified. Therefore if $\tilde{M}$ is prime, then $\tilde{M}$ is homeomorphic to $S^2 \times S^1$, and hence M^3 is homeomorphic to $P^2 \times S^1$, which one can prove using Theorem 1 in [8]. So $P^2 \times S^1$ can be said to be the unique 3-manifold in Π_c whose orientable double covering space is prime. Moreover we have a characterization of $P^2 \times S^1$ in terms of our theory, as follows:

Theorem 3.2. *Let M^3 be an element of Π_c. Then the followings* (1) *to* (3) *are equivalent:*

(1) *M^3 is homeomorphic to $P^2 \times S^1$.*

(2) *There is a double covering space $\tilde{M}$ of M^3 such that $\tilde{M}$ belongs to Π_c and $q(\tilde{M})=q(M^3)$.*

(3) *For every compact covering space $\tilde{M}$ of M^3 belonging to Π_c, $q(\tilde{M})=q(M^3)$.*

Proof. A compact covering space of $P^2 \times S^1$ is homeomorphic to either $P^2 \times S^1$ or a 2-sphere bundle over S^1. Since the latter is not an element of Π_e, (1) implies (3) immediately.

Let $P^2 \times I$ be a regular neighberhood of a projective plane P^2 in M^3, and $(M_i, P^2_{i0}, P^2_{i1})$ $(i=1, 2)$ two copies of $(M^3 - \mathrm{Int}\,(P^2 \times I), P^2 \times \{0\}, P^2 \times \{1\})$. Then the 3-manifold $\tilde{M}$ obtained from $M_1 \cup M_2$ by identifying P^2_{10} with P^2_{21} and P^2_{11} with P^2_{20} is a double covering space of M^3. Suppose (3), then $q(\tilde{M}) = q(M^3)$, so (3) implies (2).

Let g: $\tilde{M} \to M^3$ be a double covering with the covering translation τ: $\tilde{M} \to \tilde{M}$ of order two. Suppose that $\tilde{M}$ is irreducible and that $q(\tilde{M}) = q(M^3)$. Let $\{P_1{}^2, \cdots, P_q{}^2\}$ be a complete system of projective planes in M^3, where $q = q(M^3)$. By irreducibility of $\tilde{M}$, $g^{-1}(P_1{}^2 \cup \cdots \cup P_q{}^2)$ consists of $2q$ projective planes $Q_1{}^2, \cdots, Q^2_{2q}$. Since $q(\tilde{M}) = q(M^3)$, some pair are parallel and bound a submanifold E of $\tilde{M}$ homeomorphic to $P^2 \times I$, say $Q_1{}^2$ and $Q_2{}^2$. We may assume that $E \cap Q_i{}^2 = \phi$ $(i = 3, \cdots, 2q)$, by Lemma 1.1 and that $\tau(\mathrm{Int}\, E)) \cap \mathrm{Int}\,(E) = \phi$, by the fact that a single projective plane can not bound a compact 3-manifold. If $g(Q_1{}^2) \neq g(Q_2{}^2)$, then $g|E$ is an embedding and $g(Q_1{}^2)$ and $g(Q_2{}^2)$ are parallel. This contraditcs the fact that $g(Q_1{}^2)$ and $g(Q_2{}^2)$ are menbers of the complete system of projective planes in M^3. Therefore $g(Q_1{}^2) = g(Q_2{}^2)$ and $M^3 = g(E)$ is homeomorphic to $P^2 \times S^1$, so (2) implies (1).

This completes the proof.

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Department of Informational Science
Tokyo Institute of Technology,
Oh-Okayama, Meguro-ku,
Tokyo, 152 Japan