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# HOMOMORPHISMS AND MAXIMAL IDEALS IN THE ALGEBRA OF HOLOMORPHIC FUNCTIONS

By

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## 1. Introduction

We denote by $\Omega$ a domain of holomorphy in the complex Euclidean space $C^N$ and by $H(\Omega)$ the algebra of all holomorphic functions in $\Omega$, equipped with the topology of uniform convergence on each compact suset of $\Omega$.

Let $\mathscr{M}$ be the set of all maximal ideals of $H(\Omega)$. A well known theorem of Igusa [3] states:

**Theorem IG.** *For a maximal ideal* $M\in\mathscr{M}$, *the following conditions* (i)-(iv) *are equivalent*:

( i ) *M corresponds to a point of* $\Omega$.

(ii) *M is closed.*

(iii) $H(\Omega)/M\cong C$ (*the complex number field*).

(iv) *M is finitely generated.*

Non-closed maximal ideals are treated by some authors for the case $N=1$, see e.g. [1]. We considered the case $N\geqq 1$ in [6], and introduced a topology in $\mathscr{M}$ so that each $f\in H(\Omega)$ can be continuously extended to $\mathscr{M}$.

Let $\mathscr{M}^{\Delta}$ be the set of all closed maximal ideals of $H(\Omega)$. Theorem IG shows that an analytic structure can be introduced into $\mathscr{M}^{\Delta}$. Our theme in this paper is a study of behaviors of $f\in H(\Omega)$ in $\mathscr{M}-\mathscr{M}^{\Delta}$.

In the below, we suppose that $N=1$ and the domain $\Omega$ is the unit disk $U=\{|z|<1\}$, and write $T=\{|z|=1\}$, $\boldsymbol{P}=\{|z|\leqq\infty\}$ (the Riemann sphere).

Let $f\in H(\Omega)$. In §§2-3, we characterize the cluster set $C(f;\alpha)$ [2, p. 3] and the boundary cluster set $C_B(f;\alpha)$ [2, p. 81] of $f$ at $\alpha\in T$ by means of $\mathscr{M}$, and derive a fact (see (3.4)) which is an analogue of the theorem of Iversen-Gross [2, p. 91, Theorem 5.2].

In §4, we study some connections between $\mathscr{M}$ and the maximal ideal space of $H^{\infty}(U)$ (the set of all bounded holomorphic functions in $U$).

In §5, we consider analytic structures in $\mathscr{M}-\mathscr{M}^{\Delta}$.

Studies for the cases of more general plane domains or of higher dimensions will be further tasks.

For later use, we summarize here some results in [6].

(1.1) Let $M$ be a non-closed maximal ideal. We can choose functions $f_1, \cdots, f_N$, $f_j \in M$, such that $\{f_1=\cdots=f_N=0\}$ is a point sequence $Z=\{z_k\}$ in $\Omega$, infinite but not clustering in $\Omega$. Let $\Psi$ be a collection of subsets $E$ of $Z$ such that $E \in \Psi$ if there are functions $g_1, \cdots, g_h$ in $M$ with $\{f_1=\cdots=f_N=g_1=\cdots=g_h=0\}=E$. Then, $\Psi$ is a ultrafilter on $Z$. Thus, $M$ defines a pair $(Z, \Psi)$ of a sequence $Z$ and a ultrafilter $\Psi$ on $Z$.

(1.2) Let $M$ be a non-closed maximal ideal and $(Z, \Psi)$ be the pair defined by $M$. Let $I$ be the set of all positive integers and $\Phi$ be a collection of subsets $A$ of $I$ such that $A=\{n_k\} \in \Phi$ if there is $E \in \Psi$ with $E=\{z_{n_k}\}$. Let $\boldsymbol{C}^I$ be the set of all complex sequences. We say $\{a_i\} \equiv \{b_i\}$ if $\{i; a_i=b_i\} \in \Phi$. Then, $\boldsymbol{C}^*=\boldsymbol{C}^I/(\equiv)$ is a (transcendental) extension field of $\boldsymbol{C}$, and $H(\Omega)/M \cong \boldsymbol{C}^*$.

(1.3) Let $\mathscr{S}$ be the set of all infinite point sequences in $\Omega$, not clustering in $\Omega$. For a $Z \in \mathscr{S}$, let $\mathscr{U}(Z)$ be the set of all ultrafilters on $Z$, such that $\Psi \in \mathscr{U}(Z)$ contains all sets $E$ whose complements $Z-E$ are finite sets. Let $Z_1, Z_2 \in \mathscr{S}$ and $\Psi_1 \in \mathscr{U}(Z_1)$, $\Psi_2 \in \mathscr{U}(Z_2)$. We say $(Z_1, \Psi_1) \sim (Z_2, \Psi_2)$ if there is a $Z' \in \mathscr{S}$, $Z' \subset Z_1 \cap Z_2$, $Z' \in \Psi_1 \cap \Psi_2$, and $Z' \cap \Psi_1 = Z' \cap \Psi_2$.

(1.4) If $(Z_1, \Psi_1)$ and $(Z_2, \Psi_2)$ are pairs defined by the same maximal ideal $M$, as stated in (1.1), then $(Z_1, \Psi_1) \sim (Z_2, \Psi_2)$ in the sense (1.3). Conversely, for a pair $(Z, \Psi)$ put

$$M=\{f \in H(\Omega);\ Z \cap \{f=0\} \in \Psi\}\ .$$

Then $M$ is a maximal ideal, and, if $(Z_1, \Psi_1) \sim (Z, \Psi)$, then $(Z_1, \Psi_1)$ determines the same maximal ideal $M$.

(1.5) We say an equivalence class $[(Z, \Psi)]$ determines a boundary point $b_\Psi$ of $\Omega$. Put

$$\delta\Omega=\{b_\Psi;\ Z \in \mathscr{S}, \Psi \in \mathscr{U}(Z)\}\ , \quad \Omega^*=\Omega \cup \delta\Omega\ .$$

Then by (1.4), the maximal ideal space $\mathscr{M}$ of $H(\Omega)$ and $\Omega^*$ correspond each other in a one-to-one way.

(1.6) $\Omega^*$ is topologized as follows: For a point $z \in \Omega$, neighborhoods of $z$ are defined as usual. For a $b_\Psi \in \delta\Omega$, a subset $N$ of $\Omega^*$ is a neighborhood of $b_\Psi$ if

(a) $N \cap \Omega$ is an open set in $\Omega$, containing a set $E \in \Psi$, and

(b) $N \cap \delta\Omega$ consists of boundary points determined by classes of ultrafilters on point sequences ($\in \mathscr{S}$) contained in $N \cap \Omega$.
By this topology, $\Omega^*$ is a Hausdorff space, which is countably compact.

(1.7) Each $f \in H(\Omega)$ can be extended continuously on $\Omega^*$, as a map $\Omega^* \to \boldsymbol{P}$. This continuous extension of $f$ is denoted by $f^\#$.

We suppose $N=1$ and $\Omega = U$ (the unit disk) in this paper.

## 2. Homomorphisms of $H(U)$

A maximal ideal $M$ determines a point sequence $Z=\{z_k\}$ and a ultrafilter $\Psi$, which in turn determine a boundary point $b_\Psi$ (see (1.5)). The maximal ideal $M$, or equivalently the boundary point $b_\Psi$, defines a homomorphism $\phi$ of $H(U)$ onto $\boldsymbol{C}^*$ (an extension field of $\boldsymbol{C}$), $\boldsymbol{C}^* \cong H(U)/M$, see (1.2).

Conversely, let $\phi$ be a homomorphism of $H(U)$ onto an extension field $\boldsymbol{C}^* = \boldsymbol{C}^I/(\equiv)$, where $\boldsymbol{C}^I$ is the set of all complex sequences and $\equiv$ is the equivalence relation defined by a ultrafilter $\Phi$ on the set of all positive integers $I$, see (1.2). Then, the kernel $K=\ker(\phi)$ of the homomorphism $\phi$ is obviously a maximal ideal in $H(U)$. $K$ can not correspond to a point of $U$, for otherwise we would have $H(U)/K \cong \boldsymbol{C}$, in contradiction that $H(U)/K \cong \boldsymbol{C}^*$. Thus, if $f \in K$, $f$ has infinitely many zero points $Z=\{z_k\}$, see (1.1). $K$ induces a ultrafilter $\Psi$ on $Z$, which corresponds to the ultrafilter $\Phi$ on $I$, which defines the equivalency ($\equiv$). The maximal ideal $K$, or equivalently $(Z, \Psi)$, defines a boundary point. In this way, the maximal ideal space $\mathscr{M}$ of $H(U)$, which is a countable-compactification $U^*$ of $U$, can be identified with the set of homomorphisms of $H(U)$, of the type stated above.

Every $f \in H(U)$ is continuous at boundary points as a map to $\boldsymbol{P}$, see (1.7). Thus, at a boundary point $b_\Psi$, $f$ has a limit $f(b_\Psi)$, which we denote as $\phi^\#(f)$, where $\phi$ is the homomorphism defined by $b_\Psi$. Thus

$$\phi^\#(f) = f(b_\Psi) . \tag{2.1}$$

Then, $\phi^\#$ is a map of $H(U)$ to $\boldsymbol{P}$. For every $f \in H(U)$, we have

$$f^\#(\phi) = \phi^\#(f) \quad (= f(b_\Psi)) , \tag{2.2}$$

where $f^\#$ is the continuous extension of $f$ (see (1.7)).

For the function $z \in H(U)$, we use the symbol $\pi$ for $z^\#$:

$$\pi(\phi) = z^\#(\phi) , \quad \phi \in \mathscr{M} .$$

By the definitions, we have easily

**Theorem 2.1.** *The mapping $\pi$ is a continuous map of $\mathcal{M}$ onto the closed disk $\bar{U}$. Over the disk $U$, $\pi$ is one-to-one, and $\pi^{-1}$ maps $U$ homeomorphically onto the open subset $\mathcal{M}'$ of $\mathcal{M}$. $\mathcal{M}'$ is the set of all closed maximal ideals $\mathcal{M}^{\Delta}$ of $H(U)$. That is, $\pi^{-1}$ maps $\lambda \in U$ to $\phi_\lambda \in \mathcal{M}^{\Delta}$, where $\phi_\lambda$ denotes the evaluation at $\lambda$.*

It is convenient to picture $\pi$ as a projection of $\mathcal{M}$ onto the closed unit disk. By Theorem 2.1, $\pi(\mathcal{M}^{\Delta})=U$. The remainder $\mathcal{M}-\mathcal{M}^{\Delta}$ is mapped by $\pi$ onto the circumference $T$. For $\alpha \in T$, we put

(2.3) $$\mathcal{M}_\alpha=\pi^{-1}(\alpha)=\{\phi \in \mathcal{M};\ \phi^*(z)=\alpha\}\ .$$

$\mathcal{M}_\alpha$ is called the *fiber of $\mathcal{M}$ over $\alpha$*. It is a closed subset of $\mathcal{M}$. We have

(2.4) $$\mathcal{M}=\mathcal{M}^{\Delta}\cup\Big(\bigcup_{\alpha\in T}\mathcal{M}_\alpha\Big)\ ,\quad \mathcal{M}_\alpha\cap\mathcal{M}_\beta=\text{void if } \alpha\neq\beta\ .$$

For $f \in H(U)$ and $\alpha \in T$, we define the cluster set $C(f;\alpha)$ of $f$ at [2, p. 3] by

(2.5) $$\begin{aligned} C(f;\alpha)&=\{\zeta \in \boldsymbol{P};\ \text{there is a seq. } \{\lambda_n\}\subset U \text{ with } f(\lambda_n)\to\zeta\} \\ &=\bigcap_{\varepsilon>0}\overline{f(U\cap\{|z-\alpha|<\varepsilon\})}\ . \end{aligned}$$

**Theorem 2.2.** $f^*(\mathcal{M}_\alpha)=C(f;\alpha)$.

**Proof.** Let $b_\Psi \in \mathcal{M}_\alpha$. Then there is a sequence $Z=\{\lambda_n\}$ and a ultrafilter $\Psi$ on $Z$, determining $b_\Psi$. Let $\{\lambda_{n(\nu)}\}$ be a subnet of $Z$ such that $\lambda_{n(\nu)}\to b_\Psi$. Then $z^*(\lambda_{n(\nu)})\to\alpha$ and $f^*(\lambda_{n(\nu)})\to f(b_\Psi)$ and $f(b_\Psi)\in C(f;\alpha)$. Conversely, if $\zeta \in C(f;\alpha)$, there is $\lambda_n\to\alpha$ and $f(\lambda_n)\to\zeta$. Let $\Psi$ be a ultrafilter on $Z=\{\lambda_n\}$ and $b_\Psi$ be a boundary point determined by $(Z,\Psi)$. If $\{\lambda_{n(\nu)}\}$ is a subnet of $\{\lambda_n\}$ such that $\lambda_{n(\nu)}\leftarrow b_\Psi$, then $z^*(\lambda_{n(\nu)})\to\alpha$, whence $b_\Psi\in\mathcal{M}_\alpha$, and $f(\lambda_{n(\nu)})\to f(b_\Psi)$, therefore $\zeta=f(b_\Psi)\in f^*(\mathcal{M}_\alpha)$. Q.E.D.

By the way, we define the *boundary cluster set* $C_B(f;\alpha)$ [2, p. 81] of $f\in H(U)$ at $\alpha\in T$ by

(2.6) $$C_B(f;\alpha)=\bigcap_{\varepsilon>0}\overline{\big(\bigcup\{C(f;\beta);\ 0<|\beta-\alpha|<\varepsilon,\ \beta\in T\}\big)}\ .$$

### 3. Boundary fibers and boundary cluster sets

Now we put

(3.1) $$\mathcal{M}_{\alpha,\varepsilon}=\bigcup\{\mathcal{M}_\beta;\ 0<|\beta-\alpha|<\varepsilon,\ \beta\in T\}\ ,$$

(3.2) $$\mathcal{M}_\alpha^B=\bigcap_{\varepsilon>0}\text{cl}\,(\mathcal{M}_{\alpha,\varepsilon})\ ,$$

in which cl$(E)$ denotes the closure of $E$ in $\mathcal{M}$. $\mathcal{M}_\alpha^B$ is non-void, since $\mathcal{M}$ is countably compact (1.6). We call $\mathcal{M}_\alpha^B$ the *boundary fiber* of $\mathcal{M}$ at $\alpha$. $m=\mathcal{M}_\alpha^B$

can not belong to any $\mathcal{M}_\beta$ if $\beta\neq\alpha$. Hence

$$\mathcal{M}_\alpha^B\subset\mathcal{M}_\alpha . \tag{3.3}$$

$\mathcal{M}_\alpha$ and $\mathcal{M}_\alpha^B$ are closed in $\mathcal{M}$.

**Theorem 3.1.** *We have*

$$\mathcal{M}_\alpha-\mathcal{M}_\alpha^B \quad \textit{is open in} \quad \mathcal{M}-U \tag{3.4}$$

*and*

$$f^*(\mathcal{M}_\alpha^B)=C_B(f;\alpha) . \tag{3.5}$$

**Proof.** Take $m\in\mathcal{M}_\alpha-\mathcal{M}_\alpha^B$. $m$ is defined by a point sequence $Z=\{\lambda_n\}$ and a ultrafilter $\Psi$ on $Z$. There is a set $E\in\Psi$ such that $\bar{E}\cap T=\{\alpha\}$. Suppose to the contrary. Then, for any $E\in\Psi$, there would be a point $\beta_E\neq\alpha$, $\beta_E\in\bar{E}\cap T$. Take a neighborhood $N$ of $m$. $N\cap U$ contains a set $E\in\Psi$, whence $N$ intersects with $\mathcal{M}_{\beta_E}$, which implies that $m\in\mathrm{cl}\,(\mathcal{M}_{\alpha,\beta})$ for every $\varepsilon>0$, $m\in\mathcal{M}_\alpha^B$, contradicting to the hypothesis.

Thus, there is a neighborhood $N$ of $m$ such that

$$N\cap(\mathcal{M}-U)\subset\mathcal{M}_\alpha-\mathcal{M}_\alpha^B ,$$

which proves (3.4).

Now we have, by (2.6) and Theorem 2.2,

$$\begin{aligned} C_B(f;\alpha)&=\bigcap_{\varepsilon>0}\overline{(\bigcup\{f^*(\mathcal{M}_\beta);\ 0<|\beta-\alpha|<\varepsilon,\ \beta\in T\})} \\ &=\bigcap_{\varepsilon>0}\overline{(f^*(\bigcup\{\mathcal{M}_\beta;\ 0<|\beta-\alpha|<\varepsilon,\ \beta\in T\}))} \\ &\supset\bigcap_{\varepsilon>0}f^*(\mathrm{cl}\,(\mathcal{M}_{\alpha,\varepsilon}))\supset f^*(\bigcap_{\varepsilon>0}\mathrm{cl}\,(\mathcal{M}_{\alpha,\varepsilon}))=f^*(\mathcal{M}_\alpha^B) . \end{aligned}$$

Let $w_0\in C_B(f;\alpha)$. Take a sequence $\varepsilon_n\downarrow 0$. Then

$$w_0\in\overline{\bigcup\{f^*(\mathcal{M}_\beta);\ 0<|\beta-\alpha|<\varepsilon_n,\ \beta\in T\}} .$$

We can choose, for a sequence $\delta_n\downarrow 0$,

$$w_n\in\bigcup\{f^*(\mathcal{M}_\beta);\ 0<|\beta-\alpha|<\varepsilon_n,\ \beta\in T\} ,\quad |w_n-w_0|<\delta_n .$$

Then, $w_n\to w_0$. On the other hand,

$$w_n=f^*(m_n) \qquad \text{for an} \quad m_n\in\mathcal{M}_{\alpha,\varepsilon_n} .$$

By the countable compactness of $\mathcal{M}$, we have that

$$Q=\bigcap_{k>0}(\mathrm{cl}\,(\{m_n\}_{n\geq k})) \quad \text{is non-void.}$$

Take a point $m_0$ in $Q$. Then $m_0 \in \mathrm{cl}(\mathscr{M}_{\alpha,\varepsilon})$ for every $\varepsilon>0$, hence $m_0$ belongs to $\mathscr{M}_\alpha^B$. Obviously

$$f^*(m_0) \in \overline{\{f^*(m_n);\ n \geqq k\}} \qquad \text{for every} \quad k\,,$$

therefore

$$f^*(m_0)=w_0\,, \quad w_0 \in f^*(\mathscr{M}_\alpha^B)\,,$$

which implies (3.5). Q.E.D.

We denote by $\omega$ a subdomain of $U$ whose boundary $\partial\omega$ consists of a Jordan curve such that $\partial\omega \cap T=\{\alpha\}$. Put

$$\mathscr{M}_\alpha^\omega=\{m \in \mathscr{M}_\alpha;\ m \text{ is defined by a sequence } Z=\{\lambda_n\} \text{ in } \omega \text{ and a ultrafilter } \Psi \text{ on } Z\}\,.$$

**Theorem 3.2.** *We have*

$$\mathscr{M}_\alpha-\mathscr{M}_\alpha^B=\bigcup_\omega \mathscr{M}_\alpha^\omega\,, \tag{3.6}$$

*where the sum on the right is taken over all subdomains $\omega$ of the type stated above.*

**Proof.** Obviously, $\bigcup_\omega \mathscr{M}_\alpha^\omega \subset \mathscr{M}_\alpha-\mathscr{M}_\alpha^B$. Conversely, take $m \in \mathscr{M}_\alpha-\mathscr{M}_\alpha^B$. $m$ is defined by a sequence $Z=\{\lambda_n\}$ and a ultrafilter $\Psi$ on $Z$. By the proof of Theorem 3.1, there is a set $E \in \Psi$, $\bar{E} \cap T=\{\alpha\}$. Put

$$\Psi_E=E \cap \Psi=\{E \cap F;\ F \in \Psi\}\,.$$

Then, $\Psi_E$ is a ultrafilter on $E$, and $(E, \Psi_E)$ is equivalent to $(Z, \Psi)$ in the sense stated in (1.3). Thus the point $m$ is also defined by $(E, \Psi_E)$. The sequence $E$ is contained in some domain $\omega$ of the type stated above, hence $m \in \bigcup_\omega \mathscr{M}_\alpha^\omega$, which proves (3.6). Q.E.D.

Further, we have

**Theorem 3.3.** *$\mathscr{M}_\alpha-\mathscr{M}_\alpha^B$ is connected.*

**Proof.** By Theorem 3.2, it suffices to show that $\mathscr{M}_\alpha^\omega$ is connected. Suppose to the contrary, $\mathscr{M}_\alpha^\omega=A \cup B$ with open sets $A$ and $B$ in $\mathscr{M}_\alpha^\omega$, $A \cap B=$void. $A$ and $B$ are derived from some open sets in $U$, say $O_A$ and $O_B$, respectively. Since $A \cap B=$void, we may assume that $O_A \cap O_B=$void, and

$$O_A \cup O_B \supset \omega \cap \{|z-\alpha|<r\} \qquad \text{for some} \quad r>0\,,$$

because $A \cup B=\mathscr{M}_\alpha^\omega$. This is absurd by the connectivity of $\omega$. Q.E.D.

By the same method, we can show

**Theorem 3.4.** *$\mathscr{M}_\alpha$ is connected.*

## 4. Maximal ideal spaces of $H(U)$ and of $H^\infty(U)$

We denote by $H^\infty(U)$ the set of all bounded holomorphic functions in $U$. $H^\infty(U)$ is a Banach algebra with the supremum norm, and has been investigated by several authors, see e.g., [4]. We write the maximal ideal space of $H^\infty(U)$ as $\mathscr{M}(H^\infty)$. $\mathscr{M}$ denotes the maximal ideal space of $H(U)$ as before.

**Theorem 4.1.** *For any $\alpha\in T$, there is a maximal ideal $M\in\mathscr{M}_\alpha$ which does not contain any bounded holomorphic function in $U$, other than the identically zero function.*

**Proof.** Let $Z=\{\lambda_n\}$ be a point sequence in $U$ such that

(4.1) $$\lambda_n\to\alpha \quad \text{as} \quad n\to\infty\,, \qquad \text{and} \qquad \sum(1-|\lambda_n|)=\infty\,.$$

Let $\mathscr{F}$ be a family of subsets of $Z$ defined as follows: $E\subset Z$ belongs to $\mathscr{F}$ if

$$Z-E=\{\lambda_{n_k}\} \quad \text{satisfies} \quad \sum(1-|\lambda_{n_k}|)<\infty\,.$$

Let $\Psi$ be a ultrafilter containing $\mathscr{F}$. Then the maximal ideal $M$ corresponding to $(Z,\Psi)$ can not contain any bounded function other than 0. Q.E.D.

**Theorem 4.2.** *There is a continuous map $\mu$ from $\mathscr{M}$ onto $\mathscr{M}(H^\infty)$ such that for any $f\in H^\infty(U)$*

(4.2) $$f^\#(m)=f^\wedge(\mu(m))\,, \quad m\in\mathscr{M}\,,$$

*where $f^\wedge$ denotes the Gelfand representation of $f$* [4, p. 159].

**Proof.** Let $M$ be a maximal ideal in $H(U)$ and $\phi$ be the corresponding homomorphism of $H(U)$ onto $C^*$, an extension field of $C$. Then, $\phi^\#$, defined in (2.1), gives obviously a complex homomorphism of $H^\infty(U)$ onto $C$, and hence there exists a unique point $m^\#\in\mathscr{M}(H^\infty)$ corresponding to $m$, and the map $\mu$ is defined by

$$m^\#=\mu(m)\,.$$

(4.2) is clearly satisfied by the definition.

Conversely, take $m'\in\mathscr{M}(H^\infty)$. There is a point sequence $Z=\{\lambda_n\}$ in $U$ whose closure in $\mathscr{M}(H^\infty)$ contains $m'$ [5, p. 85, Corollary]. Let the subnet $\{\lambda_{n(\iota)}\}$ of $\{\lambda_n\}$ converge to $m'$.

We define a family $\mathscr{F}$ of subsets of $Z$ as follows: $E\subset Z$ belongs to $\mathscr{F}$ if there is an $\iota_0$ such that

$$\{\lambda_{n(i)};\ i\geqq i_0\}\subset E\ .$$

Then, $\mathscr{F}$ is a filter on $Z$, and a ultrafilter $\Psi$ containing $\mathscr{F}$ defines a point $m$ in $\mathscr{M}$. We will show that

$$m'=m^{\#}=\mu(m)\ . \tag{4.3}$$

Let $M'$ and $M$ be the maximal ideals in $H^{\infty}(U)$, respectively, which are represented by $m'$ and $m$ in $\mathscr{M}(H^{\infty})$ and $\mathscr{M}$, respectively. Let $\phi$ be the homomorphism corresponding to $M$. For any $f\in M'\subset H^{\infty}(U)$, we have

$$\lim_{i} f(\lambda_{(i)})=f^{\wedge}(m')=0\ .$$

It is easy to see that the closure of $\{\lambda_{n(i)}\}$ in $\mathscr{M}$ contains $m$. Hence $f(m)$ belongs to the closure of $(f(\lambda_{n(i)})\}$, therefore $f(m)=0$, $\phi^{\#}(f)=0$, which shows that

$$M'=\ker(\phi^{\#})=M^{\#}\ ,$$

and (4.3) holds. Thus, the map $\mu$ is onto $\mathscr{M}(H^{\infty})$.

Now we will prove the continuity of $\mu$. Let $\{\phi_i\}$ be a net in $\mathscr{M}$ which converges to $\phi$. Since $f\in H^{\infty}(U)\subset H(U)$ is continuous on $\mathscr{M}$, we have $f^{\#}(\phi_i)\to f^{\#}(\phi)$, hence

$$\phi_i^{\#}(f)\to\phi^{\#}(f)\ ,\quad f\in H^{\infty}(U)\ ,$$

which shows that

$$\mu(\phi_i)=\phi_i^{\#}\to\phi^{\#}=\mu(\phi)$$

in the weak* topology of $\mathscr{M}(H^{\infty})$, whence $\mu$ is continuous. Q.E.D.

$\{f^{\#};f\in H(U)\}$ separates points of $\mathscr{M}$ hence of $\mathscr{M}(H^{\infty})$, while $\{f^{\#};f\in H^{\infty}(U)\}$ separates points of $\mathscr{M}(H^{\infty})$ but not points of $\mathscr{M}$.

## 5. Analytic structures

We write

$$L_{\lambda}(z)=L(z;\ \lambda)=(z+\lambda)/(1+\bar{\lambda}z)\ ,\quad |\lambda|<1\ , \tag{5.1}$$

which transforms $U$ into $\mathscr{M}$, i.e., $L_{\lambda}\in(\mathscr{M})^{U}$.

$L_{\lambda}$ can be considered also as a map from $U$ into $\mathscr{M}(H^{\infty})$. In this point of view, we write $L_{\lambda}$ as $\mathscr{L}_{\lambda}$. That is,

$$\lambda\in U\to\mathscr{L}_{\lambda}\in(\mathscr{M}(H^{\infty}))^{U}\ . \tag{5.2}$$

By [5, p. 88, Theorem 4.3], the transformation (5.2) can be extended to $\mathscr{M}(H^{\infty})\to(\mathscr{M}(H^{\infty}))^{U}$, $m'\in\mathscr{M}(H^{\infty})\to\mathscr{L}_{m'}\in(\mathscr{M}(H^{\infty}))^{U}$. For transformation

(5.3) $$\lambda \in U \to L_\lambda \in (\mathscr{M})^U$$

we obtain, correspondingly,

**Theorem 5.1.** *The transformation* (5.3) *can be extended to a continuous transformation which assigns to each* $m \in \mathscr{M}$ *a map* $L_m \in (\mathscr{M})^U$.

*Let* $\mu$ *be the continuous map of* $\mathscr{M}$ *onto* $\mathscr{M}(H^\infty)$ *in Theorem* 4.2. *Let* $m \in \mathscr{M}$. *If* $\mu(m)$ *belongs to the closure (in* $\mathscr{M}(H^\infty)$) *of some interpolating sequence in* $U$ (*see* [5, p. 80]), *then* $L_m$ *is a one-to-one analytic map of* $U$ *onto a subset* $P(m)$ *of* $\mathscr{M}$ *such that*

$$\mu(P(m)) = \textit{the Gleason part } \mathscr{P}(\mu(m)) \textit{ of } \mu(m) \textit{ in } \mathscr{M}(H^\infty). \quad [5, \text{ p. } 75]$$

$P(m)$ *is an analytic set in the sense that for each* $f \in H^\infty(U)$, $(f^\sharp \circ L_m)(z)$ *is a holomorphic function in* $U$ (*see* [5, p. 77]).

*If* $\mu(m)$ *does not belong to the closure (in* $\mathscr{M}(H^\infty)$) *of any interpolating sequence in* $U$, *then*

$$\mu(L_m(U)) = \mu(m) ,$$

i.e., *any bounded holomorphic function is constant on* $L_m(U)$.

**Proof.** Let $m \in \mathscr{M}$ be defined by a sequence $Z=\{\lambda_n\}$ and a ultrafilter $\Psi$ on $Z$. Let $\{\lambda_{n(i)}\}$ be a subnet converging to $m$. Then $\{\lambda_{n(i)}\}$ converges in $\mathscr{M}(H^\infty)$ to $\mu(m)$, since the map $\mu$ is continuous.

By [5, p. 88, Theorem 4.3], we have in $(\mathscr{M}(H^\infty))^U$

$$\lim_i \mathscr{L}_{\lambda_{n(i)}} = \mathscr{L}_{\mu(m)} \quad \text{(see, especially, [5, p. 90])} .$$

If $\mu(m)$ is in the closure of some interpolating sequence, then $\mathscr{L}_{\mu(m)}(U)$ is the Gleason part $\mathscr{P}(\mu(m))$ in $\mathscr{M}(H^\infty)$. Take a point $z \in U$. Put

$$W=\{w_n\} , \qquad w_n = L(z;\, \lambda_n) ,$$

and, for a set $E \in \Psi$,

$$E_W = \{L(z;\, \lambda_n);\, \lambda_n \in E\} .$$

Then

$$\Psi_W = \{E_W;\, E \in \Psi\}$$

is a ultrafilter on $W$. $(W, \Psi_W)$ defines a point $m_z \in \mathscr{M}$. Then obviously,

$$w_{n(i)} = L(z;\, \lambda_{n(i)}) \to m_z \qquad \text{in } \mathscr{M} .$$

We define a map $L_m$ from $U$ into $\mathscr{M}$ by

$$L_m(z) = m_z .$$

Since $\{w_{n(i)}\}$ converges to $\mu(m_z)$ in $\mathscr{M}(H^\infty)$ by the continuity of $\mu$, we have

$$\mathscr{L}_{\mu(m)}(z)=\mu(L_m(z)) .$$

If $z_1, z_2 \in U(z_1 \neq z_2)$, then $\mathscr{L}_{\mu(m)}(z_1) \neq \mathscr{L}_{\mu(m)}(z_2)$ since $\mathscr{L}_{\mu(m)}$ is one-to-one. Hence

$$L_m(z_1) \neq L_m(z_2) ,$$

and the map $L_m$ is one-to-one. Obviously, by (4.2),

$$(f^\# \circ L_m)(z)=f^\wedge(\mu(L_m(z))=(f^\wedge \circ \mathscr{L}_{\mu(m)})(z)$$

is holomorphic on $U$ [5, p. 88, Theorem 4.3], hence $P(m)=L_m(U)$ is an analytic set. Further, by [5, p. 88, Theorem 4.3],

$$\mu(P(m))=\mu(L_m(U))=\mathscr{L}_{\mu(m)}(U)=\mathscr{P}(\mu(m))=\text{the Gleason part of } \mu(m) \text{ in } \mathscr{M}(H) .$$

Suppose, on the other hand, let $\mu(m)$ does not belong to the closure of any interpolating sequence. Then, $\mathscr{L}_{\mu(m)}$ is constant: $\mathscr{L}_{\mu(m)}(U)=\{\mu(m)\}$ ([5, p. 88, Theorem 4.3]), thus

$$\mu(L_m(U))=\mathscr{L}_{\mu(m)}(U)=\{\mu(m)\} .$$

Q.E.D.

Obviously, we have that

$$\text{if} \quad m \in \mathscr{M}_\alpha , \quad \text{then} \quad L_m(U) \subset \mathscr{M}_\alpha .$$

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