APPLICATION OF SPLINE-FUNCTIONS FOR THE CONSTRUCTION OF AN APPROXIMATE SOLUTION OF BOUNDARY VALUE PROBLEMS FOR A CLASS OF FUNCTIONAL-DIFFERENTIAL EQUATIONS

By

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Summary. The authors have constructed an approximate solution in the form of a spline-function of third degrae for a boundary value problem for functional-differential equation of second order.

In the last decades there is a vigorous advance of the theory of functional-differential equations due to the increasing circle of applications of functional-differential equations in various fields of science and technology. A detailed survey of the literature that reflects this theory is done in [1], [2] and others.

The interest towards boundary value problems for functional-differential equations is great. It is known that a basic source of such problems are the variational problems with deviating argument, the problem of optimal control, problems of ballistics and so on. Since the solution of boundary value problems as a rule is not found in closed form, then the methods for their approximate solution assumes a great importance.

The present paper proposes an efficient method for finding an approximate solution in the form of a spline-function of third degree of a boundary value problem for differential equations with deviating argument of second order.

The idea of using spline-functions when asking for an approximate solution of boundary value problems for differential equations without deviations of the arqument has been applied in a number of papers, for instance, [3]–[6].

1. Problem statement

Consider the boundary value problem

$$(1) \qquad y''(x) = f(x, y(x), y'(x), y(x-\tau_0(x)), y'(x-\tau_1(x))) , \quad x \in [a, b] ,$$

$$(2) \qquad y(x) = 0 \quad \text{for} \ \alpha \le x \le a ; \quad y'(x) = 0 \quad \text{for} \ \alpha \le x < a ; \quad y(b) = 0 .$$

Here $f(x, \xi_1, \eta_1, \xi_2, \eta_2)$ is a continuous function in the domain

$$G=[a, b] \times G_1 \times G_2 \times G_1 \times G_2 ,$$

where

$$G_1=\{\xi: \xi \in R, |\xi| \leqq Q_1\} , \qquad G_2=\{\zeta: \zeta \in R, |\zeta| \leqq Q_2\}$$

(Q_1 and Q_2 are constants), the delays $\tau_0(x) \geqq 0$ and $\tau_1(x) \geqq 0$ are continuous functions in the interval $[a, b]$;

$$\alpha=\min \{ \inf_{x \in [a,b]} (x-\tau_0(x)), \inf_{x \in [a,b]} (x-\tau_1(x))\} .$$

2. Existence and uniqueness of the solution of the boundary value problem

We ask for a smooth for $x \in (a, b)$ solution $y(x)$ of the equation (1) satisfying the conditions (2). Assume that $y(a+0)=0$, but smoothness of the solution at the point a is not assumed.

By E denote the set $\{x: x \in (a, b), x-\tau_1(x)=a\}$. We will consider the case when the set E is finite, i.e. there exists a natural number l, such that $E=\{x^{(i)}\}$, $a < x^{(i)} < b$, $i=1, 2, \cdots, l$.

The accepted assumptions yield that the second derivative of the solution $y(x)$ of the boundary value problem (1), (2) will be in general partially continuous if the right-hand side of the equation (1) really depends on $y'(x-\tau_1(x))$.

We set

$$Q=\sup \{|f(x, \xi_1, \eta_1, \xi_2, \eta_2)|: |\xi_i| \leqq Q_1, i=1, 2; |\eta_j| \leqq Q_2, j=1, 2, a \leqq x \leqq b\} ;$$

$$\delta_1=[a, x^{(1)}] , \qquad \delta_2=[x^{(1)}, x^{(2)}], \cdots, \delta_{l+1}=[x^{(l)}; b] \quad (x^{(s)} \in E, s=1, 2, \ldots, l) ;$$

$$\Omega=\{y(x): y(x) \in C[a, b]; y(x) \in C^1[a, b]; y(x) \in C^2[\delta_r], r=1, 2, \ldots, l+1;$$

$$\max_{r=1,2,\ldots,l+1} \{\sup_{\delta_r} |y''(x)|\} \leqq Q; y(x)=0, x \in [a, a] \cup \{b\}; y'(x)=0, x \in [a, a]\} .$$

Theorem 1. *Let*

1. the following condition hold:

(3)
$$b-a \leqq \min \left\{\left(\frac{8Q_1}{Q}\right)^{1/2}, \frac{2Q_2}{Q}\right\} ;$$

2. In the domain G the function $f(x, \xi_1, \eta_1, \xi_2, \eta_2)$ satisfies the Lipschitz condition

$$|f(x, \xi_1, \eta_1, \xi_2, \eta_2)-f(x, \bar{\xi}_1, \bar{\eta}_1, \bar{\xi}_2, \bar{\eta}_2)| \leqq L_1|\xi_1-\bar{\xi}_1|+L_2|\eta_1-\bar{\eta}_1|+L_3|\xi_2-\bar{\xi}_2|+L_4|\eta_2-\bar{\eta}_2| ;$$

3.
$$(L_1+L_3)\frac{(b-a)^2}{8}+(L_2+L_4)\frac{b-a}{2}<1 .$$

Then in the class of functions Ω there exists a unique solution $y(x)$ of the boundary value problem (1), (2).

Proof. Consider the space

$$(B, \|\cdot\|_B) = (C[\alpha, b] \cap C^1[\alpha, a] \cap C^1[a, b], \|\cdot\|_B)$$

with norm

(4) $\quad \|y\|_B = \max \left\{ \dfrac{8}{(b-a)^2} \max_{\alpha \leq x \leq b} |y(x)|, \ \dfrac{2}{b-a} \max \{ \max_{\alpha \leq x \leq a} |y'(x)|, \ \max_{a \leq x \leq b} |y'(x)| \} \right\}.$

Determine the operator Π, acting from the space Ω into the space B according to the formula

$$\Pi y(x) = \begin{cases} \displaystyle\int_a^b \bar{G}(x; s) f(s, y(s), y'(s), y(s-\tau_0(s)), y'(s-\tau_1(s))) ds, & x \in [a, b] \\ 0, & x \in [\alpha, a]. \end{cases}$$

Here

$$\bar{G}(x; s) = \begin{cases} G(x; s), & (a \leq x \leq b, \ a \leq s \leq b) \\ 0, & \text{otherwise} \end{cases}$$

while $G(x; s)$ is the Green function for the boundary value problem $y''(x) = 0$, $y(a) = 0$, $y(b) = 0$.

Using the consideration of the problem (1), (2) which was done in [7], when the condition (3) and the inequalities known from [8] are fulfilled

(5) $\quad \displaystyle\int_a^b |\bar{G}(x; s)| ds \leq \dfrac{(b-a)^2}{8}; \quad \int_a^b |\bar{G}_x'(x; s)| ds \leq \dfrac{b-a}{2},$

we obtain the inclusion $\Pi\Omega \subset \Omega$.

Let $y_1, y_2 \in \Omega$. Then, according to the second condition of the theorem, the inequalities (5) and the definition of the norm (4), we get

$$|\Pi y_1(x) - \Pi y_2(x)| \leq \int_a^b |\bar{G}(x; s)| \, |f(s, y_1(x), y_1'(s), y_1(s-\tau_0(s)), y_1'(s-\tau_1)))$$

$$-f(s, y_2(s), y_2'(s), y_2(s-\tau_0(s)), y_2'(s-\tau_1(s)))| ds$$

$$\leq \int_a^b |\bar{G}(x; s)| \{ L_1 |y_1(s) - y_2(s)| + L_2 |y_1'(s) - y_2'(s)|$$

$$+ L_3 |y_1(s-\tau_0(s)) - y_2(s-\tau_0(s))| + L_4 |y_1'(s-\tau_1(s)) - y_2'(s-\tau_1(s))| \} ds$$

$$\leq \left[(L_1 + L_3) \frac{(b-a)^2}{8} + (L_2 + L_4) \frac{b-a}{2} \right] \|y_1 - y_2\|_B \int_a^b |\bar{G}(x; s)| ds$$

$$\leq \left[(L_1 + L_3) \frac{(b-a)^2}{8} + (L_2 + L_4) \frac{b-a}{2} \right] \frac{(b-a)^2}{8} \|y_1 - y_2\|_B;$$

$$|(\Pi y_1)'(x)-(\Pi y_2)'(x)|\leq\left[(L_1+L_3)\frac{(b-a)^2}{8}+(L_2+L_4)\frac{b-a}{2}\right]\|y_1-y_2\|_B\int_a^b|G_x'(x;s)|ds$$

$$\leq\left[(L_1+L_3)\frac{(b-a)^2}{8}+(L_2+L_4)\frac{b-a}{2}\right]\frac{b-a}{2}\|y_1-y_2\|_B\;;$$

(6) $$\|\Pi y_1-\Pi y_2\|_B\leq\left[(L_1+L_3)\frac{(b-a)^2}{8}+(L_2+L_4)\frac{b-a}{2}\right]\|y_1-y_2\|_B\;.$$

The inequality (6) and the third condition of the theorem imply that Π is a contraction operator in the set Ω. The set Ω with the introduced norm (4) and the operator Π satisfy the Banach principle for the fixed point. Hence the operator Π has a unique fixed point in Ω, i.e. the boundary value problem has a unique solution $y(x)\in\Omega$.

3. Construction of the approximate solution

Let the following partition of the interval $[a, b]$ be given:

$$\Delta:\quad a=x_0<x_1<x_2<\cdots<x_N=b$$

and $E\subset\Delta$.

Definition 1. [8] The function $S(x)$, given in the interval $[a, b]$, is called spline-function of third degree with defect k with respect to the partition Δ, if the following conditions hold:

1. $S(x)\in P_3$ in $[x_{i-1}, x_i]$, $i=1, 2, \cdots, N$, where P_3 is the set of polynomials of degree less than or equal to three in the interval $[x_{i-1}, x_i]$.

2. $S(x)\in C^{3-k}[a, b]$.

The points x_i $(i=0, 1, \cdots, N)$ are called knots of the spline-function.

In literature [8, 9] it is accepted for the spline-functions of third degree with defect 1 to be called cubic spline-functions.

Everywhere further we will consider spline functions $S(x)$, of third degree with defect 2 which satisfy the condition

(7) $$S(x)\in C^2[\delta_r]\;,\quad r=1, 2, \cdots, l+1\;,$$

i.e. on every interval δ_r $(r=1, 2, \cdots, l+1)$, $S(x)$ is a cubic spline-function.

Let the function $y(x)$ is defined on the interval $[a, b]$.

Definition 2. The spline-function $S(x)$, satisfying the conditions

$$S(x_i)=y(x_i)=y_i\;,\quad i=0, 1, \cdots, N\;;$$
$$S''(x_0+0)=y''(x_0+0)\;,\quad S''(x_N-0)=y''(x_N-0)\;;$$
$$S''(x^{(s)}-0)=y''(x^{(s)}-0)\;,\quad S''(x^{(s)}+0)=y''(x^{(s)}+0)\;,\quad s=1, 2, \cdots, l\;,$$

will be called a splide-function interpolating to the function $y(x)$ in the nodes x_i $(i=0, 1, \cdots, N)$ and will be denoted by $S(y; x)$.

Introduce the notations:

$$h_j = x_j - x_{j-1}, \quad j=1, 2, \cdots, N;$$
$$M_{j\oplus} = S''(y; x_j+0), \quad j=0, 1, \cdots, N-1;$$
$$M_{j\ominus} = S''(y; x_j-0), \quad j=1, 2, \cdots, N.$$

Definitions 1 and 2 yield that the spline-function $S(y; x)$, interpolating to the solution $y(x)$ of the boundary value problem (1), (2), can be represented in the form

$$(8) \qquad S(y; x) = M_{(j-1)\oplus}\frac{(x_j-x)^3}{6h_j} + M_{j\ominus}\frac{(x-x_{j-1})^3}{6h_j}$$

$$+ \left(y_{j-1} - \frac{M_{(j-1)\oplus}h_j^2}{6}\right)\frac{x_j-x}{h_j} + \left(y_j - \frac{M_{j\ominus}h_j^2}{6}\right)\frac{x-x_{j-1}}{h_j},$$

$$x \in [x_{j-1}, x_j], \quad j=1, 2, \cdots, N.$$

The magnitude $M_{(j-1)\oplus}$ and $M_{j\ominus}$ $(j=1, 2, \cdots, N)$ from (8) satisfy the system of linear equations

$$(9) \qquad \begin{cases} h_{j+1}y_{j-1} - (h_j+h_{j+1})y_j + h_j y_{j+1} \\ \qquad = \frac{h_j h_{j+1}}{6}[h_j M_{(j-1)\oplus} + 2h_j M_{j\ominus} + 2h_{j+1}M_{j\oplus} + h_{j+1}M_{(j+1)\ominus}], \\ y_0 = 0, \quad y_N = 0, \quad j=1, 2, \cdots, N-1; \end{cases}$$

In view of the assumed proposition (7), the equalities, as follows, will be fulfilled:

$$M_{j\oplus} = M_{j\ominus}, \qquad x_j \overline{\in} E,$$

The derivative $S'(y; x)$ of the function $S(y; x)$ is calculated by the formula

$$(10) \qquad S'(y; x) = -M_{(j-1)\oplus}\frac{(x_j-x)^2}{2h_j} + M_{j\ominus}\frac{(x-x_{j-1})^2}{2h_j} + M_{(j-1)\oplus}\frac{h_j}{6}$$

$$-M_{j\ominus}\frac{h_j}{6} + \frac{y_j - y_{j-1}}{h_j}, \quad x \in [x_{j-1}, x_j], \quad j=1, 2, \cdots, N,$$

We set

$$A_N = \begin{bmatrix} -(h_1+h_2) & h_1 & 0 & \cdots & 0 & 0 \\ h_3 & -(h_2+h_3) & h_2 & \cdots & 0 & 0 \\ 0 & h_4 & -(h_3+h_4) & \cdots & 0 & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 & 0 & \cdots & -(h_{N-2}+h_{N-1}) & h_{N-2} \\ 0 & 0 & 0 & \cdots & h_N & -(h_{N-1}+h_N) \end{bmatrix}.$$

It can be inductively established that the matrix A_N possesses the property

(11) $$|A_N| = (-1)^{N-1} h_2 h_3 \cdots h_{N-1}(b-a) \,,$$

The matrix A_N^{-1}, inverse to the matrix A_N, satisfies the inequalities:

(12) $$\|A_N^{-1}\| = \max_i \sum_{j=1}^{N-1} |a_{ij}^{-1}| \leq \frac{K^2}{8h^3}(b-a)^2 \,;$$

(13) $$\max_{1 \leq i \leq N-2} \sum_{j=1}^{N-1} |a_{i+1,j}^{-1} - a_{ij}^{-1}| \leq \frac{K^2(b-a)}{2h^2} \,, \quad \sum_{j=1}^{N-1} |a_{ij}^{-1}| \leq \frac{K^2(b-a)}{2h^2} \,, \quad (i=1, N-1) \,,$$

where a_{ij}^{-1} are the elements of the matrix A_N^{-1}, $K = H/h$,

$$h = \min_j h_j \,, \quad (j=1, 2, \cdots, N) \,, \qquad H = \max_j h_j \,, \quad (j=1, \cdots, N) \,.$$

We will seek an approximate solution of the boundary value problem (1), (2) in the form of a spline-function $S(x)$ of third degree with defect 2 from the class

$$\Lambda = \{S(x): \ S(x) \in C[\alpha, b]; \ S(x) \in C^1[a, b]; \ S(x) \in C^2[\delta_r] \ (r=1, 2, \cdots, l+1);$$
$$S(x) = 0, \ x \in [\alpha, a] \cup \{b\}; \ \max\{\max_{x_{j-1} \leq x \leq x_j} |S''(x)|: \ j=1, \cdots, N\} \leq Q\} \,.$$

In our further considerations we will use the following theorem.

Theorem 2. [10] *If* $y(x) \in C^2[\delta_r]$, *then*

$$\|S^{(q)}(y) - y^{(q)}\| \leq D_q H^{2-q} \omega_r(y''; H) \,,$$

where $S(y) \ (=S(y; x))$ *is the cubic spline-function interpolating the function* $y(x)$ *in the knods* $\{\Lambda \cap \delta_r\}$; $q = 0, 1, 2$; $\omega_r(y''; H)$ *is the continuity modulus of the function* $y''(x)$ *for* δ_r;

$$D_0 = \frac{5}{2} \,, \qquad D_1 = 5 \,, \qquad D_2 = 5 \,; \qquad \|y\| = \max_{x \in \delta_r} |y(x)| \,.$$

Construct the iteration sequence of spline-functions in the following way.

Choose as a null approximation $S(y^{(0)}; x)$ an arbitrary function from the class Λ.

The next approximations are constructed by the scheme (14)–(18).

(14) $$M_{j\oplus}^{(k+1)} = f[x_j, \ S(y^{(k)}; x_j+0), \ S'(y^{(k)}; x_j+0), \ S(y^{(k)}; x_j+0-\tau_0(x_j+0)),$$
$$S'(y^{(k)}; x_j+0-\tau_1(x_j+0))] \,, \quad j=0, 1, \cdots, N-1 \,;$$

(15) $$M_{j\ominus}^{(k+1)} = f[x_j, \ S(y^{(k)}; x_j-0), \ S'(y^{(k)}; x_j-0), \ S(y^{(k)}; x_j-0-\tau_0(x_j-0)),$$
$$S'(y^{(k)}; x_j-0-\tau_1(x_j-0))] \,, \quad j=1, 2, \cdots, N$$

(k is the number of the iteration).

In (14) and (15) we set

$$S(y^{(k)}; x_j+0-\tau_0(x_j+0))=0 \quad \text{and} \quad S(y^{(k)}; x_j-0-\tau_0(x_j-0))=0$$

for $x_j-\tau_0(x_j)<a$;

$$S'(y^{(k)}; x_j+0-\tau_1(x_j+0))=0 \quad \text{and} \quad S'(y^{(k)}; x_j-0-\tau_1(x_j-0))=0$$

for $x_j-\tau_1(x_j)<a$.

The numbers $y_j^{(k+1)}$ are determined from the system

$$(16)\quad \begin{cases} h_{j+1}y_{j-1}^{(k+1)}-(h_j+h_{j+1})y_j^{(k+1)}+h_jy_{j+1}^{(k+1)} \\ \qquad =\dfrac{h_jh_{j+1}}{6}[h_jM_{(j-1)\oplus}^{(k+1)}+2h_jM_{j\ominus}^{(k+1)}+2h_{j+1}M_{j\oplus}^{(k+1)}+h_{j+1}M_{(j+1)\ominus}^{(k+1)}]\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2,\cdots,N-1\,, \\ y_0^{(k+1)}=y_N^{(k+1)}=0 \end{cases}$$

$S(y^{(k+1)}; x)$ and $S'(y^{(k+1)}; x)$ in the interval $[x_{j-1}, x_j]$ are determined by the formulae

$$(17)\quad S(y^{(k+1)}; x)=M_{(j-1)\oplus}^{(k+1)}\frac{(x_j-x)^3}{6h_j}+M_{j\ominus}^{(k+1)}\frac{(x-x_{j-1})^3}{6h_j}+\left(y_{j-1}^{(k+1)}-\frac{M_{(j-1)\oplus}^{(k+1)}h_j^2}{6}\right)\frac{x_j-x}{h_j}$$

$$+\left(y_j^{(k+1)}-\frac{M_{j\ominus}^{(k+1)}h_j^2}{6}\right)\frac{x-x_{j-1}}{h_j}\,,\quad x\in[x_{j-1}, x_j]\,,\quad j=1,2,\cdots,N\,;$$

$$(18)\quad S'(y^{(k+1)}; x)=-M_{(j-1)\oplus}^{(k+1)}\frac{(x_j-x)^2}{2h_j}+M_{j\ominus}^{(k+1)}\frac{(x-x_{j-1})^2}{2h_j}+M_{(j-1)\oplus}^{(k+1)}\frac{h_j}{6}-M_{j\ominus}^{(k+1)}\frac{h_j}{6}$$

$$+\frac{y_j^{(k+1)}-y_{j-1}^{(k+1)}}{h_j}\,,\quad x\in[x_{j-1}, x_j]\,,\quad j=1,2,\cdots,N\,.$$

Consider the iteration sequences $\{S(y^{(k)}; x)\}$ and $\{S'(y^{(k)}; x)\}$ determined by help of the relations (14)–(18).

Set

$$\max_{a\le x\le b}|S(y^{(k)}; x)-S(y^{(k-1)}; x)|=\alpha_k\,;$$

$$\max_{a\le x\le b}|S'(y^{(k)}; x)-S'(y^{(k-1)}; x)|=\alpha_k'\,;$$

$$\max_j|\bar{M}_j^{(k+1)}-\bar{M}_j^{(k)}|=\max\{\max_{j=1,\cdots,N}|M_{j\ominus}^{(k+1)}-M_{j\ominus}^{(k)}|;\ \max_{j=0,1,\cdots,N-1}|M_{j\oplus}^{(k+1)}-M_{j\oplus}^{(k)}|\}\,;$$

$$\lambda_1=L_1+L_3\,;\quad \lambda_2=L_2+L_4\,;\quad u=\frac{H^2}{8}+\frac{K^5}{8}(b-a)^2\,;\quad v=\frac{2}{3}H+\frac{K^5}{2}(b-a)\,;$$

$$p=\lambda_1u+\lambda_2v\,;\quad \mu=5\left(1+\frac{1}{2}L_3H^2+L_4H\right)\,.$$

We will prove the following theorem.

Theorem 3. *Let the following hold:*

1. *conditions of Theorem* 1,
2. *the inequality*

(19)
$$K^5\left[\lambda_1\frac{(b-a)^2}{8}+\lambda_2\frac{b-a}{2}\right]<1\;.$$

Then there exists a number $H_0>0$ *such that for all* $H\leqq H_0$ *the sequences* $\{S^{(i)}(y^{(k)};x)\}$ $(i=0,1)$ *are uniformly convergent in the interval* $[a,b]$ *and the estimates, as follows, hold*:

$$\|S^{(q)}(\tilde{y})-y^{(q)}\|\leqq E_q\omega(y'';H)\;,\quad q=0,1\;,$$

where $S(\tilde{y};x)=\lim_{k\to\infty}S(y^{(k)};x)$, $y(x)$ *is the exact solution of the boundary value problem* (1), (2),

$$E_0=\sup_{H\leqq H_0}\left\{\frac{u\mu}{1-p}+\frac{5}{2}H^2\right\}\;,\qquad E_1=\sup_{H\leqq H_0}\left\{\frac{v\mu}{1-p}+5H\right\}\;,$$

$$\omega(y'';H)=\max_r\{\omega_r(y'';H)\}\;,\quad r=1,2,\cdots,l+1\quad\text{and}\quad\|y\|=\max_{a\leqq x\leqq b}|y(x)|\;.$$

Proof. The determinant of the system (16), in view of (11), is different from zero. Therefore, the construction of the iteration sequence $\{S(y^{(k)};x)\}$ is possible. From the way the spline-functions $S(y^{(k)};x)$ are constructed and from the inequalities (5) it follows that they belong to the class Λ. Obviously $\{S(y^{(k)};x)\}\subset\Omega$.

From the relations (11)–(18) we get

$$|S(y^{(k+1)};x)-S(y^{(k)};x)|\leqq\frac{H^2}{8}\max_j|\bar{M}_j^{(k+1)}-\bar{M}_j^{(k)}|+\max_j|y_j^{(k+1)}-y_j^{(k)}|\;,$$

$$j=0,1,\cdots,N\;;$$

$$|S'(y^{(k+1)};x)-S'(y^{(k)};x)|$$
$$\leqq\frac{2}{3}H\max_j|\bar{M}_j^{(k+1)}-\bar{M}_j^{(k)}|+\frac{1}{h}\max_{j=1,\cdots,N}|y_j^{(k+1)}-y_{j-1}^{(k+1)}-(y_j^{(k)}-y_{j-1}^{(k)})|\;;$$

$$|y_j^{(k+1)}-y_j^{(k)}|\leqq\frac{K^5}{8}(b-a)^2\max_j|\bar{M}_j^{(k+1)}-\bar{M}_j^{(k)}|\;;$$

$$\max_{j=1,2,\cdots,N}|y_j^{(k+1)}-y_{j-1}^{(k+1)}-(y_j^{(k)}-y_{j-1}^{(k)})|\leqq\frac{K^4}{2}(b-a)H\max_j|\bar{M}_j^{(k+1)}-\bar{M}_j^{(k)}|\;;$$

$$|M_{j\oplus}^{(k+1)}-M_{j\oplus}^{(k)}|\leqq L_1|S(y^{(k)};x_j+0)-S(y^{(k-1)};x_j+0)|$$
$$+L_2|S'(y^{(k)};x_j+0)+S'(y^{(k-1)};x_j+0)|$$
$$+L_3|S(y^{(k)};x_j+0-\tau_0(x_j+0))-S(y^{(k-1)};x_j+0-\tau_0(x_j+0))|$$
$$+L_4|S'(y^{(k)};x_j+0-\tau_1(x_j+0))-S'(y^{(k-1)};x_j+0-\tau_1(x_j+0))|$$
$$\leqq\lambda_1\alpha_k+\lambda_2\alpha_k'\;,\quad j=0,1,\cdots,N-1\;;$$

$$|M_{j\ominus}^{(k+1)}-M_{j\ominus}^{(k)}|\leqq\lambda_1\alpha_k+\lambda_2\alpha_k'\;,\quad j=1,2,\cdots,N\;.$$

Therefore

(20)
$$\alpha_{k+1} \leqq u(\lambda_1 \alpha_k + \lambda_2 \alpha_k')$$

(21)
$$\alpha_{k+1}' \leqq v(\lambda_1 \alpha_k + \lambda_2 \alpha_k')$$

The inequalities (20) and (21) imply

(22)
$$\alpha_{k+1} \leqq u p^{k-1}(\lambda_1 \alpha_1 + \lambda_2 \alpha_1')$$

(23)
$$\alpha_{k+1}' \leqq v p^{k-1}(\lambda_1 \alpha_1 + \lambda_2 \alpha_1') .$$

By assumption, the condition (19) holds. Hence, there exists a number $H_0 > 0$ such that for all $H \leqq H_0$, $p < 1$ and $E_q < +\infty$, $q = 0, 1$ holds. Then for all $H \leqq H_0$ the inequalities (22) and (23) are sufficient conditions for a uniform in $[a, b]$ convergence of the series

$$\sum_{k=0}^{\infty} |S^{(q)}(y^{(k+1)}; x) - S^{(q)}(y^{(k)}; x)| , \quad q = 0, 1 .$$

This implies that the sequences $\{S^{(q)}(y^{(k)}; x)\}$ ($q = 0, 1$) when $H \leqq H_0$ are uniformly convergent in $[a, b]$.

Let $H \leqq H_0$ and $\lim_{k \to \infty} S(y^{(k)}; x) = S(\tilde{y}; x)$. Then $\lim_{k \to \infty} S'(y^{(k+1)}; x) = S'(\tilde{y}; x)$.

The function $S(\tilde{y}; x)$ is accepted to be an approximate solution of the boundary value problem (1), (2). Denote

$$\tilde{M}_{j\oplus} = S''(\tilde{y}; x_j + 0) , \quad j = 0, 1, \cdots, N-1 ;$$
$$\tilde{M}_{j\ominus} = S''(\tilde{y}; x_j - 0) , \quad j = 1, 2, \cdots, N ;$$
$$\tilde{y}_j = S(\tilde{y}; x_j) , \quad j = 0, 1, \cdots, N ,$$

It is easily seen that the following relations hold:

(24)
$$\begin{cases} S(\tilde{y}; x) = \tilde{M}_{(j-1)\oplus} \dfrac{(x_j - x)^3}{6h_j} + \tilde{M}_{j\ominus} \dfrac{(x - x_{j-1})^3}{6h_j} \\ \qquad + \left(\tilde{y}_{j-1} - \dfrac{\tilde{M}_{(j-1)\oplus} h_j^2}{6} \right) \dfrac{x_j - x}{h_j} + \left(\tilde{y}_j - \dfrac{\tilde{M}_{j\ominus} h_j^2}{6} \right) \dfrac{x - x_{j-1}}{h_j} , \\ \qquad\qquad\qquad\qquad x \in [x_{j-1}, x_j] , \quad j = 1, 2, \cdots, N ; \\ S(\tilde{y}; x) = 0 , \quad x < a ; \end{cases}$$

$$S'(\tilde{y}; x) = -\tilde{M}_{(j-1)\oplus} \frac{(x_j - x)^2}{2h_j} + \tilde{M}_{j\ominus} \frac{(x - x_{j-1})^2}{2h_j} + \tilde{M}_{(j-1)\oplus} \frac{h_j}{6} - \tilde{M}_{j\ominus} \frac{h_j}{6} + \frac{\tilde{y}_j - \tilde{y}_{j-1}}{h_j} ,$$
$$x \in [x_{j-1}, x_j] , \quad j = 1, \cdots, N ;$$

(25)
$$\tilde{M}_{j\oplus} = f[x_j, S(\tilde{y}; x_j + 0), S'(\tilde{y}; x_j + 0), S(\tilde{y}; x_j + 0 - \tau_0(x_j + 0)),$$
$$S'(\tilde{y}; x_j + 0 - \tau_1(x_j + 0))] , \quad j = 0, 1, \cdots, N-1 ;$$

(26) $\qquad \tilde{M}_{j\ominus}=f[x_j,\ S(\tilde{y};\ x_j-0),\ S'(\tilde{y};\ x_j-0),\ S(\tilde{y};\ x_j-0-\tau_0(x_j-0)),$
$$S'(\tilde{y};\ x_j-0-\tau_1(x_j-0))]\ ,\quad j=1,2,\cdots,N\ ;$$

(27) $\qquad \begin{cases} h_{j+1}\tilde{y}_{j-1}-(h_j+h_{j+1})\tilde{y}_j+h_j\tilde{y}_{j+1} \\ \qquad\qquad =\dfrac{h_jh_{j+1}}{6}[h_j\tilde{M}_{(j-1)\oplus}+2h_j\tilde{M}_{j\ominus}+2h_{j+1}\tilde{M}_{j\oplus}+h_{j+1}\tilde{M}_{(j+1)\ominus}]\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,2,\cdots,N-1\ , \\ \tilde{y}_0=\tilde{y}_N=0\ . \end{cases}$

The fulfilment of the conditions of Theorem 3 guarantees the existence of a unique in Ω solution $y(x)$ of the boundary value problem (1), (2). Let $S(y;x)$ denote the spline-function interpolating the exact solution $y(x)$ of the boundary value problem (1), (2) in the knods Δ of the interval $[a,b]$. In view of Theorems 1 and 2 the inequalities, as follows, hold:

(28) $\qquad\qquad \|S^{(q)}(y)-y^{(q)}\|\leqq D_qH^{2-q}\omega(y'';H)\ ,\quad q=0,1,2$

(here $S''(y;x_j)$ and $y''(x_j)$ for the points $x_j\in E$ denote any of the one-sided derivatives $S''(y;x_j+0)$, $S''(y;x_j-0)$ and $y''(x_j+0)$, $y''(x_j-0)$, respectively.)

We will look for a connection between $M_{j\oplus}=S''(y;x_j+0)$ and $f(x_j,S(y;x_j+0)$, $S'(y;x_j+0),\ S(y;x_j+0-\tau_0(x_j+0)),\ S'(y;x_j+0-\tau_1(x_j+0)))$.

Using the second condition of Theorem 1 and the condition (28), we get

$|M_{j\oplus}-f(x_j,\ S(y;\ x_j+0),\ S'(y;\ x_j+0),\ S(y;\ x_j+0-\tau_0(x_j+0)),\ S'(y;\ x_j+0-\tau_1(x_j+0)))|$
$\leqq|S''(y;x_j+0)-y''(x_j+0)|+|f(x_j,y(x_j),y'(x_j),y(x_j-\tau_0(x_j)),y'(x_j+0-\tau_1(x_j+0)))$
$\qquad -f(x_j,\ S(y;\ x_j+0),\ S'(y;\ x_j+0),\ S(y;\ x_j+0-\tau_0(x_j+0)),\ S'(y;\ x_j+0-\tau_1(x_j+0)))|$
$\leqq5\omega(y'';\ H)+L_1|y(x_j)-S(y;\ x_j+0)|+L_2|y'(x_j+0)-S'(y;\ x_j+0)|$
$\qquad +L_3|y(x_j-\tau_0(x_j))-S(y;\ x_j+0-\tau_0(x_j+0))|$
$\qquad +L_4|y'(x_j+0-\tau_1(x_j+0))-S'(y;\ x_j+0-\tau_1(x_j+0))|$
$\leqq5\left(1+\dfrac{1}{2}L_3H^2+\lambda_2H\right)\omega(y'';\ H)$
$=\mu\omega(y'';\ H)\ ,\quad j=0,1,\cdots,N-1\ .$

In an analogous way, we obtain

$|M_{j\ominus}-f(x_j,\ S(y;\ x_j-0),\ S'(y;\ x_j-0),\ S(y;\ x_j-0-\tau_0(x_j-0)),\ S'(y;\ x_j-0-\tau_1(x_j-0)))|$
$\qquad\leqq\mu\omega(y'';\ H)\ ,\quad j=1,2,\cdots N\ ,$

where
$$M_{j\ominus}=S''(y;x_j-0)\ ;$$

Hence,

(29) $\quad M_{j\oplus}\leqq f(x_j,\ S(y;\ x_j+0),\ S'(y;\ x_j+0),\ S(y;\ x_j+0-\tau_0(x_j+0)),\ S'(y;\ x_j+0-\tau_1(x_j+0)))$
$\qquad +\mu\omega(y'';\ H)\ ,\quad j=0,1,\cdots,N-1\ ;$

(30) $\quad M_{j\ominus}\leq f(x_j, S(y; x_j-0), S'(y; x_j-0), S(y; x_j-0-\tau_0(x_j-0)), S'(y; x_j-0-\tau_1(x_j-0)))$
$$+\mu\omega(y''; H), \quad j=1,2,\cdots,N.$$

Denote

$$\max_{a\leq x\leq b}|S^{(q)}(\tilde{y}; x)-S^{(q)}(y; x)|=\tilde{\alpha}^{(q)}, \quad q=0,1;$$

$$\max_j|\bar{\tilde{M}}_j-\bar{M}_j|=\max\{\max_{j=1,2,\cdots,N}|\tilde{M}_{j\ominus}-M_{j\ominus}|, \max_{j=0,1,\cdots,N-1}|\tilde{M}_{j\oplus}-M_{j\oplus}|\}.$$

Having in mind the relations (24)–(27), (8)–(13), (29) and (30), we get

$$|S(\tilde{y}; x)-S(y; x)|\leq\frac{H^2}{8}\max_j|\bar{\tilde{M}}_j-\bar{M}_j|+\max_j|\tilde{y}_j-y_j|, \quad j=0,1,\cdots,N;$$

$$|S'(\tilde{y}; x)-S'(y; x)|\leq\frac{2}{3}H\max_j|\bar{\tilde{M}}_j-\bar{M}_j|+\frac{1}{h}\max_{j=1,\cdots,N}|\tilde{y}_j-\tilde{y}_{j-1}-(y_j-y_{j-1})|;$$

$$|\tilde{y}_j-y_j|\leq\frac{K^5}{8}(b-a)^2\max_j|\bar{\tilde{M}}_j-\bar{M}_j|;$$

$$\max_{j=1,\cdots,N}|\tilde{y}_j-\tilde{y}_{j-1}-(y_j-y_{j-1})|\leq\frac{K^4H}{2}(b-a)\max_j|\bar{\tilde{M}}_j-\bar{M}_j|;$$

$$|\tilde{M}_{j\oplus}-M_{j\oplus}|\leq\lambda_1\tilde{\alpha}+\lambda_2\tilde{\alpha}'+\mu\omega(y''; H), \quad j=0,1,\cdots,N-1;$$

$$|\tilde{M}_{j\ominus}-M_{j\ominus}|\leq\lambda_1\tilde{\alpha}+\lambda_2\tilde{\alpha}'+\mu\omega(y''; H), \quad j=1,2,\cdots,N.$$

Therefore,

(31) $$\tilde{\alpha}\leq u(\lambda_1\tilde{\alpha}+\lambda_2\tilde{\alpha}'+\mu\omega(y''; H))$$

(32) $$\tilde{\alpha}'\leq v(\lambda_1\tilde{\alpha}+\lambda_2\tilde{\alpha}'+\mu\omega(y''; H)).$$

The inequalities (31) and (32) yield

(33) $$\tilde{\alpha}\leq\frac{u\mu}{1-p}\omega(y''; H),$$

(34) $$\tilde{\alpha}'\leq\frac{v\mu}{1-p}\omega(y''; H).$$

Then, in view of the inequalities (28), (33) and (34) we obtain the estimates

$$\|S(\tilde{y})-y\|\leq\|S(\tilde{y})-S(y)\|+\|S(y)-y\|\leq\left(\frac{u\mu}{1-p}+\frac{5}{2}H^2\right)\omega(y''; H)\leq E_0\omega(y''; H);$$

$$\|S'(\tilde{y})-y'\|\leq\|S'(\tilde{y})-S'(y)\|+\|S'(y)-y'\|\leq\left(\frac{v\mu}{1-p}+5H\right)\omega(y''; H)\leq E_1\omega(y''; H).$$

The Theorem is proved.

Theorem 3 and the definition (4) of the norm $\|\cdot\|_B$ imply the estimate

$$\|S(\tilde{y})-y\|_B \leq \max\left\{\frac{8E_0}{(b-a)^2}, \frac{2E_1}{b-a}\right\}\omega(y''; H) .$$

Remark. When solving particular boundary value problems of the type (1), (2) by the described approximate method, practically $S(y^{(k)}; x)$ is taken as an approximate solution for k sufficiently large in order to reach a definite proximity between $S(y^{(k+1)}; x)$ and $S(y^{(k)}; x)$. The error in this case may be estimated on the grounds of the inequalities (22) and (23) in the following way:

(35)
$$\|S(y^{(k+i)})-S(y^{(k)})\| \leq u(\lambda_1\gamma_1+\lambda_2\alpha_1')p^{k-1}\frac{1-p^i}{1-p}$$
$$\|S'(y^{(k+i)})-S'(y^{(k)})\| \leq v(\lambda_1\alpha_1+\lambda_2\alpha_1')p^{k-1}\frac{1-p^i}{1-p} .$$

For $H \leq H_0$ and $i \to \infty$, from the inequalities (35) we obtained the inequalities

$$\|S(\tilde{y})-S(y^{(k)})\| \leq u(\lambda_1\alpha_1+\lambda_2\alpha_1')\frac{p^{k-1}}{1-p}$$
$$\|S'(\tilde{y})-S'(y^{(k)})\| \leq v(\lambda_1\alpha_1+\lambda_2\alpha_1')\frac{p^{k-1}}{1-p} .$$

Therefore, for an arbitrary $\varepsilon > 0$ for all $H \leq H_0$ a number of iteration k_0 might be exactly pointed, such that for all numbers $k \geq k_0$

$$\|S^{(q)}(\tilde{y})-S^{(q)}(y^{(k)})\| \leq \varepsilon , \quad q=0,1 ,$$

will be fulfilled. Then, for $k \geq k_0$ in the conditions of Theorem 1, the inequalities

$$\|S^{(q)}(y^{(k)})-y^{(q)}\| \leq \varepsilon+E_q\omega(y''; H) , \quad q=0,1$$

hold.

Example 1. Consider the boundary value problem

$$y''(x)=\alpha y'\left(x-\frac{1}{2}\right)+e^x , \quad x \in [0,1] , \quad |\alpha|<1$$

$$\left.\begin{array}{l}y(x)=0\\y'(x)=0\end{array}\right\} \quad x<0 , \quad y(0)=y(1)=0 .$$

Here $E=\{1/2\}$. $Q_2=3$, $Q=3|\alpha|+e$; $L_1=L_2=L_3=0$, $L_4=|\alpha|$.

It is easily seen that the conditions of Theorem 1 hold. Hence, there exists a unique solution of the boundary value problem belonging to the set Ω, corresponding to the data of the problem. By checking it can be found that this solution is as follows:

$$y(x)=\begin{cases}(a-1)x+e^x-1\,, & 0\leqq x\leqq\dfrac{1}{2}\,,\\[2mm]a\left(x+\alpha\,\dfrac{4x^2-4x+1}{8}\right)-1-\dfrac{5}{8}\alpha-\left(1+\dfrac{\alpha}{2}\right)x-\dfrac{\alpha}{2}x^2+\alpha e^{x-(1/2)}+e^x\,, & \dfrac{1}{2}\leqq x\leqq 1\,,\end{cases}$$

where

$$a=\frac{16-8\alpha\sqrt{e}+13\alpha-8e}{8+\alpha}\,.$$

For the construction of the approximate solution we use the uniform division of the interval $[0,1]$ with length of the subintervals $(1/2^n)(N=(b-a)2^n)$. Then the point $0,5$ turns out to be a knod of the corresponding spline-functions, while the condition for convergence of the iteration sequence of spline-function (19) coincides with the third condition of the theorem for existence and uniqueness of the solution.

The example is calculated at $\alpha=0,5$ on a ES 1020 computer. Calculations are done with double accuracy. The program is written in FORTRAN.

The obtained results are given on Table 1. The following notations are used in the tables:

Table 1 $(\alpha=0,5)$

x_j	$|y_j-\tilde{y}_j|$			$|y_j'-\tilde{y}_j'|$		
	$N+1=9$	$N+1=33$	$N+1=129$	$N+1=9$	$N+1=33$	$N+1=129$
0,00	0,00	0,0	0,0	9,88 (-4)	6,2 (-5)	4,0 (-6)
0,25	2,02 (-4)	1,3 (-5)	1,3 (-6)	6,18 (-4)	3,8 (-5)	2,0 (-6)
0,50	3,00 (-4)	1,9 (-5)	1,0 (-6)	1,44 (-4)	9,0 (-6)	1,0 (-6)
0,75	2,53 (-4)	1,6 (-5)	1,0 (-6)	5,50 (-4)	3,5 (-5)	2,0 (-6)
1,00	0,00	0,0	0,0	1,52 (-3)	9,5 (-5)	6,0 (-6)

y_j—the value of the exact solution in x_j,

$\tilde{y}_j$—the value of the approximate solution in x_j,

y_j'—the value of the derivative of the exact solution in x_j,

$\tilde{y}_j'$—the value of the derivative of the approximate solution in x_j,

$N+1$—number of the nodes.

The maximal number of iterations to obtain the shown results is 8.

Example 2. Consider the boundary value problem

$$y''(x)=\frac{1}{4}y'\left(x-\frac{\pi}{2}\right)+\sin x\,, \quad 0\leqq x\leqq\pi\,,$$

$$\left.\begin{array}{l}y(x)=0\\y'(x)=0\end{array}\right\} \quad x<0\,, \quad y(0)=y(\pi)=0\,.$$

Here $E=\{\pi/2\}$. For $Q_2=\pi$, $Q=2$, $L_1=L_2=L_3=0$, $L_4=1/4$.

The conditions of the theorem for existence and uniqueness of the solution of the boundary value problem are fulfilled. By checking it is found that the unique solution of the boundary value problem is as follows:

$$y(x)=\begin{cases}\dfrac{8x}{\pi(32+\pi)}-\sin x , & 0\leqq x\leqq\dfrac{\pi}{2} , \\[2mm] \dfrac{x^2+(8-\pi)x}{\pi(32+\pi)}-\dfrac{3}{4}\sin x-\dfrac{8}{32+\pi} , & \dfrac{\pi}{2}\leqq x\leqq\pi .\end{cases}$$

Example 2 is also calculated on a ES 1020 computer, with double accuracy the program is also written in FORTRAN. The obtained results are shown on Table 2. The maximal number of iterations to obtain the shown results is 54.

Table 2

x_j	$|y_j-\tilde{y}_j|$			$|y_j'-\tilde{y}_j'|$		
	$N+1=9$	$N+1=33$	$N+1=129$	$N+1=9$	$N+1=33$	$N+1=129$
0,00	0,00	0,00	0,0	1,10 (−2)	6,87 (−4)	4,3 (−5)
$\Pi/4$	7,57 (−3)	4,77 (−4)	3,0 (−5)	7,26 (−3)	4,52 (−4)	2,8 (−5)
$\Pi/2$	9,85 (−3)	6,21 (−4)	3,9 (−5)	1,84 (−3)	1,16 (−4)	7,0 (−6)
$3\Pi/4$	6,39 (−3)	4,03 (−4)	1,5 (−5)	6,80 (−3)	4,26 (−4)	2,9 (−5)
Π	0,00	0,00	0,0	9,05 (−3)	5,64 (−4)	3,5 (−5)

4. Influence of the errors at the rounding off in the process of calculations

Practically, with the use of a computer, the numbers $M_{j\oplus}^{(k)}$, $j=0,1,\cdots,N-1$; $M_{j\ominus}^{(k)}$, $j=1,2,\cdots,N$ and $y_j^{(k)}$, $j=0,1,\cdots,N$, $k=1,2,\cdots$ are calculated approximately. Hence the question about the influence of the error that is done at each iteration on the result obtained after the k-th iteration.

By T_A we denote the operator acting from A into A by the scheme (14)–(18). We will prove the following lemma.

Lemma 1. *If conditions of Theorem 3 hold, then there exists a number $H_0>0$, such that for all $H\leqq H_0$ the sequences $\{S^{(q)}(y^{(k)};x)\}$, $q=0,1$, are uniformly convergent in $[a,b]$ and the operator T_A is contractive in the space $A\subset B$.*

Proof. Theorem 3 implies that for sufficiently small values of H the sequences $\{S^{(q)}(y^{(k)};x)\}$, $q=0,1$, are uniformly convergent in $[a,b]$.

Let $S_1(y^{(0)};x)$ and $S_2(y^{(0)};x)$ be two functions from A. Then, by reasoning analogous to that for finding (20) and (21), we obtain the following inequalities:

$$|(T_\Lambda S_1)(y^{(0)}; x) - (T_\Lambda S_2)(y^{(0)}; x)|$$

$$= |S_1(y^{(1)}; x) - S_2(y^{(1)}; x)| \leq u\left[\lambda_1 \frac{(b-a)^2}{8} + \lambda_2 \frac{b-a}{2}\right]\|S_1 - S_2\|_B \ ;$$

$$|(T_\Lambda S_1)'(y^{(0)}; x) - (T_\Lambda S_2)'(y^{(0)}; x)|$$

$$= |S_1'(y^{(1)}; x) - S_2'(y^{(1)}; x)| \leq v\left[\lambda_1 \frac{(b-a)^2}{8} + \lambda_2 \frac{b-a}{2}\right]\|S_1 - S_2\|_B \ .$$

Whence,

$$(36) \qquad \|T_\Lambda S_1 - T_\Lambda S_2\|_B \leq \max\left\{\frac{8u}{(b-a)^2}, \frac{2v}{b-a}\right\}\left[\lambda_1 \frac{(b-a)^2}{8} + \lambda_2 \frac{b-a}{2}\right]\|S_1 - S_2\|_B \ .$$

By assumption, $K^5[\lambda_1((b-a)^2/8) + \lambda_2((b-a)/2)] < 1$. Then, for sufficiently small values of H, the inequality

$$(37) \qquad \max\left\{\frac{8u}{(b-a)^2}, \frac{2v}{b-a}\right\}\left[\lambda_1 \frac{(b-a)^2}{8} + \lambda_2 \frac{b-a}{2}\right] < 1$$

holds.

Therefore, there exists a number $H_0 > 0$, such that for all $H \leq H_0$, the sequences $\{S^{(q)}(y^{(k)}; x)\}$, $q = 0, 1$, are uniformly convergent in $[a, b]$ and the inequality (37) holds. Let

$$\sigma = \max\left\{\left(\frac{8H_0^2}{3(b-a)^2} + K^5\right), \left(\frac{5H_0}{3(b-a)} + K^5\right)\right\}\left[\lambda_1 \frac{(b-a)^2}{8} + \lambda_2 \frac{b-a}{2}\right] .$$

Then, the inequality (36) yields

$$\|T_\Lambda S_1 - T_\Lambda S_2\|_B \leq \sigma \|S_1 - S_2\|_B \ ,$$

where $\sigma < 1$.

Thus, Lemma 1 is proved.

The iteration process (14)–(18) may be written by help of the operator T_Λ in the following way:

$$T_\Lambda S(y^{(k)}; x) = S(y^{(k+1)}; x) .$$

Under the conditions of Theorem 3, the function $S(\tilde{y}; x)$ is a fixed point of the operator T_Λ.

Theorem 4. *Let conditions of Theorem 3 hold and let, when each function $S(y^{(k)}; x) = S_k(x)$ is calculated, an error be done, which, by the norm in B, does not exceed some number Δ_0, i.e.*

$$(39) \qquad \|\tilde{T}_\Lambda S_k - T_\Lambda S_k\|_B \leq \Delta_0$$

where $\widetilde{T}_A S_k(x) = \widetilde{T}_A S(y^{(k)}; x)$ *is the approximately calculated value of* $T_A S(y^{(k)}; x)$.

Then, if the approximately calculated by formula (38) *consecutive approximations* $\widetilde{S}(y^{(1)}; x), \widetilde{S}(y^{(2)}; x), \cdots, \widetilde{S}(y^{(k)}; x)$ *belong to the set* Λ, *then for the deviation of the calculated after the k-th iteration approximation* $\widetilde{S}(y^{(k)}; x) = \widetilde{\widetilde{S}}_k(x)$ *from* $S(\bar{y}; x) = \widetilde{S}(x)$, *the estimate as follows holds*:

$$\|\widetilde{\widetilde{S}}_k - \widetilde{S}\|_B \leq \sigma^k \|S_0 - \widetilde{S}\|_B + \frac{\Delta_0}{1-\sigma}(1 - \sigma^k)$$

where $S_0(x) = S(y^{(0)}; x)$.

Proof. The inequality (39) and Lemma 1 imply the chain of inequalities

$$\|\widetilde{\widetilde{S}}_k - \widetilde{S}\|_B \leq \sigma \|\widetilde{\widetilde{S}}_{k-1} - \widetilde{S}\|_B + \Delta_0 \leq \cdots$$

$$\leq \sigma^k \|\widetilde{\widetilde{S}}_0 - \widetilde{S}\|_B + \Delta_0(1 + \sigma + \cdots + \sigma^{k-1})$$

$$\leq \sigma^k \|S_0 - \widetilde{S}\|_B + \Delta_0 \frac{1 - \sigma^k}{1 - \sigma}.$$

Thus, Theorem 4 is proved.

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