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THREE-POINT BOUNDARY VALUE PROBLEMS—EXISTENCE AND UNIQUENESS

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ABSTRACT. This paper gives a criterion for the existence and uniqueness of solutions to three-point boundary value problems associated with the third order non-linear differential equations. A matching technique is developed to match solutions of two, two-point boundary value problems which yield a unique solution of a certain class of three-point boundary value problems.

1. Introduction.

The study of three-point boundary value problems is an interesting area of current research and a great deal of work has been done by many authors in the recent years ([1], [2], [3], [4], [5]). This paper gives a guarentee for the existence and uniqueness of solutions of three-point boundary value problems associated with the differential equation

$$y'''=f(x, y, y', y'') \tag{1.1}$$

where $f(x, y, y', y'')$ is assumed to be continuous on a subset of R^4, solutions to initial value problems associated with (1.1) exist, are unique, and extend throughout a fixed subinterval of R. In this paper a matching technique is developed to match solutions of two, two-point boundary value problems which yields a unique solution of three-point boundary value problems.

In Section 2 a monotonicity restriction on f ensures that the following boundary value problems:

$$(1.2)_i \qquad \begin{array}{l} y'''=f(x, y, y', y'') \\ y(x_1)=y_1\,, \qquad y(x_2)=y_2\,, \qquad y^{(i)}(x_2)=m \quad (i=1, 2) \end{array}$$

$$(1.3)_i \qquad \begin{array}{l} y'''=f(x, y, y', y'') \\ y(x_2)=y_2\,, \qquad y^{(i)}(x_2)=m\,, \qquad y(x_3)=y_3 \quad (i=1, 2) \end{array}$$

have solutions and with added hypothesis a unique solution of the following three-point boundary value problem:

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$$y'''=f(x, y, y', y'') \tag{1.4}$$
$$y(x_1)=y_1, \quad y(x_2)=y_2, \quad y(x_3)=y_3$$

is constructed. This is accomplished by matching solutions of (1.2_2) and (1.3_2).

We need the following conditions for our latter discussion

(A) f: $[x_1, x_3]\times R^3 \to R$ and g: $[x_1, x_3]\times R^3 \to R$ are continuous functions.

(B) for all $w_1, w_2 \in R$ $f(x, y_1, z_1, w_1)-f(x, y_2, z_2, w_2)>g(x, y_1-y_2, z_1-z_2, w_1-w_2)$ for all $x\in(x_1, x_2]$ if $y_1\leqq y_2$, $z_1\geqq z_2$ and $f(x, y_1, z_1, w_1)-f(x, y_2, z_2, w_2)>g(x, y_1-y_2, z_1-z_2, w_1-w_2)$ for all $x\in[x_2, x_3)$ if $y_1\geqq y_2$, $z_1\geqq z_2$ where $g(x, u, v, w)$ satisfies.

(C) the initial value problems $z'''=g(x, z, z', z'')$, $z(c)=0$, $z'(c)=0$, $z''(c)=\delta$ or $z(c)=0$, $z'(c)=\delta$, $z''(c)=0$ where $c\geqq x$, has a solution defined for all $x\geqq c$ (δ arbitrary).

(D) there exists a number $h>0$ such that no non-trivial solution $z(x)$ of (C) may satisfy $z(c)=z(d)=0$ with $0<|d-c|<h$.

(E) for any p, $g(x, z_1, v_1, p)\geqq g(x, z_2, v_2, p)$ either $z_1\leqq z_2$, $v_1\geqq v_2$, $x\in(x_1, x_2]$ or $g(x, z_1, v_1, p)\geqq g(x, z_2, v_2, p)$ if $z_1\geqq z_2$, $v_1\geqq v_2$, $x\in[x_2, x_3)$. Note that the functions defined by $f(x, y, y', y'')=xy$ and $g(x, z, z', z'')=-1$ satisfies all the above conditions at 0 on any interval (a, b) where $a<0<b$.

The major advantage of this study is to find existence and uniqueness of solutions of more general class of three-point boundary value problems, which are not covered in [1] and also includes some of the cases of [1] when $g=0$.

2. Existence and uniqueness of solutions to three-point boundary value problems.

In this section we intend to find some criteria under which solutions of (1.1) which satisfy boundary conditions at two-points may be matched to obtain a unique solution of three-point boundary value problems. We assume in the sequel that initial value problems of the type (C) exists and are unique in the interval $[x_1, x_3]$. Theorem 2.1 displays the idea of matching of solutions of three-point boundary value problems associated with (1.4). We now prove the following lemmas which will be used in our subsequent discussion.

Lemma 2.1. *Assume that conditions* (A), (B), (C), (D) *and* (E) *hold. Then if* $(x_2-x_1)\leqq h$ *(or* $(x_3-x_2)\leqq h$*) and for each* $y_1, y_2, y_3, m\in R$, *there exists at most one solution of either of the following boundary value problems*

$$y'''=f(x, y, y', y'') \tag{2.1}$$
$$y(x_1)=y_1, \quad y(x_2)=y_2, \quad y'(x_2)=m$$

$$(2.2)\qquad \begin{aligned} &y'''=f(x, y, y', y'') \\ &y(x_2)=y_2\,, \qquad y'(x_2)=m\,, \qquad y(x_3)=y_3\,. \end{aligned}$$

Proof. The proof of the uniqueness of solution of (2.2) will be given. Similar arguments will hold for (2.1). Suppose $\theta(x)$ and $\eta(x)$ be solutions of (2.2) and write $\phi(x)=\theta(x)-\eta(x)$. Without loss of generality we can assume that there exists c and d such that $x_2\leqq c<d\leqq x_3$, $\phi(c)=\phi'(c)=\phi(d)=0$ and $\phi(x)>0$ for all $x\in(c, d)$. Since $\phi(c)=0=\phi(d)$, it follows that there exists an $r\in(c, d)$ such that $\phi'(r)=0$ and $\phi'(x)>0$ for all $x\in(c, r)$. Consider the solution of (C) with initial conditions $z(c)=0$, $z'(c)=0$ and $z''(c)=\phi''(c)$. Let $\psi(x)=\phi(x)-Z(x)$. Clearly $\psi(c)=0=\psi'(c)=\psi''(c)$, $\psi(d)=-Z(d)<0$ and

$$\begin{aligned} \psi'''(c)&=\phi'''(c)-Z'''(c) \\ &=\theta'''(c)-\eta'''(c)-Z'''(c) \\ &=f(c, \theta(c), \theta'(c), \theta''(c))-f(c, \eta(c), \eta'(c), \eta''(c))-g(c, Z(c), Z'(c), Z''(c)) \\ &>g(c, \theta(c)-\eta(c), \theta'(c)-\eta'(c), \theta''(c)-\eta''(c)-g(c, Z(c), Z'(c), Z''(c)) \\ &=g(c, 0, 0, \theta''(c)-\eta''(c))-g(c, 0, 0, Z''(c)) \\ &=0\,. \end{aligned}$$

Hence it follows that there exists an $r\in(c, d)$ such that $\psi(r)=0$ and $\psi(t)>0$ for all $t\in(c, r)$. Since $\psi(c)=0$, $\psi(r)=0$ it follows that there exists a $t_0\in(c, r)$ such that $\psi'(t)>0$ $\forall t\in(c, t_0)$ and $\psi'(t_0)=0$. This together with $\psi'(c)=0$ gives that there exists a $t_1\in(c, t_0)$ such that $\psi''(t_1)=0$ and $\psi'''(t_1)\leqq 0$ and $\psi''(t)>0$ for all $t\in(c, t_1)$. But

$$\begin{aligned} \psi'''(t_1)&=\phi'''(t_1)-Z'''(t_1) \\ &=f(t_1, \theta(t_1), \theta'(t_1), \theta''(t_1))-f(t_1, \eta(t_1), \eta'(t_1), \eta''(t_1))-Z'''(t_1) \\ &>g(t_1, \phi(t_1), \phi'(t_1), \phi''(t_1))-g(t_1, Z(t_1), Z'(t_1), Z''(t_1)) \\ &\geqq 0\,. \end{aligned}$$

Lemma 2.2. *Assume that conditions* (A), (B), (C), (D) *and* (E) *hold. Then if* $(x_3-x_2)\leqq h$ *and for each* y_1, y_2, y_3, $m\in R$ *there exists atmost one solution of each of the following boundary value problems*:

$$\begin{aligned} &y'''=f(x, y, y', y'') \\ &y(x_2)=y_2\,, \qquad y''(x_2)=m\,, \qquad y(x_3)=y_3 \end{aligned}$$

or

$$y(x_1)=y_1\,, \qquad y(x_2)=y_2\,, \qquad y''(x_2)=m\,.$$

Proof. The proof is analogous as in Lemma 2.1.

Theorem 2.1. *Let $f(x, y, z, w)$ satisfy conditions* (A), (B), (C), (D) *and* (E) *and suppose that for each $m \in R$ there exist solutions of* $(1.2)_i$ *and* $(1.3)_i$ $(i=1, 2)$. *Then there exists a unique solution of* (1.4).

Proof. From Lemmas 2.1 and 2.2 the solutions of $(1.2)_i$ and $(1.3)_i$ $(i=1, 2)$ whenever exists are unique. Let θ be a solution of (1.2_2) with second derivative m at $x=x_2$.

Claim. $\theta'(x_2, m)$ is an increasing function of m with range R. Write $\phi=\theta(\cdot, m_1)-\theta(\cdot, m_2)$. If $m_1<m_2$, $\phi''(x_2)<0$. Clearly $\phi(x_1)=0$, $\phi(x_2)=0$. Without loss of generality we can assume that $\phi'(x_2)>0$ since $\phi(x_1)=0$, $\phi(x_2)=0$ there exists an $r \in (x_1, x_2) \ni \phi'(r)=0$, $\phi'(x)>0$ $\forall x \in [r, x_2]$. Clearly $\phi(x)<0$ in $[r, x_2)$. Consider the solution of initial value problem $z'''=g(x, z, z', z'')$, $z(x_2)=0$, $z'(x_2)=\delta$, $z''(x_2)=0$. Write $\psi=\phi-Z$. Clearly $\psi(x_2)=0$, $\psi'(x_2)>0$, $\psi''(x_2)<0$, $\psi'(r)\leqq 0$. Hence there exists an $r_1 \in (r, x_2)$ such that $\psi''(r_1)=0$ and $\psi''(x)<0$ in $(r_1, x_2]$. These properties imply that $\psi'''(r_1)\leqq 0$. Now $\psi'(x)>0$ in $[r_1, x_2]$ and $\psi(x)<0$ in $[r_1, x_2)$. Consider

$$\begin{aligned}\psi'''(r_1)=\phi'''(r_1)-Z'''(r_1)&=f(r, \theta(r_1, m_1), \theta'(r_1, m_1), \theta''(r_1, m_1))\\&\quad -f(r, \theta(r_1, m_2), \theta'(r_1, m_2), \theta''(r_1, m_2))-g(r_1, Z(r_1), Z'(r_1), Z''(r_1))\\&\quad > g(r_1, \phi(r_1, m_1), \phi'(r_1, m_1), \phi''(r_1, m_1))-g(r_1, Z(r_1), Z'(r_1), Z''(r_1))\\&\quad >0\,. \text{ Hence a contradiction.}\end{aligned}$$

Hence $\phi'(x_2)\leqq 0$.

Let $m \in R$. Now the problem (1.2_1) has a unique solution in $[x_1, x_2]$ and let it be ψ_m. Since ψ_m and $\theta(\cdot, \psi_m''(x_2))$ are solutions of the problem $y'''=f(x, y, y', y'')$, $y(x_1)=y_1$, $y(x_2)=y_2$, $y''(x_2)=\psi_m''(x_2)$ in $[x_1, x_2]$, we must have $\theta(\cdot, \psi_m''(x_2))=\psi_m$. Hence $\theta'(x_2, \psi_m''(x_2))=\psi_m'(x_2)=m$. Thus R is the range of $\theta'(x_2, \cdot)$. Hence the claim. The remainder of the proof of existence follows as in Theorem 2.1 in [1].

Now to establish uniqueness, let ϕ and ψ be solutions.

Claim. $\phi''(x_2)=\psi''(x_2)$.

Suppose to the contrary $\phi''(x_2)\neq\psi''(x_2)$. Without loss of generality assume that $\phi''(x_2)<\psi''(x_2)$. Let $\phi''(x_2)=m_1$ and $\psi''(x_2)=m_2$. Then ϕ is a solution of the problem (1.2_2) with second derivative m_1 at $x=x_2$ and ψ is a solution of the problem (1.3_2) with second derivative m_2 at $x=x_2$. Hence we must have

$$\left.\begin{aligned}\phi(x)&=\theta(x, m_1)\\ \psi(x)&=\theta(x, m_2)\end{aligned}\right\} \quad \text{for all} \quad x \in [x_1, x_2]\,.$$

Therefore

$$\phi^{(i)}(x_2)=\theta^{(i)}(x_2, m_1) \quad (i=1,2)\,; \qquad \psi^{(i)}(x_2)=\theta^{(i)}(x_2, m_2) \quad (i=1,2)\,.$$

Now $m_1<m_2 \Rightarrow \theta'(x_2, m_1)<\theta'(x_2, m_2)$ i.e.,

$$\phi'(x_2)<\psi'(x_2)\,. \tag{2.3}$$

Similarly ϕ and ψ are solutions of the problem (1.3_2). Therefore, we must have

$$\left.\begin{array}{l}\phi(x)=\eta(x, m_1)\\ \psi(x)=\eta(x, m_2)\end{array}\right\} \quad \text{for all} \quad x\in[x_2, x_3]\,.$$

Hence

$$\phi^{(i)}(x_2)=\eta^{(i)}(x_2, m_1)\,; \qquad \psi^{(i)}(x_2)=\eta^{(i)}(x_2, m_2) \quad (i=1,2).$$

Now $m_1<m_2 \Rightarrow \eta'(x_2, m_1)>\eta'(x_2, m_2)$ i.e.,

$$\phi'(x_2)>\psi'(x_2)\,. \tag{2.4}$$

(2.3) and (2.4) contradict each other. Hence the claim. Thus uniqueness is established.

The next theorem establishes validity of hypothesis (ii) in Theorem 2.1.

Theorem 2.2. *Let f: $[x_1, x_3]\times R^3\to R$ with $x_1<x_2<x_3$ and suppose there exists a constant $N>0$ such that $|f(x, y, z, w)|\leqq N$ for all $x\in[x_1, x_3]$, $-\infty<y, z, w<\infty$ then there exist solutions of the problems $(1.2)_i$ and $(1.3)_i$ $(i=1, 2)$.*

Proof. The proof is analogous as in Theorem 2.3 in [5].

References

[1] D. Barr and T. Sherman: *Existence and uniqueness of solutions of three-point boundary value problems*, J. Differential Equations, 13, 197-212 (1973).

[2] P. Bailey, L. Shampine and P. Waltman: *Non-linear two-point boundary value problems*, Academic Press, New York and London (1968).

[3] L. Jackson and K. Schrader: *Existence and uniqueness of solutions of boundary value problems for third order differential equations*, J. Differential Equations, 9, 46-54 (1971).

[4] A. Lasota and Z. Opial: *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second order differential equations*. Colloq. Math.,18, 1-5 (1967).

[5] K. N. Murty: *Existence and uniqueness of solutions of three-point boundary value problems associated with system of differential equations*, J. Math. Phys. Sci., Vol. 11, No. 3, 265-272 (1977).

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