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UNIVERSALLY WEAKLY INNER ONE-PARAMETER AUTOMORPHISM GROUPS OF SIMPLE C^*-ALGEBRAS

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ABSTRACT. Every universally weakly inner one-parameter automorphism group of a simple C^*-algebra with identity is shown to be uniformly continuous (so that it is inner by Sakai's theorem).

E. C. Lance has shown in [3] that every universally weakly inner automorphism of a UHF algebra is inner. His method can be generalized to yield the result that such an automorphism of a separable simple AF algebra is inner (e.g., in non-unital case, implemented by a unitary multiplier). But it remains open for a general simple C^*-algebra.

G. A. Elliott has given in [2] a characterization of universally weakly inner one-parameter automorphism groups of separable C^*-algebras. (In particular, those are approximately inner.) We may ask the following question; whether such automorphism groups are inner if the C^*-algebra is simple. (If the C^*-algebra is a separable simple AF algebra, it follows from Lance's result that this is the case.) We shall answer this affirmatively; every universally weakly inner one-parameter automorphism group of a simple C^*-algebra is inner, i.e., implemented by a norm-continuous unitary group in the algebra if it has an identity (Theorem 7), and by a unitary group of multipliers, which is continuous in the strict topology, if it does not have an identity (Theorem 8).

For a (one-parameter) automorphism group of a C^*-algebra we may think of the condition, which is apparantly weaker than universally weak innerness, that the dual action of the automorphism group on the dual of the C^*-algebra is strongly continuous, i.e., the extension of the automorphism group to the second dual of the C^*-algebra is weak*-continuous. In fact what we shall show in Theorem 7 is that an automorphism group satisfying this condition is inner (if the C^*-algebra is simple and unital).

In the appendix we shall discuss the case of AF algebras above-mentioned; the results there are more or less known.

We shall give a series of lemmas to prove the main lemma, Lemma 6.

Lemma 1. *Let A be a C^*-algebra acting on a Hilbert space $\mathfrak{H}$ and α a strongly continuous one-parameter automorphism group of A implemented by a weakly continuous unitary group V on $\mathfrak{H}$. Then*

$$I=\{x\in A;\ t\mapsto xV_t \text{ is norm-continuous}\}$$

is a closed two-sided ideal of A.

Proof. It is obvious that I is a left ideal. If $x\in I$ and $y\in A$, then the identity $xyV_t=xV_t\alpha_{-t}(y)$ shows $xy\in I$. Hence I is a two-sided ideal. If $x_n\in I$ converges to $x\in A$, then x_nV_t converges to xV_t uniformly in t, so $x\in I$. Thus I is closed.

q.e.d.

Let H be the (self-adjoint) generater of V, and $E(\ )$ the spectral projections of H.

There is a sequence (f_n) of positive continuous functions on $\mathbf{R}$ such that $0\leqq \hat{f}_n\leqq 1=\hat{f}_n(0)$, and f_n converges to the Dirac function at the origin (in the dual of $C_b(\mathbf{R})$).

Lemma 2. *Under the assumption in Lemma 1, adopting the functions (f_n) given above; for any $a\in A$,*

$$[\hat{f}_n(H),\ a]=\hat{f}_n(H)a-a\hat{f}_n(H)$$

converges to zero as n tends to infinity.

Proof. Let $g(t)=\|\alpha_t(a)-a\|$. Then g is continuous and bounded by $2\|a\|$, and vanishes at $t=0$. Now by a simple calculation,

$$\begin{aligned}\|[\hat{f}_n(H),\ a]\| &\leqq \int f_n(t)\|V_ta-aV_t\|dt\\ &=\int f_n(t)g(t)dt\ .\end{aligned}$$

The last term goes to zero as n tends to infinity. q.e.d.

Lemma 3. *Under the assumption in Lemma 1, $a\in I$ if and only if*

$$\lim_{N\to\infty}\{1-E(-N,\ N)\}a=0\ ,$$

Proof. If $a\in I$, then $g(t)=\|V_ta-a\|$ is continuous. As in the proof of Lemma 2 we show that

$$\|\{1-\hat{f}_n(H)\}a\|$$

converges to zero. Since $1-\hat{f}_n$ is close to 1 uniformly except for a compact set, we get the conclusion.

The converse implication follows from:

$$\|V_t a-a\|\leqq 2\|\{1-E(-N, N)\}a\|+\|V_t E(-N, N)-E(-N, N)\|\,\|a\| . \qquad \text{q.e.d.}$$

Lemma 4. *Under the assumption of Lemma* 1, *let* $e, a\in A$ *satisfy* $e=e^*$, $ea=a$. *If there is* $r<1$ *such that*

$$\lim_{N\to\infty}\|\{1-E(-N, N)\}e\|<r ,$$

then $a\in I$.

Proof. Notice that $\|\{1-E(-N, N)\}e\|$ is monotonely decreasing so that the limit exists.

For large enough N, we have

$$r^2>\|\{1-E(-N, N)\}e^2\{1-E(-N, N)\}\| .$$

Since $\hat{f}_n$ converges to 1 on every compact set,

$$\|\{1-\hat{f}_n(H)\}\{1-E(-N, N)\}-\{1-\hat{f}_n(H)\}\|$$

converges to zero as $n\to\infty$. Hence, as $\|1-\hat{f}_n(H)\|\leqq 1$, for large n,

$$r^2>\|\{1-\hat{f}_n(H)e^2\{1-\hat{f}_n(H)\}\| .$$

By Lemma 2 this implies that for large n,

$$r^2>\|\{1-\hat{f}_n(H)\}^2e^2\| .$$

Again we can replace $\{1-\hat{f}_n(H)\}^2$ by $1-E(-M, M)$ with large enough M and still get the same inequality. Hence

$$\lim_{N\to\infty}\|\{1-E(-N, N)\}e^2\|<r^2 .$$

By repeating this argument we obtain

$$\lim_{N\to\infty}\|\{1-E(-N, N)\}e^k\|<r^k$$

for any $k=1, 2, \ldots$. Since

$$\begin{aligned}\|\{1-E(-N, N)\}a\|&=\|\{1-E(-N, N)\}e^k a\|\\ &\leqq\|\{1-E(-N, N)\}e^k\|\,\|a\|\end{aligned}$$

we have that

$$\lim \|\{1-E(-N, N)\}a\|<r^k\|a\|$$

for any k. Hence we get the conclusion. q.e.d.

Lemma 5. *Suppose that A is acting irreducibly, besides the assumption of Lemma 1. If there are $r<1$ and $b\in A$ with $\|b\|=1$ such that*

$$\sup_{x\in A,\|x\|\leq 1}\|b(\alpha_t(x)-x)b^*\|<r \tag{*}$$

is satisfied for sufficiently small, t, then $I\neq(0)$.

Proof. We have (*) with b^*b in place of b. If we replace b by an element $e\in A$ with the property that $\|b^*b-e\|$ is sufficiently small, we still get (*), possibly by replacing r too by a slightly larger $r'<1$.

Hence we may assume that there are e and a in A such that $\|e\|=\|a\|=1$, $0\leqq e\leqq 1$, $ea=a$ and

$$\sup_{x\in A,\|x\|\leqq 1}\|e(\alpha_t(x)-x)e\|<r\ (<1)$$

is satisfied for sufficiently small t.

For any $\varepsilon>0$ with $r+4\varepsilon<1$, since $\|e\|=1$, there is a unit vector $\Phi\in H$ such that $\|e\Phi\|>1-\varepsilon$. Since

$$\begin{aligned}\|(eV_t-e)xe\Phi\| &\leqq \|eV_txV_t^*e\Phi-exe\Phi\|+\|x\|\,\|V_t^*e\Phi-e\Phi\|\\ &\leqq \|e\alpha_t(x)e-exe\|+\|x\|\,\|V_t^*e\Phi-e\Phi\|\ ,\end{aligned}$$

we know that for sufficiently small t,

$$\sup_{x\in A,\|x\|\leqq 1}\|(eV_t-e)xe\Phi\|<r+\varepsilon\ .$$

For any unit vector $\Psi\in H$, by applying Kaplansky's density theorem to $\|e\Phi\|^{-2}\Psi\otimes e\Phi\in A''$, we have a sequence $x_n\in A$ with $\|x_n\|\leqq\|e\Phi\|^{-1}$ such that $\lim x_ne\Phi=\Psi$. Thus, for sufficiently small t,

$$\|(eV_t-e)\Psi\|<\|e\Phi\|^{-1}(r+\varepsilon)<r+4\varepsilon\ ,\quad \text{i.e.,}\quad \|eV_t-e\|<r+4\varepsilon\ .$$

By using the functions f_n given in Lemma 2,

$$\|\hat{f}_n(H)e-e\|<1$$

for large n. Hence

$$\lim\|\{1-E(-N,N)\}e\|<1\ .$$

By Lemma 4 we have $a\in I$. q.e.d.

Lemma 6. *Under the assumption of Lemma 5, suppose that $I=(0)$. Then there is a state φ of A such that $t\mapsto\varphi\circ\alpha_t$ is not norm-continuous.*

Proof. By Lemma 5, for any $b \in A$ with $\|b\|=1$, the set of $t>0$ which satisfies

$$\sup_{x \in A, \|x\| \leq 1} \|b(\alpha_t(x)-x)b^*\| > 2/3$$

has 0 as an accumulation point. Hence there are $t_1 \in (0, 1)$ and $x_1 = x_1^* \in A$ with $\|x_1\|=1$ such that

$$\|\alpha_{t_1}(x_1)-x_1\| > 1/3 .$$

Otherwise, for all $y \in A$ with $\|y\|=1$ and $t \in (0, 1)$ we have

$$\|\alpha_t(y)-y\| \leqq 2/3$$

by decomposing y into the self-adjoint and anti-self-adjoint parts, which is a contradiction.

We may suppose that $\mathrm{Sp}(\alpha_{t_1}(x_1)-x_1) \cap [1/3, 2] \neq \emptyset$. Let f and g be continuous functions on $\mathbf{R}$ such that

$$f(t)=\begin{cases} 1 & t \geqq 1/6 \\ 6t & 0 \leqq t \leqq 1/6 \\ 0 & t \leqq 0 \end{cases}$$

$$g(t)=\begin{cases} 1 & t \geqq 1/3 \\ 6t-1 & 1/6 \leqq t \leqq 1/3 \\ 0 & t \leqq 1/6 . \end{cases}$$

Set $a_1 = f(\alpha_{t_1}(x_1)-x_1)$ and $b_1 = g(\alpha_{t_1}(x_1)-x_1)$. Then $0 \leqq a_1 \leqq 1$, $0 \leqq b_1 \leqq 1$, $\|a_1\| = \|b_1\| = 1$, and $a_1 b_1 = b_1$.

Next there are $t_2 \in (0, t_1/2)$ and $x_2 = x_2^* \in A$ with $\|x_2\|=1$ such that

$$\|\alpha_{t_2}(b_1)-b_1\| < 1/24 ,$$
$$\mathrm{Sp}(b_1(\alpha_{t_2}(x_2)-x_2)b_1) \cap [1/3, 2] \neq \emptyset .$$

Set $a_2 = f(b_1(\alpha_{t_2}(x_2)-x_2)b_1)$ and $b_2 = g(b_1(\alpha_{t_2}(x_2)-x_2)b_1)$. Then the pair (a_2, b_2) satisfies the same properties as (a_1, b_1). Furthermore $a_1 a_2 = a_2$. (If f is a polynomial with $f(0)=0$, this follows from $a_1 b_1 = b_1 a_1 = b_1$.)

Now we repeat this argument, i.e., assuming that we have constructed t_k, x_k, a_k, b_k up to $n-1$, we have $t_n \in (0, t_{n-1}/2)$ and $x_n = x_n^* \in A$ with $\|x_n\|=1$ such that

$$\|\alpha_{t_n}(b_{n-1})-b_{n-1}\| < 1/24 ,$$
$$\mathrm{Sp}(b_{n-1}(\alpha_{t_n}(x_n)-x_n)b_{n-1}) \cap [1/3, 2] \neq \emptyset ,$$

and set a_n and b_n as before. Notice that (a_n) has the property: If $n>m$, $a_n a_m = a_n$.

Let φ_n be a state of A such that $\varphi_n(a_n)=1$. Then $\varphi_n(a_k)=1$ for any $k \leqq n$. Let

φ be an accumulation point of (φ_n). Then $\varphi(a_n)=1$ for all n. We want to show that $t\mapsto\varphi\circ\alpha_t$ is not norm-continuous.

Since $\varphi(a_n)=1$, the restriction of φ to the C^*-algebra generated by $b_{n-1}(\alpha_{t_n}(x_n)-x_n)b_{n-1}$ has support in the closed subset $\{p;\, p(a_n)=1\}$ of its spectrum. Hence $\varphi(b_{n-1}(\alpha_{t_n}(x_n)-x_n)b_{n-1})\geqq 1/6$. Thus

$$\begin{aligned}&\varphi(\alpha_{t_n}(b_{n-1}x_nb_{n-1})-b_{n-1}x_nb_{n-1})\\ &\qquad\geqq\varphi(b_{n-1}\alpha_{t_n}(x_n)b_{n-1}-b_{n-1}x_nb_{n-1})-\|\alpha_{t_n}(b_{n-1}x_nb_{n-1})-b_{n-1}\alpha_{t_n}(x_n)b_{n-1}\|\\ &\qquad\geqq 1/6-2\|\alpha_{t_n}(b_{n-1})-b_{n-1}\|>1/12\,.\end{aligned}$$

This implies that $\|\varphi\circ\alpha_{t_n}-\varphi\|>1/12$. q.e.d.

Now we come to the main theorems. Remark that if α^{**} on A^{**} is σ-weak continuous, α^{**} fixes a minimal central projection, and that any σ-weak continuous one-parameter automorphism group of $B(\mathfrak{H})$ is covariant.

Theorem 7. *Let A be a simple C^*-algebra with identity and α a (strongly continuous) one-parameter automorphism group of A. If α^* on A^* is strongly continuous, then α is uniformly continuous, so that it is implemented by a norm-continuous unitary group of A.*

Proof. By the assumption there are an irreducible representation π of A and a weakly continuous unitary group V on $\mathfrak{H}_\pi$ such that $\pi\circ\alpha_t(x)=V_t\pi(x)V_t^*$, $x\in A$. Hence we can apply the preceding lemmas to $\pi(A)$ which is identified with A. Since I (in Lemma 1) is an ideal of A, I is either (0) or A. If $I=(0)$, Lemma 6 yields a state φ of A with the property that $t\mapsto\varphi\circ\alpha_t$ is not norm-continuous. This is a contradiction. Hence $I=A$. Since $A\ni 1$, $t\mapsto V_t$ is norm-continuous. Thus α is uniformly continuous, and so $V_t\in A$ by Sakai's theorem. q.e.d.

Remark that the strong continuity of α follows from α^* being strongly continuous.

We say that α is universally weakly inner if for any representation π of A, there is a weakly continuous unitary group V of $\pi(A)''$ such that $\pi\circ\alpha_t(x)=V_t\pi(x)V_t^*$, $x\in A$.

Theorem 8. *Let A be a simple C^*-algebra without identity, and α a (strongly continuous) one-parameter automorphism group of A. If α is universally weakly inner, then α is implemented by a unitary group of multipliers which is continuous in the strict topology.*

Proof. By the argument in the proof of Theorem 7, we can show that for any irreducible representation π of A, the weakly continuous unitary group V^π on $\mathfrak{H}_\pi$ which implements $\pi\circ\alpha_t\circ\pi^{-1}$ (and which is unique up to phase factors) satisfies that $t\mapsto V_t\pi(x)$ is norm-continuous for any $x\in A$. Now we have to show that $V_t\pi(x)\in\pi(A)$, $x\in A$.

The proof is essentially the same as Sakai's in [4, 4.1.9–11]. Let D_π be the C^*-algebra generated by $V_t^\pi\pi(x)$, $x\in A$, $t\in\mathbf{R}$. We divide the proof into several steps.

Step 1. For any non-zero $\lambda\in\mathbf{R}$, there is not an endomorphism Φ of D_π which satisfies that $\Phi(x\hat{f}(H^\pi))=x\hat{f}(H^\pi+\lambda))$, $x\in\pi(A)$, $f\in L^1(\mathbf{R})$, where H^π is the generator of V^π.

This is because

$$\lim_{|n|\to\infty}\|x\hat{f}(H^\pi+n\lambda)\|=0\,.$$

Step 2. D_π is independent of π, i.e., for two irreducible representations π_1 and π_2 of A, there is an isomorphism Φ of D_{π_1} onto D_{π_2} such that $\Phi(\pi_1(x)\hat{f}(H^{\pi_1}))=\pi_2(x)\hat{f}(H^{\pi_2}+\lambda)$, $x\in A$, $f\in L^1$ (with some $\lambda\in\mathbf{R}$).

To show this first notice that V_t^π is a multiplier of D_π. Let $\pi_2=\pi_\omega$ with a pure state ω of A. We extented the state $\omega\circ\pi_1^{-1}$ of $\pi_1(A)$ to a pure state of D_{π_1}, say $\bar{\omega}$. In the representation $\pi_{\bar{\omega}}\circ\pi_1$ of A there is a weakly continuous unitary group U in $\pi_{\bar{\omega}}\circ\pi_1(A)''$ such that $U_t\pi_{\bar{\omega}}\circ\pi_1(x)U_t^*=\pi_{\bar{\omega}}\circ\pi_1\circ\alpha_t(x)$, $x\in A$. Hence $\pi_{\bar{\omega}}(V_t)U_t^*=U_t^*\pi_{\bar{\omega}}(V_t)$ is in the commutant of $\pi_{\bar{\omega}}\circ\pi_1(A)$, hence in the center of $\pi_{\bar{\omega}}(D_{\pi_1})''$ which is trivial. Thus $\pi_{\bar{\omega}}\circ\pi_1$ is irreducible, i.e., $\pi_{\bar{\omega}}\circ\pi_1=\pi_2$ and $U=V^{\pi_2}$. Since

$$\pi_{\bar{\omega}}(V_t^{\pi_1})=e^{i\lambda t}V_t^{\pi_2}$$

with some $\lambda\in\mathbf{R}$, $\pi_{\bar{\omega}}$ is a homomorphism of D_{π_1} onto D_{π_2} with the prescribed property. By interchanging π_1 with π_2, we get such a homomorphism of D_{π_2} onto D_{π_1}; the composition of those must be trival by Step 1.

Step 3. D_π is simple.

If J is a proper closed ideal of D_π, then $J\cap\pi(A)=(0)$. Let ρ be an irreducible representation of D_π/J, and q be the qoutient map of D_π onto D_π/J. Then $\rho\circ q$ map D_π onto $D_{\rho\circ q\circ\pi}$ satisfying the prescribed property, so $\rho\circ q$ must be an isomorphism.

Now fixing such an irreducible representation π, we identify $\pi(A)$ with A and let $D=D_\pi$.

Step 4. If two factorial states of D coincide on A, they are equal.

Let φ_1 and φ_2 be these two states which are different. First we can show that $\pi_{\varphi_i}(A)''=\pi_{\varphi_i}(D)''$. Since $\varphi_1=\varphi_2$ on A, there is an isomorphism Φ of $\pi_{\varphi_1}(A)''$ onto $\pi_{\varphi_2}(A)''$ such that

$$\Phi\circ\pi_{\varphi_1}(x)=\pi_{\varphi_2}(x)\ ,\quad x\in A\ .$$

As in the proof of Step 2 we can show that this isomorphism induces an automorphism of D, which is of the type prohibited in Step 1.

The final step is to combine the result in Step 4 with Sakai's arguments in [4, 4.1.9, 10]. His arguments yield, by supposing $D\supsetneqq A$, two different factorial states which coincide on A, a contradiction. q.e.d.

Our final result is a simple remark on the assumption which has appeared in Theorem 7:

Proposition 9. *Let A be a C^*-algebra and α a one-parameter automorphism group of A. If $t\mapsto\varphi\circ\alpha_t$ is norm-continuous for any pure state φ of A, then $t\mapsto\varphi\circ\alpha_t$ is norm-continuous for any state φ of A.*

Proof. Suppose that there is a state φ of A such that $\|\varphi\circ\alpha_t-\varphi\|$ does not converge to zero as $t\to 0$; there are $\delta>0$ and a sequence (t_n) converging to zero such that $\|\varphi\circ\alpha_{t_n}-\varphi\|>\delta$. Then there is a sequence (x_n) in A with $\|x_n\|=1$ such that

$$|\varphi\circ\alpha_{t_n}(x_n)-\varphi(x_n)|>\delta\ . \tag{*}$$

Let G be the subgroup of $\mathbf{R}$ generated by t_n, $n\in\mathbf{N}$. Then G is countable. Let B be the C^*-subalgebra generated by

$$\{\alpha_t(x_n);\ t\in G,\ n\in\mathbf{N}\}\ .$$

Then B is separable and α_t-invariant for $t\in G$.

Any pure state ω of B has an extension to a pure state $\bar\omega$ of A. Since, for $t\in G$,

$$\|\omega\circ\alpha_t-\omega\|\leqq\|\bar\omega\circ\alpha_t-\bar\omega\|\ ,$$

$\|\omega\circ\alpha_t-\omega\|$ converges to zero as $t\to 0$ in G. For the restriction ψ of φ to B, there is a Radon measure ν on the set E of pure states of B such that

$$\psi=\int_E\omega d\nu(\omega)\ .$$

Let (y_n) be a dense sequence in the unit ball of B. Then for $t\in G$,

$$\|\omega\circ\alpha_t-\omega\|=\sup_n|\omega\circ\alpha_t(y_n)-\omega(y_n)|\ .$$

Thus $E \in \omega \mapsto \|\omega\circ\alpha_{t_n}-\omega\|$ is measurable, and

$$\|\psi\circ\alpha_{t_n}-\psi\| \leqq \int_E \|\omega\circ\alpha_{t_n}-\omega\| d\nu(\omega) .$$

Since $E \in \omega \mapsto \|\omega\circ\alpha_{t_n}-\omega\|$ is bounded by 2, and goes to zero as $n\to\infty$, theL ebesgue theorem implies that $\|\psi\circ\alpha_{t_n}-\psi\|\to 0$ which contradicts (*). q.e.d.

Appendix

Let A be a C^*-algebra and α an automorphism of A. If π is a representation of A such that α maps the kernel of π onto itself, then α induces an automorphism of $\pi(A)$. If this automorphism extends to an automorphism of the weak closure of $\pi(A)$, then α is said to be extendible in the representation π (cf. [3]).

Theorem A1. *Let A be a separable simple AF algebra and α an automorphism of A. If α is extendible in every irreducible representation of A, then α is inner.*

Proof. Let (A_n) be a generating increasing sequence of finite-dimensional subalgebras of A. Since A is simple, we may assume that each factor direct summand of A_n is mapped into each one of A_{n+1} under the natural imbedding for $n=1, 2, \dots$.

An inspection of the proof of Lemma 2.6 in [1] shows that α there can be chosen as an inner automorphism. Hence there is an inner automorphism σ of A such that

$$\sigma\circ\alpha(\bigcup_n A_n)=\bigcup_n A_n .$$

Therefore by taking $\sigma\circ\alpha$ instead of α, and by choosing a subsequence of (A_n) we may assume that $\alpha(A_n)\subset A_{n+1}$ and $\alpha^{-1}(A_n)\subset A_{n+1}$.

We assert that for each projection e_0 of A_{n_0},

$$\|e_0(\alpha-\iota)|A_n{}^c\| \equiv \sup_{x\in A_n{}^c, \|x\|\leq 1} \|e_0(\alpha(x)-x)\|$$

converges to zero as $n\to\infty$, where $A_n{}^c=A\cap A_n{}'$ and ι is the identity automorphism.

If it is false, there are $\delta>0$, a subsequence (n_k) of integers and $x_k=x_k{}^*\in A_{n_k}\cap A_{n_{k-1}+3}{}'$ with $\|x_k\|=1$ such that $n_k>n_{k-1}+3$ and $\|e_0(\alpha(x_k)-x_k)\|>\delta$. We may assume that the positive part z_k of $\alpha(x_k)-x_k$ satisfies that $\|e_0z_k\|>\delta$. Set $y_k=e_0z_k$.

Since $\alpha(x_k)-x_k\in A_{n_k+1}\cap A_{n_{k-1}+2}{}'$, all z_k are mutually commuting. There are minimal central projections e and p in $B_k=e_0A_{n_k+1}e_0$ and $C_{k-1}=e_0A_{n_{k-1}+2}e_0$ respectively, such that $\|y_kep\|=\|y_k\|$. Since

$$e\cdot pB_kp \cong ep(B_k \cap C'_{k-1}) \otimes pC_{k-1} ,$$

and

$$pB_{k-1} \cong B_{k-1} ,$$

it follows that

$$\|y_1 \cdots y_k\| \geqq \|y_1 \cdots y_k ep\| = \|y_1 \cdots y_{k-1} p\| \, \|y_k ep\| = \|y_1 \cdots y_k\| \, \|y_k\| .$$

Hence, for any k, $\|y_1 \cdots y_k\| = \|y_1\| \cdots \|y_k\|$.

Let

$$S = \{\varphi;\ \text{state of } A,\ \varphi(y_k) = \|y_k\|, \text{ for any } k\} .$$

Then S is a compact face. Let φ_k be a state of A such that

$$\varphi_k(y_1 \cdots y_k) = \|y_1 \cdots y_k\| .$$

Then $\varphi_k(y_i) = \|y_i\|$ for $i \leqq k$. Since any weak limit point of (φ_k) belongs to S, S is not empty. Let φ be an extremal point of S, which is a pure state of A.

In the representation π_φ, since x_k is a central sequence in A, we may assume that $\pi_\varphi(x_k)$ converges weakly, say to z, which is a multiple of the identity. Since $\varphi(\alpha(x_k) - x_k) = \|y_k\| > \delta$, and $\pi_\varphi \circ \alpha \circ \pi_\varphi^{-1}$ extends to an automorphism, say $\tilde{\alpha}$, of $\pi_\varphi(A)''$, we obtain

$$(\Omega_\varphi, (\tilde{\alpha}(z) - z)\Omega_\varphi) \neq 0 .$$

which contradicts $\tilde{\alpha}(z) = z$.

Hence $\lim \|e_0(\alpha - \iota) | A_n^c\| = 0$. The rest of the proof procedes as the proof of Lemma 3.1 in [3].

Assuming that A is acting irreducibly on a Hilbert space $\mathfrak{H}$, we find a unitary V on $\mathfrak{H}$ such that

$$\alpha(x) = VxV^* , \quad x \in A .$$

For $y \in A_n^c$ with $\|y\| \leqq 1$,

$$\|e_0 VyV^* - e_0 y\| \leqq \|e_0(\alpha - \iota) | A_n^c\| \equiv \varepsilon_n .$$

Hence for any unitary u in A_n^c with $n \geqq n_0$, we have

$$\|e_0 V - ue_0 Vu^*\| \leqq \varepsilon_n .$$

Since A_n^c is an AF algebra and $(A_n^c)' = A_n$, there is $x \in A_n$ such that $\|e_0 V - x\| \leqq \varepsilon_n$. Thus $e_0 V \in A$, and $\alpha(e_0 V) = Ve_0 \in A$. Since A is generated by projections in $\bigcup A_n$, this implies that V is a multiplier of A. q.e.d.

Corollary A2. *Let A be a separable simple AF algebra without identity and α a one-parameter automorphism group of A. If α^* on A^* is strongly continuous, then α is implemented by a unitary group of multipliers which is continuous in the strict topology.*

Proof. It follows from the assumption that each α_t is extendible in every irreducible representation. Hence there is a family (v_t) of unitary multipliers such that $\alpha_t = \mathrm{Ad}\, v_t$, $t \in \mathbf{R}$. It is routine to construct a group with the required continuity property from (v_t). q.e.d.

By using the fact that A has an approximate identity consisting of projections, we can give a proof to this result as a corollary of Theorem 7. First we can form, as an approximate identity, an increasing sequence (e_n) of projections which are all in the domain of the generator δ of α. Then, by using

$$\delta(e_n) = [\delta(e_n)e_n - e_n\delta(e_n),\, e_n] ,$$

we repeat inner perturbations to get another one-parameter automorphism group β of A such that

$$\beta_t(x) = \mathrm{Ad}\, v_t \circ \alpha_t(x) , \quad x \in A ,$$
$$\beta_t(e_n) = e_n , \quad n = 1, 2, \cdots ,$$

where (v_t) is a continuous cocycle of unitary multipliers. Since β satisfies the same property as α does, a successive application of Theorem 7 to e_nAe_n yields that β is inner.

Hence we may conjecture the following; the conclusion of Theorem 8 is true under the weaker assumption on α that α^* on A^* is strongly continuous.

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Note added in proof. After completion of this work I have shown that every universally weakly inner automorphism of a separable simple C^*-algebra is inner and that the conjecture at the end of Appendix is true. See Commun. Math. Phys. 81, 429–435 (1981) for details.

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