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ON THE ORDER OF COMPLETE REGULARITY FOR A WEAKLY STATIONARY RANDOM SEQUENCE

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In their book, I. A. Ibragimov and Yu. A. Rozanov [3] have given the necessary and sufficient condition for the regularity coefficient of a weakly stationary random sequence to have some order of its going to zero in terms of smoothness of the spectral density function. In this paper the speed of approaching zero for the regularity coefficient will be evaluated using the modulus and L^p mean modulus of continuity of the spectral density function in a slightly more general way. Some remarks will be given in the final section, which tells us that we can evaluate it in a similar fashion to the above one by using the outer factor of the spectral density function.

1. Introduction

Consider a complex valued weakly stationary random sequence $\{\xi(t,\omega): t=0, \pm 1, \cdots\}$ with $E\xi(t,\omega)=0$, $t=0, \pm 1, \pm 2, \cdots$ and the covariance function

$$B(u)=E\xi(t+u,\omega)\overline{\xi(t,\omega)} = \int_{-\pi}^{\pi} e^{iu\lambda}dF(\lambda), \quad t=0, \pm 1, \pm 2, \cdots; \quad u=0, \pm 1, \pm 2, \cdots, \tag{1.1}$$

where $F(\lambda)$ is the spectral distribution function.

Let H be closed (in the mean square sense) linear manifold generated by $\{\xi(t,\omega); t=0, \pm 1, \pm 2, \cdots\}$. As usual, introducing a scalar product in H by $(h_1, h_2)=Eh_1\overline{h_2}$, we have a Hilbert space H. We also denote by $H_{(ab)}$ the subspace of H, which is generated by $\{\xi(t,\omega); a\leqq t\leqq b, \infty\leqq a\leqq b\leqq\infty\}$.

The random sequence $\xi(t,\omega)$ is said to be *linear regular* if the subspace $H_{(-\infty,-\infty)}\equiv\bigcap_t H_{(-\infty,t)}$ contains only the random variable which is zero with probability one. For the linear regularity of a weakly stationary sequence $\xi(t,\omega)$, it is necessary and sufficient that its spectral distribution function $F(\lambda)$ is absolutely continuous and has the spectral density $f(\lambda)\equiv F'(\lambda)$ satisfying the condition

$$\int_{-\pi}^{\pi} \ln f(\lambda)d\lambda > -\infty . \tag{1.2}$$

It is also known ([3], Chapter IV, Section 1) that the condition (1.2) holds if and only if $f(\lambda)$ admits a factorization

$$f(\lambda)=|g(e^{i\lambda})|^2, \tag{1.3}$$

where $g(z)$ belongs to the Hardy class H^2 and $g(e^{i\lambda})$ is the boundary function $\lim_{r\to 1-0} g(re^{i\lambda})$, which is defined for almost all λ. A function $g(z)$ is said to belong to the class H^p $(0<p\leqq\infty)$ if it is analytic in the unit disc and satisfies the condition:

$$\sup_{0\leqq r<1}\int_{-\pi}^{\pi}|g(re^{i\lambda})|^p d\lambda<\infty\ , \qquad 0<p<\infty\ , \tag{1.4}$$

$$\operatorname*{ess\,sup}_{\substack{0\leqq\lambda<2\pi\\ 0\leqq r<1}}|g(re^{i\lambda})|<\infty\ , \qquad p=\infty\ . \tag{1.5}$$

For such a function $g(z)$, the boundary function $g(e^{i\lambda})$ exists a. e. The regularity coefficient of the random sequence $\xi(t,\omega)$ is defind by

$$\rho(t)=\sup|E\eta_1\bar{\eta}_2|, \qquad t\geqq 0\ , \tag{1.6}$$

where the supremum is taken over all random variables $\eta_1\in H_{(-\infty,0)}$, $\eta_2\in H_{(t,\infty)}$ with $E|\eta_1|^2=E|\eta_2|^2=1$. $\rho(t)$ is thought of as the measure of degree of dependence between the past and the future of the stationary random sequence. The behavior of $\rho(t)$ has been studied in the context of a linear prediction problem by many authors.

Moreover a random sequence $\xi(t,\omega)$ is said to be *completely regular* if

$$\lim_{t\to\infty}\rho(t)=0\ . \tag{1.7}$$

We know that if $\xi(t,\omega)$ is completely regular, then it is also linearly regular. We shall assume from now on throughout the paper that $\xi(t,\omega)$ is completely regular. It is known in this case that the regularity coefficient (1.7) can be also written in the form

$$\rho(t)=\sup\left|\int_{-\pi}^{\pi}\theta(e^{i\lambda})f(\lambda)e^{i\lambda t}d\lambda\right|\ , \tag{1.8}$$

where the supremum is taken over all θ's which belong to the class H^1 satisfying the condition

$$\int_{-\pi}^{\pi}|\theta(e^{i\lambda})|f(\lambda)d\lambda\leqq 1\ , \tag{1.9}$$

([3] Chapter V Section 1). We also denote $\rho(t)$ in (1.8) by $\rho(t;f)$ when the spectral density function is particularly signified.

As to the speed of approaching zero for $\rho(t)$ exhaustive studies were made by Ibragimov and Rozanov [3], who gave the conditions for $\rho(t)$ to have some order of its going to zero in terms of the Lipschitz's condition for $f^{(k)}(\lambda)$, the k-th derivative of $f(\lambda)$. One of their results is stated as Theorem A below.

Let the modulus of continuity of a periodic function $\phi(\lambda)$ with period 2π, be

(1.10) $$\omega(\delta, \phi)=\max_{0<|h|\leqq\delta}\|\phi(\lambda+h)-\phi(\lambda)\|,$$

where

(1.11) $$\|\phi(\lambda)\|=\max_{|\lambda|\leqq\pi}|\phi(\lambda)|.$$

Write $\phi\in\Lambda_\alpha$, when $\omega(\delta, \phi)=O(\delta^\alpha)$ as δ tends to zero, $0<\alpha\leqq 1$. If a stationary random sequence $\xi(t)$ is completely regular, then this sequence has the spectral density $f(\lambda)$ representable as

(1.12) $$f(\lambda)=w(\lambda)|P(e^{i\lambda})|^2,$$

where $P(z)$ is a polynomial of some order ([3] p. 147). Using this $w(\lambda)$, we have the following

Theorem A. ([3] p. 181) *Suppose $w(\lambda)\geqq c>0$ for some constant c. Then for an integer $k\geqq 0$*

(1.13) $$\rho(t;f)=O(t^{-k-\alpha}), \qquad 0<\alpha<1,$$

as $t\to\infty$ if and only if $w(\lambda)$ is k-times differentiable and its k-th derivative satisfies

(1.14) $$w^{(k)}(\lambda)\in\Lambda_\alpha.$$

The purpose of this paper is to give conditions which enables us to have some order of completely regularity, in terms of the modulus and the mean modulus of continuity of $w(\lambda)$ and $w^{(k)}(\lambda)$ in a slightly more general way. We shall show in Section 4 that the conditions similar to these are given in terms of the outer factor $g(e^{i\lambda})$.

The mean modulus of continuity of a periodic function whth period 2π in L^p norm is defined by

(1.15) $$\omega_p(\delta, \phi)=\max_{0<|h|\leqq\delta}\|\phi(\lambda+h)-\phi(\lambda)\|_p,$$

where $\phi(\lambda)$ is not necessarily continuous and

$$\|\phi(\lambda)\|_p=\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}|\phi(\lambda)|^p d\lambda\right)^{1/p}, \qquad 1\leqq p<\infty . \tag{1.16}$$

We shall write $\phi\in\Lambda_\alpha^p$, when $\omega_p(\delta,\phi)=O(\delta^\alpha)$, as δ tends to zero, $0<\alpha\leqq 1$.

2. Theorems

We have assumed that our stationary random sequence is completely regular. Hence the spectral density $f(\lambda)$ of this sequence is representable as in (1.12).

Theorem 1. *Suppose that for an integer* $k\geqq 1$

(i) *the k-th derivative* $w^{(k)}(\lambda)$ *exists,*

(2.1) (ii) $w(\lambda)\geqq c$, a.e.

for some constants $c>0$ *and*

(2.2) (iii) $w^{(k)}(\lambda)\in L^p(-\pi,\pi)$, $1\leqq p<\infty$.

Then we have

$$\rho(t;f)=O\left(\sum_{u=t}^{\infty}u^{-k-1+1/p}\omega_p(1/u,w^{(k)})\right), \tag{2.3}$$

as $t\to\infty$.

(2.3) *is still true, when* $k=0$, *if the condition* (ii) *and* (iii) *are satisfied and* $t^{1/p}\omega_p(1/t,w)$ *monotonically goes to zero as* $t\to\infty$, *the condition* (i) *being not needed.*

For $p=\infty$, we also have a theorem analogous to Theorem 1.

Theorem 2. *Suppose that for an integer* $k\geqq 0$

(2.4) (i) $w(\lambda)$ *has the continuous k-th derivative* $w^{(k)}(\lambda)$
(ii) $w(\lambda)\geqq c$, a.e.

for some constant $c>0$. *Then we have*

$$\rho(t;f)=O\left(\sum_{u=t}^{\infty}u^{-k-1}\omega(1/u,w^{(k)})\right). \tag{2.5}$$

The following corollary is an immediate consequence of Theorem 1.

Corollary 1. *Suppose that for an integer* $k\geqq 0$

(2.6) (i) *the k-th derivative* $w^{(k)}(\lambda)$ *exists,*
(ii) $w(\lambda)\geqq c$, a.e.,

for some constant $c>0$ *and*

(2.7) (iii) $w^{(k)}(\lambda)\in\Lambda_\alpha^p$, $1\leqq p<\infty$, $1/p<\alpha<1$,

Then we have

$$\rho(t;f)=O(t^{-k-\alpha+1/p}) . \tag{2.8}$$

Theorem A and Corollary 1 yield the subsequent interesting result. When $P\equiv 1$ in (1.12), we have

Corollary 2. *For an integer* $k\geqq 0$,

(i) $f^{(k)}(\lambda)$ *exists*

and

(ii) $f(\lambda)\geqq c$, a.e.

for some constant $c>0$, *then*

$$f^{(k)}\in\Lambda_\alpha^p \ \textit{implies}\ f^{(k)}\in\Lambda_{\alpha-1/p},$$

for $1/p<\alpha<1$.

This corollary resembles Hardy-Littlewood's theorem:

if $g\in H^p$ *then* $g\in\Lambda_\alpha^p$ *implies* $g\in\Lambda_{\alpha-1/p}$, $1/p<\alpha<1$. ([1], p. 91)

The proofs of Theorems are given in the following section. The corollaries are simple consequences from the theorems.

3. Proofs of Theorems

We shall prove only Theorem 1. Theorem 2 can be proved in exactly the same way.

Because of $\rho(t+n;f)\leqq\rho(t,w)$ for n, the order of the polynomial in (1.12), we may suppose $P\equiv 1$ ([3], p. 147), and for the general case, we have (2.3) with $\rho(t+n;f)$ in place of $\rho(t;f)$. But we easily verify the validity of (2.3) itself, once (2.3) holds for $\rho(t+n;f)$ since n is a fixed integer. Then we assume $P\equiv 1$ in what follows.

Let $S_n(\lambda)$ be the partial sum of the Fourier series of the spectral density function $f(\lambda)$:

$$S_n(\lambda)=\sum_{k=-n}^{n}c_k e^{i\lambda k}, \tag{3.1}$$

where

$$c_k=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\lambda)e^{-ik\lambda}d\lambda, \qquad k=0,\pm 1,\pm 2,\cdots .$$

Write

$$\sigma_n(\lambda)=\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)c_k e^{ik\lambda}\,, \tag{3.2}$$

and

$$\begin{aligned}\tau_n(\lambda)&=2\sigma_{2n-1}(\lambda)-\sigma_{n-1}(\lambda)\\&=\sum_{n\leqq|k|\leqq 2n-1}\left(2-\frac{|k|}{n}\right)c_k e^{i\lambda k}+S_{n-1}(\lambda)\ .\end{aligned} \tag{3.3}$$

(See [4], p. 116). In order to prove our theorem we need the following inequality relation.

$$\operatorname*{ess\,sup}_{|\lambda|\leqq\pi}|f(\lambda)-\tau_n(\lambda)|\leqq C\sum_{\nu=2n}^{\infty}\omega_p(1/\nu,\ f^{(k)})/\nu^{k+1-1/p}\ . \tag{3.4}$$

The proof of this fact is carried out in the following way.

Let

$$\begin{aligned}T_n(\lambda)&=\tau_{2n}(\lambda)-\tau_n(\lambda)\\&=\sum_{2n\leqq|k|\leqq 4n-1}\left(2-\frac{|k|}{2n}\right)c_k e^{ik\lambda}-\sum_{n\leqq|k|\leqq 2n-1}\left(1-\frac{|k|}{n}\right)c_k e^{ik\lambda}\end{aligned} \tag{3.5}$$

and write the Féjèr kernel by

$$K_n(u)=\frac{1}{2\pi(n+1)}\left(\frac{\sin\,(n+1)u/2}{\sin u/2}\right)^2\ . \tag{3.6}$$

Then because of orthogonality of the system $\{e^{ik\lambda};\ |\lambda|\leqq\pi\}_k$, and $\int_{\pi}^{\pi}K_n(u)\,du=1$, we easily see

$$T_n(\lambda)=\int_{-\pi}^{\pi}(T_n(\lambda)-T_n(\lambda+u))K_{n-1}(u)\,du\ . \tag{3.7}$$

Also we see that $C_1>0$, $C_2>0$ can be chosen in such a way that

$$\int_{E_n^c}K_{n-1}(u)\,du=1-\int_{E_n}K_{n-1}(u)\,du\leqq 1/4\ , \tag{3.8}$$

where

$$E_n=\left\{u;\ \frac{C_1}{n}\leqq|u|\leqq\frac{C_2}{n}\right\}\ .$$

From (3.7) and (3.8), we have

$$\begin{aligned}|T_n(\lambda)|\leq&\int_{E_n}|T_n(\lambda)-T_n(\lambda+u)|K_{n-1}(u)\,du\\&+\int_{E_n^c}\{|T_n(\lambda)|+|T_n(\lambda+u)|\}K_{n-1}(u)\,du\ ,\end{aligned} \tag{3.9}$$

and hence

$$\sup_{|\lambda|\le\pi}|T_n(\lambda)|\leqq \sup_{\substack{|\lambda|\le\pi\\ u\in E_n}}|T_n(\lambda)-T_n(\lambda+u)|+1/2\sup_{|\lambda|\le\pi}|T_n(\lambda)|\,, \tag{3.10}$$

from which

$$\sup_{|\lambda|\le\pi}|T_n(\lambda)|\leqq 2\sup_{\substack{|\lambda|\le\pi\\ u\in E_n}}|T_n(\lambda+u)-T_n(\lambda)| \tag{3.11}$$

([2], p. 93). On the other hand, we can write ([4], p. 116)

$$\begin{aligned} T_n(\lambda)&=\frac{2}{\pi}\int_0^\infty \left\{f\left(\lambda+\frac{v}{2n}\right)+f\left(\lambda-\frac{v}{2n}\right)-f\left(\lambda+\frac{v}{n}\right)-f\left(\lambda-\frac{v}{n}\right)\right\}H_0(v)\,dv\\ &=\frac{2}{\pi}\int_0^\infty \left\{(1/2n)^k f^{(k)}\left(\lambda+\frac{v}{2n}\right)+(-1/2n)^k f^{(k)}\left(\lambda-\frac{v}{2n}\right)\right.\\ &\quad\left.-(1/n)^k f^{(k)}\left(\lambda+\frac{v}{n}\right)-(-1/n)^k f^{(k)}\left(\lambda-\frac{v}{n}\right)\right\}H_k(v)\,dv\,, \end{aligned} \tag{3.12}$$

where

$$H_0(u)=h(u)/u^2\,, \tag{3.13}$$

$$H_k(u)=\int_u^\infty H_{k-1}(v)\,dv\,,\qquad k=1,2,3,\cdots \tag{3.14}$$

$$h(u)=\sin^2 u-\sin^2 u/2\,, \tag{3.15}$$

and it is easily seen that for each k, $H^{(k)}(u)$ is bounded for $0\leqq u<\infty$ and is $O(u^{-2})$ near $u=\infty$. Therefore we have

$$\begin{aligned} &|T_n(\lambda+u)-T_n(\lambda)|\\ &\leqq\frac{1}{n^k}\left(\left|\int_0^\infty \left\{f^{(k)}\left(\lambda+\frac{v}{2n}+u\right)-f^{(k)}\left(\lambda+\frac{v}{2n}\right)\right\}H_k(v)\,dv\right|\right.\\ &\quad+\left|\int_0^\infty \left\{f^{(k)}\left(\lambda-\frac{v}{2n}+u\right)-f^{(k)}\left(\lambda-\frac{v}{2n}\right)\right\}H_k(v)\,dv\right|\\ &\quad+\left|\int_0^\infty \left\{f^{(k)}\left(\lambda-\frac{v}{n}+u\right)-f^{(k)}\left(\lambda+\frac{v}{n}\right)\right\}H_k(v)\,dv\right|\\ &\quad\left.+\left|\int_0^\infty \left\{f^{(k)}\left(\lambda-\frac{v}{n}+u\right)-f^{(k)}\left(\lambda-\frac{v}{n}\right)\right\}H_k(v)\,dv\right|\right)\\ &=I_1+I_2+I_3+I_4\,, \end{aligned} \tag{3.16}$$

say. Changing variables we have

$$I_1=\frac{1}{n^k}\left|\int_0^\infty \left\{f^{(k)}\left(\lambda+\frac{v}{2n}+u\right)-f^{(k)}\left(\lambda+\frac{v}{2n}\right)\right\}H_k(v)\,dv\right| \tag{3.17}$$

$$\leqq \frac{2}{n^{k-1}}\left(\int_0^{2\pi} |f^{(k)}(\lambda+v+u)-f^{(k)}(\lambda+v)|\,|H_k((2nv)|dv\right.$$
$$\left.+\sum_{\nu=1}^{\infty}\int_{2\nu\pi}^{2(\nu+1)\pi} |f^{(k)}(\lambda+v+u)-f^{(k)}(\lambda+v)|\,|H_k(2nv)|dv\right).$$

By the Hölder's inequality, we easily see

$$\text{(3.18)*}\qquad \int_0^{2\pi} |f^{(k)}(\lambda+v+u)-f^{(k)}(\lambda+v)|\,|H_k(2nv)|dv$$
$$\leqq \frac{C}{n^{1/q}}\omega_p(u, f^{(k)}), \qquad (1/p+1/q=1,\ 1\leqq p\leqq\infty)$$

and

$$\text{(3.19)}\qquad \int_{2\nu\pi}^{2(\nu+1)\pi} |f^{(k)}(\lambda+v+u)-f^{(k)}(\lambda+v)|\,|H_k(2nv)|dv \leqq \frac{C}{(n\nu)^2}\omega_p(u, f^{(k)}).$$

Hence

$$\text{(3.20)}\qquad I_1\leqq\left(\frac{C}{n^{k-1/p}}+\frac{C}{n^{k+1}}\right)\omega_p(u, f^{(k)})$$
$$\leqq\frac{C}{n^{k-1/p}}\omega_p(u, f^{(k)}).$$

The same estmiates are obtained also for I_2, I_3, and I_4 and thus

$$\text{(3.21)}\qquad I_j\leqq\frac{C}{n^{k-1/p}}\omega_p(u, f^{(k)}), \qquad j=1, 2, 3, 4,$$

which from (3.16) gives us

$$\text{(3.22)}\qquad \sup_{|\lambda|\leqq\pi}|T_n(\lambda+u)-T_n(\lambda)|\leqq\frac{C}{n^{k-1/p}}\omega_p(u, f^{(k)}),$$

and in turn from (3.11),

$$\text{(3.23)}\qquad \sup_{|\lambda|\leqq\pi}|T_n(\lambda)|\leqq\frac{C}{n^{k-1/p}}\omega_p(1/n, f^{(k)}).$$

We can now prove (3.4) using (3.23): Since $\tau_m(\lambda)$ converges to $f(\lambda)$ almost all λ, we have

$$\text{(3.24)}\qquad \operatorname*{ess\,sup}_{|\lambda|\leqq\pi}|f(\lambda)-\tau_m(\lambda)|=\operatorname*{ess\,sup}_{|\lambda|\leqq\pi}|\lim_{n\to\infty}\tau_{2^n m}(\lambda)-\tau_m(\lambda)|$$
$$\leqq\operatorname*{ess\,sup}_{|\lambda|\leqq\pi}\sum_{l=0}^{\infty}|\tau_{2^{l+1}m}(\lambda)-\tau_{2^l m}(\lambda)|$$

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* C's are constants which do not depend on λ, n and u and may be different from place to place.

$$\leqq \sum_{l=0}^{\infty} \sup_{|\lambda|\leqq\pi} |T_{2^l m}(\lambda)|$$

$$\leqq \sum_{l=0}^{\infty} \frac{C}{(2^l m)^{k-1/p}} \omega_p(1/2^l m, f^{(k)}) .$$

The last series is rewritten in the following way

$$\frac{1}{(2n)^{k-1/p}} \omega_p(1/2n, f^{(k)}) \leqq 2 \sum_{\nu=n+1}^{2n} \omega_p(1/\nu, f^{(k)})/\nu^{k+1-1/p} , \tag{3.25}$$

and putting here $n=2^{j-1}m$, summing over $j=1, 2, 3, \cdots$ and adding to it the term for $j=0$, we have

$$\sum_{j=0}^{\infty} \frac{1}{(2^j m)^{k-1/p}} \omega_p\left(\frac{1}{2^j m}, f^{(k)}\right) \leqq 2 \sum_{\nu=m}^{\infty} \omega_p\left(\frac{1}{\nu}, f^{(k)}\right) \Big/ \nu^{k+1-1/p} . \tag{3.26}$$

Putting this together with (3.24) and using the fact that

$$\omega_p(\alpha\delta, \phi) \leqq (\alpha+1)\omega_p(\delta, \phi) , \tag{3.27}$$

for a real $\alpha \geqq 0$, we have (3.4).

Once (3.4) is obtained, we can easily get the conclusion of the theorem if we note the following fact:

$$\int_{-\pi}^{\pi} \theta(e^{i\lambda}) \tau_m(\lambda) e^{2mi\lambda} d\lambda = 0 \tag{3.28}$$

for any $\theta(e^{i\lambda}) \in H^1$. Actually this is obvious since the Fourier series of $\theta(e^{i\lambda})$ contains only the Fourier coefficient of nonegative exponents. Then from (1.8) and (3.4), we obtain

$$\rho(2m) = \sup_{\substack{\theta \in H \\ \|\theta\|_f^{(1)}=1}} \left| \int_{-\pi}^{\pi} \theta(e^{i\lambda}) f(\lambda) e^{2m\lambda i} d\lambda \right| \tag{3.29}$$

$$\leqq \sup \left| \int_{-\pi}^{\pi} \theta(e^{i\lambda})(f(\lambda) - \tau_m(\lambda)) e^{2m\lambda i} d\lambda \right|$$

$$\leqq C \operatorname*{ess\,sup}_{|\lambda|\leqq\pi} |f(\lambda) - \tau_m(\lambda)|$$

$$\leqq C \sum_{\nu=2m}^{\infty} \omega_p\left(\frac{1}{\nu}, f^{(k)}\right) \Big/ \nu^{k+1-1/p} .$$

Now for $t=2m+1$, we have using (3.27),

$$\rho(2m+1) \leqq \rho(2m)$$

$$\leqq C \sum_{\nu=2m}^{\infty} \omega_p(1/\nu, f^{(k)})/\nu^{k+1-1/p}$$

$$\leqq 2C \sum_{\nu=2m+1}^{\infty} \omega_p(1/\nu, f^{(k)})/\nu^{k+1-1/p}$$

which completes the proof.

4. Remark

We can prove the theorems similar to those in Section 2 in terms of $g^{(k)}(e^{i\lambda})$, the k-th derivative of the outer factor of the spectral density function $f(\lambda)$.

Theorem 3. *Suppose that*

(4.1) (i) $g(z)\in H^{\infty}$ *and* $|g(e^{i\lambda})|>c$ a.e.

for some constant $c>0$.

(4.2) (ii) $g^{(j)}(e^{i\lambda})\in\Lambda_1^p$ $(j=0,1,2,\cdots,k-1)$

for a positive integer k and hence
$g^{(k)}(e^{i\lambda})$ *exists* a.e. *and*

(4.3) $$g^{(k)}(e^{i\lambda})=(ie^{i\lambda})^k\lim_{r\to1-0}g^{(k)}(re^{i\lambda}),\quad \text{a.e.}$$

Then we have

(4.4) $$\rho(t;f)=O\Bigl(\sum_{u=t}^{\infty}u^{-k-1+1/p}\omega_p\Bigl(\frac{1}{u},\,g^{(k)}\Bigr)\Bigr)$$

for $1\leqq p$.

(4.4) is still true, when $k=0$, if (i) is satisfied and $t^{1/p}\omega_p(1/t,g)$ monotonically decreases to zero as $t\to\infty$, the condition (ii) being not needed. (For the meaning of $g^{(k)}(e^{i\lambda})$ and the justification of (4.3), see [1], p. 42, p. 70). For $p=\infty$, we also have

Theorem 4. *Suppose that*

(4.5) (i) *there is an* $h(\lambda)$ *such that* $g(e^{i\lambda})=h(\lambda)$ *for almost all* λ,
(ii) $|h(\lambda)|>c$

for some constant $c>0$ *and*

(iii) $h(\lambda)$ *has the continuous k-th derivative for some integer* $k=0,1,2,\cdots$.

Then we have

(4.6) $$\rho(t,f)=O\Bigl(\sum_{u=t}^{\infty}u^{-k-1}\omega(1/u,\,h^{(k)})\Bigr).$$

For the proof of these theorems, we put in place of (3.1) and (3.2),

(4.7) $$S_n(\lambda)=\sum_{k=0}^{n}g_ke^{ik\lambda},$$

(4.8) $$\sigma_n(\lambda)=\sum_{k=0}^{n}\left(1-\frac{k}{n+1}\right)g_k e^{ik\lambda},$$

where g_k is the Fouries coefficient of $g(e^{i\lambda})$,

(4.9) $$g_k=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\lambda})e^{-ik\lambda}d\lambda$$

and replace the second and the third lines of the equation (3.29) by

(4.10) $$\int_{-\pi}^{\pi}\theta(e^{i\lambda})f(\lambda)\overline{\left\{1-\frac{\tau_n(\lambda)}{g(e^{i\lambda})}\right\}}e^{2mi\lambda}d\lambda\leqq C\operatorname*{ess\,sup}_{|\lambda|\leqq\pi}|g(e^{i\lambda})-\tau_m(\lambda)|$$

and the other is almost the same as the proof of Theorem 1.

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