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ON THE MAXIMUM TERM AND RANK OF ANALYTIC FUNCTIONS AND THEIR DERIVATIVES

By

G. P. KAPOOR and K. GOPAL

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1. Introduction and notations

Let H_R, $0<R<\infty$, denote the class of functions $f(z)=\sum_{n=0}^{\infty}a_nz^n$ analytic in $D_R\equiv\{z: |z|<R\}$. The maximum term $\mu(r)$ and its rank $\nu(r)$, for a function f in H_R, are defined as $\mu(r)\equiv\mu(r,f)=\max_{n\geq 0}\{|a_n|r^n\}$ and $\nu(r)\equiv\nu(r,f)=\max\{n: \mu(r)=|a_n|r^n\}$. Let $\mu_k(r)\equiv\mu(r,f^{(k)})$ and $\nu_k(r)=\nu(r,f^{(k)})$, where $f^{(k)}$, $k=1,2,\cdots$, denotes the kth-derivative of f. We reckon $\nu_k(r)$, $k=0,1,2,\cdots$, from the first term of the series of f. For the uniformity in the notation we write $\mu_0(r)\equiv\mu(r)$ and $\nu_0(r)\equiv\nu(r)$. We denote the kth-derivateve of $\mu(r)$ by $\mu^{(k)}(r)$ at the point of its existence in $(0,R)$. For the difference $\nu_k(r)-\nu(r)$, we have the notation

$$\phi(r,k)\equiv\nu_k(r)-\nu(r)\,, \qquad k=1,2,\cdots. \tag{1.1}$$

We assume throughout this paper that $\mu(r)\to\infty$ as $r\to R$. It is easily seen that the functions $\mu_k(r)$ and $\nu_k(r)$, $k=1,2,\cdots$, are positive, non-decreasing and unbounded function of r in $(0,R)$, have only ordinary discontinuities and $\nu_k(r)\geq\nu(r)$.

For a function f in H_R we use the following definitions of the order ρ, the lower order λ, the type T and the lower type t:

$$\begin{matrix}\rho\\ \lambda\end{matrix}=\lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^+\log^+M(r)}{\log(R/(R-r))} \tag{1.2}$$

and

$$\begin{matrix}T\\ t\end{matrix}=\lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^+M(r)}{(R/(R-r))^{\rho}} \qquad (0<\rho<\infty) \tag{1.3}$$

where $M(r)\equiv M(r,f)=\max_{|z|=r}|(f(z)|$, $0<r<R$ and $\log^+x=\max(\log x,0)$.

A function f, in H_R, having order ρ and lower order λ is said to be of regular growth if $0\leqq\lambda=\rho<\infty$.

For a function f in H_1 and $0<\rho<\infty$, Sons [3, Lemma 2] has shown that

$$1+\rho=\limsup_{r\to 1}\frac{\log\nu(r)}{-\log(1-r)}\,. \tag{1.4}$$

For $0<\rho<\infty$, she also proved [3, p. 301] that

$$1+\lambda \geqq \liminf_{r\to 1} \frac{\log \nu(r)}{-\log(1-r)} . \tag{1.5}$$

However, the equality in (1.5) need not hold in general. For, consider the following example due to Paul V. Reichelderfer:

Let

$$F(z)=\sum_{i=0}^{\infty} \exp(k_i) z^{k_{i+1}}$$

where k_0 is any integer greater than one, and $k_{i+1}=k_i^2$. Then, it is easily seen that F is analytic in D_1 and has lower order 1/4 while

$$\liminf_{r\to 1} \frac{\log \nu(r)}{-\log(1-r)}=1 .$$

Thus, we are led to consider the functions in the class H_R^* consisting of functions in H_R and satisfying

$$1+\lambda=\liminf_{r\to R} \frac{\log \nu(r)}{\log(R/(R-r))} . \tag{1.6}$$

For the functions in H_R, in the present paper, we find a precise measure of the rates of growth of $\{\mu_k(r)/\mu(r)\}$, $\{\mu^{(k)}(r)/\mu(r)\}$ and $\phi(r,k)$ as $r\to R$ in terms of the parameters defined in (1.2) and (1.3). We observe that the growth formulae in terms of order, type and lower type are found for the whole class H_R while the growth formulae in terms of the lower order hold for the class H_R^*. Our results give necessary and sufficient conditions such that f in H_R^* is of regular growth. Some of our results include the results in [2] and [1].

2. Statements of results

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be in H_R and have order ρ. Then,*

$$1+\rho=\limsup_{r\to R} \frac{\log(r\{\mu_k(r)/\mu(r)\}^{1/k})}{\log(R/(R-r))}, \qquad k=1,2,\cdots . \tag{2.1}$$

Further, if f belongs to H_R^* and is of lower order λ, then

$$1+\lambda=\liminf_{r\to R} \frac{\log(r\{\mu_k(r)/\mu(r)\}^{1/k})}{\log(R/(R-r))}, \qquad k=1,2,\cdots . \tag{2.2}$$

Theorem 2. *Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be in H_R and have order ρ. Let $\phi(r,k)$ be defined by* (1.1). *Then, for $0<r_0<r<R$ and $k=1,2,\cdots$,*

$$1+\rho=\limsup_{r\to R}\left\{\frac{1}{k}\left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}\,dx\right\}. \tag{2.3}$$

Further, if f belongs to H_R^* and is of lower order λ, then, for $0<r_0<r<R$ and $k=1, 2, \cdots$,

$$1+\lambda=\liminf_{r\to R}\left\{\frac{1}{k}\left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}\,dx\right\}. \tag{2.4}$$

Corollary. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be in H_R^* and have regular growth and order ρ. Then, as $r\to R$,*

$$\int_{r_0}^{r}\frac{\phi(x,k)}{x}\,dx\sim\log\left(\frac{R}{R-r}\right)^{k(\rho+1)}, \qquad k=1,2,\cdots. \tag{2.5}$$

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be in H_R and have order ρ. Let $\phi(r,k)$ be defined by* (1.1). *Set*

$$\begin{matrix}\alpha_k\\ \beta_k\end{matrix}=\lim_{r\to R}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\phi(r,k)}{R/(R-r)}, \qquad k=1,2,\cdots. \tag{2.6}$$

Then,

$$k(\rho+1)\leqq\alpha_k. \tag{2.7}$$

Further, if f belongs to H_R^* and is of lower order λ, then

$$\beta_k\leqq k(\lambda+1). \tag{2.8}$$

Remark 1. If $\beta_k=\infty$, then $\lambda=\rho=\infty$.

Remark 2. If $\lim_{r\to R}\{\phi(r,k)/(R/(R-r))\}$ exists and is finite for a function f in H_R^*, then f is of regular growth and

$$k(\rho+1)=\lim_{r\to R}\left\{\frac{\phi(r,k)}{R/(R-r)}\right\}. \tag{2.9}$$

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be in H_R^* and have finite order ρ. Let $(R-r)\cdot\phi(r,k)$ be monotonic in $(0,R)$, where $\phi(r,k)$ is defined by* (1.1). *Then, for $k=1,2,\cdots$,*

(i) *$\phi(r,k)/(R/(R-r))$ is bounded in $(0,R)$;*

(ii) *f is of regular growth;*

and

(iii) $\lim_{r\to R}\{\phi(r,k)/(R/(R-r))\}=k(\rho+1)$.

Corollary. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be in* H_R^* *and have order* ρ, $1<\rho<\infty$. *Let* $(R-r)\phi(r,k)$ *be monotonic, where* $\phi(r,k)$ *is defined by* (1.1). *Then, for* $k=1,2,\cdots$, *as* $r\to R$

$$\mu_k(r)\sim\mu(r)\left(\frac{\nu(r)}{r}\right)^k . \tag{2.10}$$

Theorem 5. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be in* H_R *and have order* ρ. *Then, for* $k=1,2,\cdots$, *and for almost all values of* r *satisfying* $0<r_0\leq r<R$

$$r\frac{\mu^{(k)}(r)}{\mu^{(k-1)}(r)}+k-1=\nu(r) \tag{2.11}$$

$$1+\rho=\limsup_{r\to R(E')}\frac{\log(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k})}{\log(R/(R-r))} . \tag{2.12}$$

Further, if f belongs to H_R^*, *then*

$$1+\lambda=\liminf_{r\to R(E')}\frac{\log(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k})}{\log(R/(R-r))} \tag{2.13}$$

where $r\to R(E')$ *implies that* $r\to R$ *through values of* r *excluding a set of measure zero for which* $\mu^{(k)}(r)$ *does not exist.*

Corollary 1. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be in* H_R. *Then, for* $k=1,2,\cdots$, *as* $r\to R(E')$

$$\frac{\mu^{(k)}(r)}{\mu^{(k-1)}(r)}\sim\frac{\nu(r)}{r} \quad \text{and} \quad \frac{\mu^{(k)}(r)}{\mu(r)}\sim\left(\frac{\nu(r)}{r}\right)^k . \tag{2.14}$$

Corollary 2. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *be in* H_R^* *and have order* ρ. *Then, f is of regular growth if and only if*

$$1+\rho=\lim_{r\to R(E')}\frac{\log(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k})}{\log(R(R-r))}<\infty . \tag{2.15}$$

Theorem 6. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *belongs to* H_R^* *and have order* ρ $(0<\rho<\infty)$, *type* T *and lower type* t $(0<t\leqq T<\infty)$ *and let* $(R-r)\phi(r,k)$ *be monotonic where* $\phi(r,k)$ *is defined by* (1.1). *Then, at the points of existence of* $\mu^{(k)}(r)$, $k=1,2,\cdots$,

$$1/R\leqq\limsup_{r\to R}\frac{\{\mu^{(k)}(r)/\mu(r)\}^{1/k}}{\rho T(R/(R-r))^{\rho+1}}\leqq\left(\frac{\rho+1}{\rho}\right)^{\rho+1}\Big/R \tag{2.16}$$

$$1/R\leqq\limsup_{r\to R}\frac{\{\mu_k(r)/\mu(r)\}^{1/k}}{\rho T(R/(R-r))^{\rho+1}}\leqq\left(\frac{\rho+1}{\rho}\right)^{\rho+1}\Big/R \tag{2.17}$$

and

$$\liminf_{r\to R} \frac{\{\mu^{(k)}(r)/\mu(r)\}^{1/k}}{\rho t(R/(R-r))^{\rho+1}} \leqq \liminf \frac{\{\mu_k(r)/\mu(r)\}^{1/k}}{\rho t(R/(R-r))^{\rho+1}} \leqq 1/R \,. \tag{2.18}$$

3. Proof of theorems

Proof of Theorem 1. Since the order of a function and its derivative are the same, in view of (1.4), we have

$$1+\rho = \limsup_{r\to R} \frac{\log \nu_k(r)}{\log (R/(R-r))}\,, \qquad k=0,1,2,\cdots\,. \tag{3.1}$$

Now for $k=0,1,2,\cdots$, let $f^{(k)}(z)=\sum_{n=k}^{\infty} A_n z^n$, $\nu_k(r)=N$ and $\nu_{k+1}(r)=N_1$, then,

$$\begin{aligned}\mu_{k+1}(r) &= N_1 |A_{N_1}| r^{N_1-1}\\ &= \frac{N_1}{r} |A_{N_1}| r^{N_1} \leqq \frac{\nu_{k+1}(r)}{r} \mu_k(r)\,.\end{aligned}$$

This implies,

$$r\frac{\mu_{k+1}(r)}{\mu_k(r)} \leqq \nu_{k+1}(r)\,, \qquad k=0,1,2,\cdots\,. \tag{3.2}$$

Further, for $k=0,1,2,\cdots$,

$$\mu_k(r) = |A_N| r^N = \frac{r}{N} N |A_N| r^{N-1} \leqq \frac{r}{\nu_k(r)} \mu_{k+1}(r)\,.$$

It follows for $k=0,1,2,\cdots$, that

$$\nu_k(r) \leqq r\frac{\mu_{k+1}(r)}{\mu_k(r)}\,. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\nu_k(r) \leqq r\frac{\mu_{k+1}(r)}{\mu_k(r)} \leqq \mu_{k+1}(r)\,, \qquad k=0,1,2,\cdots\,.$$

The above inequality, after a simple calculation, yields

$$\nu(r) \leqq r\left\{\frac{\mu_k(r)}{\mu(r)}\right\}^{1/k} \leqq \nu_k(r)\,, \qquad k=0,1,2,\cdots\,. \tag{3.4}$$

Taking logarithm throughout, dividing by $\log(R/(R-r))$ and proceeding to limits as $r\to R$, (3.4) gives that for $k=1,2,\cdots$,

$$\begin{aligned}\lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu(r)}{\log (R/(R-r))} &\leqq \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log (r\{\mu_k(r)/\mu(r)\}^{1/k})}{\log (R/(R-r))}\\ &\leqq \lim_{r\to R} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \nu_k(r)}{\log (R/(R-r))}\,.\end{aligned} \tag{3.5}$$

The relations (3.1) and (3.5) lead to (2.1). Further, if the function f belongs to H_R^*, then (1.6) and (3.5) gives (2.2). This completes the proof of Theorem 1.

Proof of Theorem 2. We have, for $k=1, 2, \cdots$,

$$f^{(k)}(z)=\sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)a_n z^{-n-k}$$

and

$$\mu_k(r)=\nu_k(r)(\nu_k(r)-1) \cdots (\nu_k(r)-k+1)|a_{\nu_k}(r)|r^{\nu_k(r)-k} .$$

The functions $\nu_k(r)$ and $|a_{\nu_k(r)}|$ are constants in intervals, have at most an enumerable number of discontinuities and so their derivatives vanish almost every where except possibly at a set of measure zero. Taking logarithm of both the sides, differentiating with respect to r and denoting the derivative of $\mu_k(r)$ by $\mu_k'(r)$ at the point of its existence, we have for almost all values of r in (r_0, R).

$$\frac{\mu_k'(r)}{\mu_k(r)}=\frac{\nu(r)-k}{r}, \qquad k=1, 2, \cdots .$$

This implies, for $k=1, 2, \cdots$ and r sufficiently close to R,

$$\log \mu_k(r)=\int_{r_0}^{r} \frac{\nu_k(x)-k}{x}dx+O(1) . \tag{3.6}$$

Following Valiron [49, p. 201], for $0<r_0<r<R$, we have

$$\log \mu(r)=\log \mu(r_0)+\int_{r_0}^{r} \frac{\nu(x)}{x}dx . \tag{3.7}$$

Combining (3.6) and (3.7), we have for r sufficiently close to R,

$$\log \left(r\left\{\frac{\mu_k(r)}{\mu(r)}\right\}^{1/k}\right)=\frac{1}{k}\int_{r_0}^{r} \frac{\nu_k(x)-\nu(x)}{x}dx+O(1) . \tag{3.8}$$

In view of (2.1) and (2.2) the equation (3.8) yields (2.3) and (2.4) respectively. This completes the proof of Theorem 2.

Proof of Corollary to Theorem 2. Since the function f belonging to H_R is of regular growth and order ρ, by (2.3) and (2.4), we have for $0<r_0<r<R$ and $k=1, 2, \cdots$,

$$1+\rho=\lim_{r\to R} \left\{\frac{1}{k}\left(\log \frac{R}{R-r}\right)^{-1}\int_{r_0}^{r} \frac{\phi(x, k)}{x}dx\right\} . \tag{3.9}$$

Now, by (3.9), the corollary follows immediately.

Proof of Theorem 3. First, let $\alpha_k<\infty$ and $\beta_k>0$. By (2.6), for any $\varepsilon>0$

and $r>r_0$ $(0<r_0=r_0(\varepsilon)<r<R)$, we have

$$\frac{R}{R-r}(\beta_k-\varepsilon)<\phi(r,k)<(\alpha_k+\varepsilon)\frac{R}{R-r}\ .$$

The above inequality, after a simple transformation yields

$$\begin{aligned}(3.10)\qquad \frac{(\beta_k-\varepsilon)}{\log(R/(R-r))}\int_{r_0}^{r}\frac{R}{x(R-x)}dx&<\left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}dx\\ &<\frac{(\alpha_k+\varepsilon)}{\log(R/(R-r))}\int_{r_0}^{r}\frac{R}{x(R-x)}dx\ .\end{aligned}$$

Now, by Theorem 2 and (3.10), we have, for f in H_R, $k=1,2,\cdots$,

$$(3.11)\qquad k(\rho+1)=\limsup_{r\to R}\left\{\left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}dx\right\}\leqq\alpha_k\ .$$

The inequality (3.11) is obviously satisfied if $\alpha_k=\infty$. If, f belongs to H_R^*, then by Theorem 2 and (3.10), we have, for $k=1,2,\cdots$,

$$(3.12)\qquad \beta_k\leqq\liminf_{r\to R}\left\{\left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}dx\right\}=k(\lambda+1)\ .$$

This completes the proof of Theorem 3 since (3.11) and (1.12) are obvious if $\alpha_k=$ = or $\beta_k=0$.

Proof of Theorem 4. Since ρ is finite, by hypothesis of the theorem, using (3.12), we have for $k=1,2,\cdots$,

$$(3.13)\qquad \left(\log\frac{R}{R-r}\right)^{-1}\int_{r_0}^{r}\frac{\phi(x,k)}{x}dx=O(1)\qquad\text{as}\quad r\to R\ .$$

If possible, let $\phi(r,k)/(R/(R-r))$ be unbounded, then, since $(R-r)\phi(r,k)$ is monotonic and non-negative, we can find $r_0(k)$ such that

$$\frac{\phi(r,k)}{R/(R-r)}>k$$

for every r satisfying $0<r_0(k)<r<R$. Thus,

$$\int_{r_0}^{r}\frac{\phi(x,k)}{x}dx>k\left(\log\frac{r}{R-r}-\log\frac{r_0}{R-r_0}\right)\ .$$

Since, k can be made arbitrarily large, we obtain a contradiction of (3.13) Thus $\phi(r,k)/(R/(R-r))$ is bounded in $(0,R)$.

Further, as $(R-r)\phi(r,k)$ is monotonic and bounded, it must tend to a limit and so $\alpha_k=\beta_k<\infty$. Thus, in view of Theorem 3, it follows that f is of regular growth and $\lim_{r\to R}\{\phi(r,k)/(R/(R-r))\}=k(\rho+1)$. This completes the proof of Theorem 4.

Proof of the corollary to Theorem 4. By Theorem 4, $f(z)$ is of regular growth and

$$(R-r)\nu(r)[(\nu_k(r)/\nu(r))-1]\sim kR(\rho+1) \quad \text{as} \quad r\to R\,.$$

Following Sons [3, lemma 2], one easily gets

$$\rho=\lim_{r\to R}(\log\log\mu(r)/\log(R/(R-r))) \qquad \text{and} \qquad \lim_{r\to R}(\log\mu(r)/(R/(R-r)))=\infty$$

follows since $1<\rho<\infty$. Now, by (3.7), $\log\mu(r)<O(1)+\nu(r)(\log R-\log r_0)$ so that $\lim_{r\to R}(R-r)\nu(r)=\infty$. Thus, the above asymptotic relation gives $\nu_k(r)\sim\nu(r)$ as $r\to R$. The corollary now immediately follows from the inequalities (3.4).

Proof of Theorem 5. Since $\mu(r)=|a_{\nu(r)}|r^{\nu(r)}$ is differentiable everywhere except at an enumerable set of points of discontinuities of $|a_{\nu(r)}|$ and $\nu(r)$, we have, at the points of existence of $\mu^{(1)}(r)$,

$$r\frac{\mu^{(1)}(r)}{\mu(r)}=\nu(r)\,, \tag{3.14}$$

for the derivatives of $|a_{\nu(r)}|$ and $\nu(r)$ vanish almost everywhere. Differentiating (3.14) at the points of existence of $\mu^{(1)}(r)$ and $\mu^{(2)}(r)$, we get

$$\{r\mu^{(2)}(r)+\mu^{(1)}(r)\}\mu(r)-r\{\mu^{(1)}(r)\}^2=0\,.$$

This, on using (3.14), implies

$$r\frac{\mu^{(2)}(r)}{\mu^{(1)}(r)}=\nu(r)-1\,.$$

On repeating the differentiation j times, we get

$$r\frac{\mu^{(j)}(r)}{\mu^{(j-1)}(r)}=\nu(r)-j+1\,, \qquad j=1,2,\cdots\,. \tag{3.15}$$

This proves (2.11). Now, writing (3.15) for $j=1,2,\cdots,k$ and then multiplying the k-inequalities thus obtained gives

$$\begin{aligned} r^k\left\{\frac{\mu^{(k)}(r)}{\mu(r)}\right\}&=\nu(r)(\nu(r)-1)\cdots(\nu(r)-k+1) \\ &=(\nu(r))^k(1-o(1))\,. \end{aligned} \tag{3.16}$$

Thus,

$$\frac{\log\nu(r)}{\log(R/(R-r))}+o(1)=\frac{\log(r\{\mu^{(k)}(r)/\mu(r)\}^{1/k})}{\log(R/(R-r))}\,. \tag{3.17}$$

Now, if f belongs to H_R then (3.1) and (3.17) imply (2.12) and if f belongs to H_R^* then (1.5) and (3.17) imply (2.13), on proceeding to limits as $r \to R(E')$. This completes the proof of Theorem 5.

Proof of corollaries to Theorem 5. The asymptotic relations in (2.14) follows from (3.15) and (3.16). Further, if f is of regular growth then $\rho=\lambda$ and so (2.12) and (2.13) imply (2.15). If (2.15) holds, then (2.12) and (2.13) imply that f is of regular growth.

Proof of Theorem 6. First, we prove the inequalities on the left hand sides of (2.16) and (2.17). Since $\mu^{(k)}(r) \leqq \mu_k(r)$ for r satisfying $0<r_0<r<R$ and $k=0, 1, 2, \cdots$, it is sufficient to prove the inequality related to $\mu^{(k)}(r)$. Let, if possible, for r satisfying $0<r_0<r<R$ and a fixed positive ε,

$$\frac{\{\mu^{(k)}(r)/\mu(r)\}^{1/k}}{\rho T(R/(R-r))^{\rho+1}}<\frac{1}{R}-\varepsilon ,$$

then, on using (2.14), we have, for r sufficiently close to R,

$$\frac{\nu(r)}{r}<\left(\frac{1}{R}-\varepsilon\right)\rho T\left(\frac{R}{R-r}\right)^{\rho+1} .$$

The above inequality, on using (3.7), implies

$$\frac{\log \mu(r)}{(R/(R-r))^{\rho}}<\left(\frac{1}{R}-\varepsilon\right)TR .$$

This is inconsistent with the fact that $f(z)$ is of order ρ and type T. It proves the left hand side inequalities in (2.16) and (2.17).

Now, concerning the inequalities on the right hand side of (2.16) and (2.17), again by virtue of the relation $\mu^{(k)}(r) \leqq \mu_k(r)$, it is sufficient to prove the inequality relating to $\mu_k(r)$.

Proceeding on the lines of proof of [2, (1.11)], one can prove

$$\limsup_{r\to R} \frac{\nu(r)/r}{\rho T(R/(R-r))^{\rho+1}} \leqq \left(\frac{\rho+1}{\rho}\right)^{\rho+1} \Big/ R . \tag{3.18}$$

Using (2.14), the inequality (3.18) yields

$$\limsup_{r\to R} \frac{\{\mu_k(r)/\mu(r)\}^{1/k}}{\rho T(R/(R-r))^{\rho+1}} = \limsup_{r\to R} \frac{\nu(r)/r}{\rho T(R/(R-r))^{\rho+1}} \leqq \left(\frac{\rho+1}{\rho}\right)^{\rho+1} \Big/ R .$$

This completes the proof of (2.16) and (2.17). Again proceeding on the lines of proofs of [2, (1.8)], it is easily seen that

(3.19) $$\liminf_{r\to R} \frac{\nu(r)/r}{\rho t(R/(R-r))^{\rho+1}} \leqq 1/R \,.$$

Using (2.14), the inequality (3.19) gives (2.18).

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Department of Mathematics
Indian Institute of Technology
Kanpur 208016
India
and
Department of Mathematics
V. V. Post Graduate College
Shamli 247776
India