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REDUCTION OF THE CODIMENSION FOR SPECIAL SUBMANIFOLDS OF A SPACE FORM $R^m(k)$

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1. Introduction

In this paper we give some theorems in connection with the reduction of the codimension of submanifolds of a complete simply-connected space form $R^m(k)$. Sect'on 3 treats about surfaces which have asymptotic lines. In section 4 we look for submanifolds (and especially surfaces) for which the first normal space is 1-dimensional at each point. Finally, in section 5 we prove the main theorem: if a connected n-dimensional submanifold N of $R^m(k)$ has two independent asymptotic distributions, which are respectively $n-1$ and 1-dimensional, and if the first normal space of N has everywhere constant dimension n_1 $(0\leqq n_1\leqq n-1)$ then N is contained in an $(n+n_1)$-dimensional totally geodesic subspace of $R^m(k)$.

A few special submanifolds are treated in this connection. Some of the propositions and corollaries (and especially the main theorem) are extensions of results which we gave in earlier papers ([4], [5], [6] and [7]).

2. Preliminaries

We shall assume throughout that all manifolds, maps, vector fields, etc.... are differentiable of class C^∞.

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold R^m. If $\bar{D}$ (resp. D) is the Riemannian connection of R^m (resp. N) and if X and Y are tangent vector fields of N, then we find by decomposing $\bar{D}_XY$ into a tangent and a normal component

$$\bar{D}_XY=D_XY+V(X, Y)\ .$$

V is a 2-covariant symmetric vector-valued tensor field on N. A vector Z of N at the point p is called an asymptotic vector (and then determines an asymptotic direction at p if $Z\neq 0$) if $V(Z, Z)=0$. A line on N is an asymptotic line if the tangent vector T of the curve is at each point an asymptotic vector of N. Remark that since we have in this case $\bar{D}_TT=D_TT$, the first principal normal vector of an asymptotic line is at each point a vector of the tangent space of N.

A distribution δ of N is said to be asymptotic if at each point p of N each vector of δ_p is an asymptotic vector of N. Suppose again that X is a vector field of N. If ξ is a normal vector field on N, then we find the Weingarten equation by decomposing $\bar{D}_X\xi$ into a tangent and a normal component:

$$\bar{D}_X\xi = -A_\xi(X) + D_X^\perp\xi .$$

$A_\xi(X)$ is at a point p of N only dependent on ξ_p and X_p, and A_ξ determines at p a self-adjoint linear map $N_p \to N_p$. $D^\perp$ is a metric connection in the normal bundle $N^\perp$.

We also have, if $\langle\ ,\ \rangle$ denotes the metric tensor of R^m (we use the same notation for the induced metric on N) and if X and Y are N-vector fields

$$\langle A_\xi(X), Y\rangle = \langle V(X, Y), \xi\rangle . \tag{2.1}$$

A normal vector field ξ on N is called parallel in the normal bundle $N^\perp$ if $D_X^\perp\xi = 0$ for each N-vector field X. A subbundle F of $N^\perp$ is said to be parallel in $N^\perp$ if for each vector field η of F and each vector field X of N, $D_X^\perp\eta$ is again a vector field of F.

The first normal space N_1 of N is by definition the subbundle of $N^\perp$, at each point p given by the orthogonal complement in $N_p^\perp$ of

$$\{\sigma \in N_p^\perp \,\|\, A_\sigma = 0\} .$$

The curvature tensor $K^\perp$ of the connection $D^\perp$ is given by

$$K^\perp(X, Y) = D_X^\perp D_Y^\perp - D_Y^\perp D_X^\perp - D_{[X,Y]}^\perp .$$

If $K^\perp$ vanishes identically, then N is said to have flat normal connection.

Suppose that $e_1, \cdots, e_n$ is an orthonormal base field of N and that $\xi_1, \cdots, \xi_{m-n}$ is a normal orthonormal base field on N in R^m, then the mean curvature vector field H of N is in the intersection of the domains of these base fields given by

$$H = \frac{1}{n}\sum_{i=1}^{n} V(e_i, e_i) = \frac{1}{n}\sum_{j=1}^{m-n}(\mathrm{sp}\, A_{\xi_j})\xi_j . \tag{2.2}$$

If $H=0$ at each point of N, then N is called minimal. If for each normal vector field ξ the map A_ξ is at each point proportional to the identity transformation, then N is said to be totally umbilical.

Suppose that ξ_p is a unit normal vector on N, then det A_{ξ_p} is the Lipschitz-Killing curvature of N in R^m in the direction of ξ_p.

From now on we assume that N is a submanifold of a complete simply-connected space form $R^m(k)$ with constant curvature k ($k>0$, $k=0$ or $k<0$). It is well known that in this case the normal connection of N is flat if and only if all the second fundamental tensors A_ξ are simultaneously diagonalizable (or for every normal orthonormal base field $\xi_1, \cdots, \xi_{m-n}$ on N, the linear maps A_{ξ_i} must pairwise commute at each point of the domain of the base field).

If X and Y are unit orthogonal vectors of N_p, then, from the Gauss equation, we know that the Riemanncurvature $K(X, Y)$ of N in the two-dimensional direction of N_p spanned by X and Y, is given by

$$K(X, Y)=k-\langle V(X, Y), V(X, Y)\rangle+\langle V(X, X), V(Y, Y)\rangle . \tag{2.3}$$

3. Surfaces with asymptotic directions in a space form $R^m(k)$

From now on we consider only submanifolds (and in this section especially surfaces) which are minimal ($H=0$ at each point of N) or everywhere not minimal ($H\neq 0$ at each point of N). Moreover, from now on we consider only submanifolds for which the first normal space N_1 has at each point constant dimension.

Theorem 1. *Suppose that N is a connected surface in $R^m(k)$. If N has at each point two different asymptotic directions, then N is a surface in a 3-dimendional totally geodestic submanifold $R^3(k)$ of $R^m(k)$.*

Proof. Take an orthonormal base field e_1, e_2 of N such that e_1 determines at each point of the domain of the field an asymptotic direction of N at that point. Then we have $V(e_1, e_1)=0$. Suppose that $x=ae_1+be_2$ is the vector field of N which gives at each point the other asymptotic direction of N at that point, then $V(x, x)=a^2V(e_1, e_1)+2abV(e_1, e_2)+b^2V(e_2, e_2)=0$ or, since $b\neq 0$, because the asymptotic directions are different at each point, $2aV(e_1, e_2)+bV(e_2, e_2)=0$. This means that $V(e_1, e_2)$ and $V(e_2, e_2)$ are linearly dependent.

There are three cases:

(1) If $V(e_1, e_2)=0$ at each point, then we must have that also $V(e_2, e_2)=0$ at each point, which gives that N is totally geodesic in $R^m(k)$.

(2) Suppose that $V(e_1, e_2)\neq 0$ and $V(e_2, e_2)=0$ at each point.

Then N is a minimal surface of $R^m(k)$, with at each point two orthogonal asymptotic directions. It is clear that the first normal space N_1 of N is spanned by $V(e_1, e_2)$. So, N_1 is everywhere one-dimensional.

This means that N has flat normal connection and it is known that a connected minimal surface with flat normal connection in a space form $R^m(k)$ is contained in a 3-dimensional totally geodesic submanifold of $R^m(k)$ (e.g. [1], p. 115 Remark 2.1.).

(3) Suppose that $V(e_1, e_2)\neq 0$ and $V(e_2, e_2)\neq 0$ at each point.

Then N is not minimal. We must now have that $V(e_1, e_2)//V(e_2, e_2)$. So, the first normal space N_1 of N is again one-dimensional. It follows that N has flat normal connection. Thus we can choose an orthonormal base field e_1', e_2' of N such that $V(e_1', e_2')=0$ at each point. Take an orthonormal base field $\xi_1, \cdots, \xi_{m-2}$ of $N^\perp$ such that ξ_1 spans at each point the first normal space. If we set $V(X, Y)=\sum_{i=1}^{m-2} V^i(X, Y)\xi_i$ for each two N-vector fields X and Y, then we find

$$V^2(X, Y)=\cdots=V^{m-2}(X, Y)=0\,. \tag{3.1}$$

Since $\det A_{\xi_1}\neq 0$ (because $V(e_1, e_2)\neq 0$) at each point, we have that $V^1(e_1', e_1')V^1(e_2', e_2')\neq 0$ and hence $V^1(e_1', e_1')\neq 0$ and $V^1(e_2', e_2')\neq 0$ at each point.

If $\bar{R}$ is the curvature tensor of $R^m(k)$ and if X, Y, Z are N-vector fields, then the Codazzi equation says

$$\begin{aligned}(\bar{R}(X, Y)Z)^\perp=&\sum_{i=1}^{m-2}\{(D_X V^i)(Y, Z)-(D_Y V^i)(X, Z)\}\xi_i\\ &+\sum_{i=1}^{m-2} V^i(Y, Z)D_X{}^\perp\xi_i-\sum_{i=1}^{m-2} V^i(X, Z)D_Y{}^\perp\xi_i=0\,.\end{aligned}$$

From (3.1), we have that

$$\begin{aligned}(\bar{R}(e_i', e_j')e_i')^\perp=&\{D_{e_i'}V^1)(e_j', e_i')-(D_{e_j'}V^1)(e_i', e_i')\}\xi_1\\ &+V^1(e_j', e_i')D^\perp_{e_i'}\xi_1-V^1(e_i', e_i')D^\perp_{e_j'}\xi_1=0 \qquad i, j=1, 2 \quad i\neq j\,.\end{aligned}$$

But $V^1(e_j', e_i')=0$ $i\neq j$, while $D^\perp_{e_j'}\xi_1\perp\xi_1$ and $V^1(e_i', e_i')\neq 0$.

So we find

$$D^\perp_{e_j'}\xi_1=0 \qquad j=1, 2, .$$

This tells us that $D_X{}^\perp\xi_1=0$ for each vector field X of N, which means that ξ_1 and thus also the first normal space N_1 (which is one-dimensional) of N is parallel in the normal bundle $N^\perp$. This completes the proof.

From part (2) of the proof of theorem 1, we have at once

Corollary 1. *If N is a connected minimal surface in $R^m(k)$, with at each point an asymptotic direction, then N is a minimal surface in an $R^3(k)$.*

From this corollary we find at once the known result that a minimal ruled surface in the euclidiean space E^m is necessarily a surface in an E^3 (thus a part of a plane or an helicoid; see also [4]).

Corollary 2. *If the connected surface N of $R^m(k)$ has at each point an asymptotic direction and if the Lipschitz-Killing curvature of N is at each point non-zero minimal in the direction of the mean curvature vector $H \neq 0$ of N, then N is a surface in some 3-dimensional totally geodesic submanifold of $R^m(k)$.*

Proof. Take an orthonormal base field e_1, e_2 of N such that e_1 determines at each point of the domain of the field the given asymptotic direction of N, i.e. $V(e_1, e_1)=0$. Since the Lipschitz-Killing curvature is not for every normal zero, we have because of (2.1) that $V(e_1, e_2) \neq 0$. Now it is clear that if $\xi_1, \cdots, \xi_{m-2}$ is an orthonormal base field of $N^\perp$ such that $\xi_1 // V(e_1, e_2)$, then $\det A_{\xi_1}<0$ and $\det A_{\xi_r}=0$ $r=2, \cdots, m-2$. From this we get $H // V(e_1, e_2)$ or, since $2H=V(e_1, e_1)+V(e_2, e_2)$, we find $V(e_2, e_2) // V(e_1, e_2)$, which gives us the situation of case (3) of the proof of Theorem 1.

Remark. The condition that the Lipschitz-Killing curvature is non-zero is necessary: a non-trivial developable (i.e. flat) ruled surface in E^m has for every normal at each point Lipschitz-Killing curvature zero and can have arbitrary codimension.

Consider on the surface N of $R^m(k)$ a curve σ with unit tangent vector field T. Recall that if σ is a geodesic line of $R^m(k)$, then $\bar{D}_T T=D_T T+V(T, T)=0$ and hence $V(T, T)=0$, which means that σ is an asymptotic line of N. So we have from theorem 1.

Corollary 3. *If N is a connected surface of $R^m(k)$ such that at each point of N there are two (different) geodesic lines of $R^m(k)$ through p on N, then N is necessarily a surface in a 3-dimensional totally geodesic submanifold of $R^m(k)$.*

If $k=0$, then it is well-known that the one-sheet hyperboloids and the hyperbolic paraboloids are the only non-trivial surfaces in E^m with two different families of straight lines on it.

4. Submanifolds in $R^m(k)$ with one-dimensional first normal space

Theorem 2. *Suppose that N is an n-dimensional connected submanifold of $R^m(k)$ $(n \geqq 2)$, for which the first normal space N_1 is everywhere one-dimensional. If moreover the Lipschitz-Killing curvature of N in the direction of N_1 is never zero, then N is contained in an $(n+1)$-dimensional totally geodesic subspace of $R^m(k)$.*

Proof. In the same way as part (3) of the proof of theorem 1, we have here: since the first normal space is one-dimensional, N has flat normal connection and so it is possible to choose an orthonormal base field $e_1, \cdots, e_n$ of N such that $V(e_i, e_j)=0$ $i\neq j$ $i, j=1, \cdots, n$. Take an orthonormal base field $\xi_1, \cdots, \xi_{m-n}$ of $N_p^\perp$ such that $\xi_1 // N_1$. Then we find now

$$V^2(X, Y)=\cdots=V^{m-n}(X, Y)=0 \tag{4.1}$$

for each two N-vector fields X and Y. Since the Lipschitz-Killing curvature of N in the direction of N_1 is not zero, we have

$$\det A_{\xi_1}=V^1(e_1, e_1)\cdots V^1(e_n, e_n)\neq 0$$

at each point, and so

$$V^1(e_1, e_1)\neq 0, \cdots, V^1(e_n, e_n)\neq 0 .$$

If $\bar{R}$ is the curvature tensor of $R^m(k)$, we find from the Codazzi equation and from (4.1)

$$\begin{aligned}(\bar{R}(e_i, e_j)e_i)^\perp=&\{(D_{e_i}V^1)(e_j, e_i)-(D_{e_j}V^1)(e_i, e_i)\}\xi_1\\ &+V^1(e_j, e_i)D^\perp_{e_i}\xi_1-V^1(e_i, e_i)D^\perp_{e_j}\xi_1=0 ,\end{aligned}$$

which gives for $i, j=1, \cdots, n$ $i\neq j$; since $V^1(e_j, e_i)=0$, $V^1(e_i, e_i)\neq 0$ and $D^\perp_{e_j}\xi_1\perp\xi_1$, that $D^\perp_{e_j}\xi_1=0$ $j=1, \cdots, n$. Thus ξ_1 and hence also the first normal space N_1 is parallel in the normal bundle $N^\perp$. This completes the proof.

Suppose that N is a totally umbilical submanifold of $R^m(k)$ and that $\xi_1, \cdots, \xi_{m-n}$ is an orthonormal base field of $N^\perp$. Then there exist functions λ_i such that $A_{\xi_i}=\lambda_i I$ $i=1, \cdots, m-n$ where I is the identity transformation. So, from (2.1) and (2.2), we get

$$\begin{aligned}V(X, Y)&=\sum_{i=1}^{m-n}\langle V(X, Y), \xi_i\rangle\xi_i=\langle X, Y\rangle\sum_{i=1}^{m-n}\lambda_i\xi_i\\ &=\frac{1}{n}\langle X, Y\rangle\sum_{i=1}^{m-n}(\operatorname{tr} A_{\xi_i})\xi_i\end{aligned}$$

or

$$V(X, Y)=\langle X, Y\rangle H .$$

This means that the first normal space is at most one-dimensional. If $H=0$, then N is totally geodesic and if $H\neq 0$, then we find from theorem 2 the well-known result that an n-dimensional totally umbilical submanifold is contained in an $(n+1)$-dimensional totally geodesic subspace of $R^m(k)$ (e.g. see [1], proposition 3.2).

Remark that theorem 2 is in fact an extension of theorem 1, because the assumptions of theorem 2 follow from these of theorem 1 in the case $n=2$ (and N not totally geodesic). Here again the non-zero condition for the Lipschitz-Killing curvature is necessary: a not totally geodesic monosystem (i.e. a submanifold generated by a one-parameter family of linear spaces) in the euclidean space E^m, which is total developable (i.e. which is flat) has $\dim N_1=1$, Lipschitz-Killing curvature zero everywhere, and can have arbitrary codimension (e.g. the osculating planes of any curve in E^m $(m>3)$ generate such (3-dimensional) total developable monosystem).

Corollary 4. *If a connected surface N of $R^m(k)$ has everywhere non-zero Lipschitz-Killing curvature in the direction of the mean curvature vector $H\neq 0$ and zero Lipschitz-Killing curvature in each normal direction orthogonal to H, then N is contained in an $R^3(k)$.*

Proof. Take an orthonormal base field $\xi_1, \cdots, \xi_{m-2}$ of $N^\perp$ such that $\xi_1 // H$. Then $\det A_{\xi_1}\neq 0$, while $\det A_{\xi_i}=0$ $i=2, \cdots, m-2$. But $\operatorname{tr} A_{\xi_i}=0$ and so $A_{\xi_i}=0$ $i=2, \cdots, m-2$. We thus have $\dim N_1=1$ and theorem 2 gives our result.

Remark. If a surface N in $R^m(k)$ has flat normal connection and at each point an asymptotic direction, then it is clear that $\dim N_1\leqq 1$ at each point. Conversely, if $\dim N_1=1$ at each point, then N has flat normal connection and if we choose an orthonormal base field e_1, e_2 such that $V(e_1, e_2)=0$, we find from (2.3) for the Gauss curvature G

$$G=k+\langle V(e_1, e_1), V(e_2, e_2)\rangle.$$

But $V(e_1, e_1) // V(e_2, e_2)$ and so we have an asymptotic direction only in the case $G\leqq k$ at each point of N. If always $G<k$, then there are at each point two distinct asymptotic directions and thus N is a surface in an $R^3(k)$. In fact we have more:

Corollary 5. *If a connected n-dimensional submanifold of $R^m(k)$ has at each point* $\dim N_1=1$ *and nowhere Riemann curvature k, then N is a hypersurface in an $R^{n+1}(k)$. Especially, if a connected surface N in $R^m(k)$ has everywhere $G\neq k$ and* $\dim N_1=1$, *then N is contained in an $R^3(k)$.*

Proof. Use the same base field $e_1, \cdots, e_n$ as in the proof of theorem 2. Since we have nowhere Riemann curvature k, we see from (2.3) that the normal fields $V(e_1, e_1), \cdots, V(e_n, e_n)$ are never zero. Thus the Lipschitz-Killing curvature of N in the direction of N_1 is always different from zero and theorem 2 completes the proof.

Remarks.

1. If a connected surface N in $R^m(k)$ has everywhere $\dim N_1=1$ and lies not entirely in an $R_3(k)$, then there are points on it where $G=k$.
2. A not totaly geodesic surface in $R^m(k)$ with constant curvature k and with flat normal connection can have arbitrary codimension, even if $\dim N_1=1$: e.g. a not totally geodesic developable ruled surface in E^m.

5. n-dimensional submanifolds in $R^m(k)$ with an $(n-1)$-dimensional asymptotic distribution

Lemma. *Suppose that δ is a l-dimensional asymptotic distribution of the submanifold N, then we have $V(X, Y)=0$ for each two vectors X and Y of δ_p at each point p of N.*

Proof. Take an orthonormal base field $e_1, \cdots, e_n$ of N such that δ is in the domain of the field determined by $e_1, \cdots, e_l$. Since V is linear in his arguments, it is sufficient to prove that $V(e_i, e_j)=0$ $i, j=1, \cdots, l$ $i\neq j$ and this follows from $O=V(e_i+e_j, e_i+e_j)=V(e_i, e_i)+V(e_i, e_j)+V(e_j, e_i)+V(e_j, e_j)=2V(e_i, e_j)$.

One can extend theorem 1 for n-dimensional submanifolds in the following way.

Theorem 3. *If the n-dimensional connected submanifold N of $R^m(k)$ $(n\geqq 2)$ has two asymptotic distributions δ and σ, respectively $n-1$ and 1-dimensional, which are independent (i.e. $\delta_p\oplus\sigma_p=N_p$, $\forall p\in N$) and if the first normal space N_1 of N has everywhere constant dimension n_1 $(0\leqq n_1\leqq n-1)$, then N is contained in an $(n+n_1)$-dimensional totally geodesic subspace $R^{n+n_1}(k)$ of $R^m(k)$.*

Proof. If we choose a base field $e_1', \cdots, e_n'$ of N such that $e_1', \cdots, e_{n-1}'$ determine at each point of the field the distribution δ and such that e_n' determines σ, we see at once from the lemma that the first normal space N_1 of N is spanned by $V(e_1', e_n'), \cdots, V(e_{n-1}', e_n')$ and hence is at most $(n-1)$-dimensional.

If N_1 is at each point 0-dimensional, then N is totally geodesic in $R^m(k)$. Suppose that $\dim N_1=n_1\neq 0$ $(n_1\leqq n-1)$ at each point of N. Take an orthonormal base field $e_1, \cdots, e_n$ such that $e_1, \cdots, e_{n-1}$ determine at each point of the domain of the field the distribution δ. There are two cases:

(1) If $V(e_n, e_n)=0$ at each point, then we may assume that e_n determines the distribution σ (remark that σ is not necessarily unique; see also theorem 4) and moreover N is minimal.

(2) If $V(e_n, e_n)\neq 0$ then there exist functions $\lambda_1, \cdots, \lambda_n$ such that $V(\lambda_1 e_1+$

$\cdots+\lambda_n e_n, \lambda_1 e_1+\cdots+\lambda_n e_n)=\sum_{i=1}^{n-1} 2\lambda_i\lambda_n V(e_i, e_n)+\lambda_n^2 V(e_n, e_n)=0$, with $\lambda_n\neq 0$ at each point because δ and σ are independent. Thus

$$\sum_{i=1}^{n-1} 2\lambda_i V(e_i, e_n)+\lambda_n V(e_n, e_n)=0 \quad \lambda_n\neq 0 ,$$

and this means that $V(e_n, e_n)$ is a linear combination of $V(e_1, e_n), \cdots, V(e_{n-1}, e_n)$. So, in both cases we have that the first normal space N_1 is spanned by $V(e_1, e_n), \cdots, V(e_{n-1}, e_n)$.

Consider an orthonormal base field $\xi_1, \cdots, \xi_{m-n}$ of $N^\perp$ such that $\xi_1, \cdots, \xi_{n_1}$ is a base field of the normal subbundle N_1. If we set $V(X, Y)=\sum_{t=1}^{m-n} V^t(X, Y)\xi_t$ for each two N-vector fields X and Y, then it is clear that

$$V^{n_1+1}(X, Y)=\cdots=V^{m-n}(X, Y)=0 \text{ at each point .} \tag{4.1}$$

Suppose that $\bar{R}$ is the curvature tensor of $R^m(k)$, then we have for three N-vector fields X, Y, Z the Codazzi equation

$$(R(X, Y)Z)^\perp=\sum_{t=1}^{m-n}\{(D_X V^t)(Y, Z)-(D_Y V^t)(X, Z)\}\xi_t + \sum_{t=1}^{m-n} V^t(Y, Z)D_X{}^\perp\xi_t-\sum_{t=1}^{m-n} V^t(X, Z)D_Y{}^\perp\xi_t=0 . \tag{4.2}$$

In the intersection of the domains of the fields $e_1, \cdots, e_n$ and $\xi_1, \cdots, \xi_{m-n}$, we put

$$D^\perp_{e_r}\xi_s=\sum_{h=1}^{n_1} B^h_{rs}\xi_h+\sum_{q=n_1+1}^{m-n} B^q_{rs}\xi_q \qquad r=1, \cdots, n ; \quad s=1, \cdots, n_1 . \tag{4.3}$$

From (4. 1) and (4. 2) we have

$$(\bar{R}(e_i, e_n)e_j)^\perp=\sum_{s=1}^{n_1}\{\cdots\}\xi_s+\sum_{s=1}^{n_1} V^s(e_n, e_j)D^\perp_{e_i}\xi_s - \sum_{s=1}^{n_1} V^s(e_i, e_j)D^\perp_{e_n}\xi_s=0 \qquad i, j=1, \cdots, n-1 . \tag{4.4}$$

But $V(e_i, e_j)=0$ $i, j=1, \cdots, n-1$ and so we find from (4. 3) and (4. 4)

$$\sum_{s=1}^{n_1} V^s(e_n, e_j)B^q_{is}=0 \qquad i, j=1, \cdots, n-1 ; \quad q=n_1+1, \cdots, m-n . \tag{4.5}$$

For each fixed i $(i=1, \cdots, n-1)$ and q $(q=n_1+1, \cdots, m-n)$, (4. 5) gives for $j=1, \cdots, n-1$ a system of $n-1$ homogeneous linear equations with n_1 unknowns B^q_{is} $(s=1, \cdots, n_1)$. But, since the first normal space N_1 is spanned by $V(e_1, e_n), \cdots, V(e_{n-1}, e_n)$, the rank of the matrix

$$[V^s(e_n, e_j)]_{\substack{s=1,\dots,n_1\\ j=1,\dots,n-1}}$$

is at each point equal to n_1. So, from (4.5) we find at once

(4.6) $\qquad B^q_{is}=0 \qquad i=1,\cdots,n-1\,;\quad q=n_1+1,\cdots,m-n\,;\quad s=1,\cdots,n_1\,.$

We also have

$$(\bar{R}(e_n, e_i)e_n)^{\perp}=\sum_{s=1}^{n_1}\{\cdots\}\xi_s+\sum_{s=1}^{n_1}V^s(e_i, e_n)D^{\perp}_{e_n}\xi_s$$
$$-\sum_{s=1}^{n_1}V^s(e_n, e_n)D^{\perp}_{e_i}\xi_s=0 \qquad i=1,\cdots,n-1\,.$$

But because of (4.6) we see that $D^{\perp}_{e_i}\xi_s$ $i=1,\cdots,n-1,$ $s=1,\cdots,n_1$ has no component in the complementary subbundle $N_1{}^{\perp}$ of N_1 in $N^{\perp}$. So, from the last equation we find

$$\sum_{s=1}^{n_1}V^s(e_i, e_n)B^q_{ns}=0 \qquad i=1,\cdots,n-1\,;\quad q=n_1+1,\cdots,m-n\,.$$

This gives in the same way

(4.7) $\qquad B^q_{ns}=0 \qquad q=n+1,\cdots,m-n\,;\quad s=1,\cdots,n_1\,.$

From (4.3), (4.6) and (4.7) we see that the first normal space is parallel in the normal bundle $N^{\perp}$ and this completes the proof.

Corollary 6. *If a connected minimal n-dimensional submanifold N of $R^m(k)$ $(n\geqq 2)$ has an $(n-1)$-dimensional asymptotic distribution and if the first normal space N_1 has constant dimension n_1, $(0\leqq n_1\leqq n-1)$, then N is a submanifold of an $R^{n+n_1}(k)$.*

Proof. Since N is minimal, we must have, if we use the same base field $e_1,\cdots,e_n$ as in the proof of theorem 3, $V(e_n, e_n)=0$ and this gives us the situation of case (1) of the proof of theorem 3.

We can say more about the codimension than in the statement of theorem 3. In fact we have:

Theorem 4. *If the connected n-dimensional submanifold N of $R^m(k)$ $(n\geqq 2)$, with constant dimensional first normal space N_1, has an $(n-1)$-dimensional asymptotic distribution δ and an l-dimensional asymptotic distribution σ $(l\geqq 1)$, which span the tangent spaces (i.e. $\delta_p+\sigma_p=N_p$, $\forall p\in N$), then N is contained in a $(2n-l)$-dimensional totally geodesic subspace of $R^m(k)$. Especially, if N contains two (at each point) different $(n-1)$-dimensional asymptotic distributions, then N is a hypersurface in an $R^{n+1}(k)$.*

Proof. Suppose that $e_1, \cdots, e_n$ is a (not necessarily orthonormal) base field such that $e_1, \cdots, e_{n-1}$ determine δ at each point of the domain of the base field and that e_n gives an asymptotic vector of σ (which do not belong to δ since $e_1, \cdots, e_n$ are linearly independent) at each point. If there is an other asymptotic vector field given by $e=\sum_{i=1}^{n}\lambda_i e_i$ (not belonging to δ and different from e_n at each point), then we must have, since $\lambda_n \neq 0$

$$\lambda_1 V(e_1, e_n)+\cdots+\lambda_{n-1}V(e_{n-1}, e_n)=0\,. \tag{4.8}$$

Remark that if such field e exists, then each vector of the vectorplanes $\alpha e+\beta e_n$, α, $\beta\in R$ is an asymptotic vector. If $\xi_1, \cdots, \xi_{m-n}$ is an orthonormal base field of $N^{\perp}$ such that $\xi_1, \cdots, \xi_{n_1}$ span N_1 at each point and if we set again $V(X, Y)=\sum_{t=1}^{m-n}V^t(X, Y)\xi_t$, then we find from (4.8)

$$\sum_{j=1}^{n-1}\lambda_j\sum_{s=1}^{n_1}V^s(e_j, e_n)\xi_s=0$$

or

$$\sum_{j=1}^{n-1}\lambda_j V^s(e_j, e_n)=0 \qquad s=1, \cdots, n_1\,.$$

This gives us a system of n_1 homogeneous linear equations with $n-1$ unknowns $\lambda_1, \cdots, \lambda_{n-1}$. Moreover, since the first normal space is spanned by $V(e_1, e_n), \cdots, V(e_{n-1}, e_n)$, we have that

$$\operatorname{rank}\,[V^s(e_j, e_n)]_{\substack{s=1,\dots,n_1\\ j=1,\dots,n-1}}=n_1 \quad \text{at each point.}$$

From all this we get (if we assume that σ is not a subdistribution of a higher-dimensional asymptotic distribution)

$$\dim\sigma=l=n-n_1 \qquad \text{or} \qquad \dim\sigma+\dim N_1=n \quad 0\leqq n_1\leqq n-1\,. \tag{4.9}$$

(Remark: consider at each point p the linear transformation f: $N_p\to N_p{}^{\perp}$; $x\to V(x, (e_n)_p)$. The second equality of (4.9) is in fact nothing else than the relation: nullity+rank$=\dim N_p=n$ for f at each point of N).

Theorem 3 together with (4.9) completes the proof.

Remark. If under the assumption of theorem 3 (or 4), N has constant curvature and $n>2$, then N is totally geodesic in $R^m(k)$: to see this, use the same base field $e_1, \cdots, e_n$ as in the proof of theorem 3, then, because of (2.3), we get at once that N must have constant curvature k and that $V(e_1, e_n)=\cdots=V(e_{n-1}, e_n)=0$. Since there is at each point p an asymptotic vector which is not a linear combination of $(e_1)_p, \cdots, (e_{n-1})_p$, we find immediately that $V(e_i, e_j)=0$ $i, j=1, \cdots, n$ and hence N is totally geodesic.

Example. Suppose that the n-dimensional submanifold N ($n \geqq 3$) of $R^m(k)$ is foliated by $(n-1)$-dimensional totally geodesic submanifolds of $R^m(k)$. If $\bar{D}$, D and D' are respectively the Riemannian connections of $R^m(k)$, N and the leave B_p through the point p of N and if X and Y are vector fields which are tangent to B_p, then we get

$$\bar{D}_X Y = D_X Y + V(X, Y) \quad \text{and} \quad \bar{D}_X Y = D'_X Y$$

and so we have $V(X, Y) = 0$. This means that the tangent spaces of the leaves of such totally geodesic foliation determine an $(n-1)$-dimensional asymptotic distribution on N and thus we can apply theorems 3 and 4. Especially corollary 6 gives a strong reduction of the codimension in the minimal case (see also [5], [6] and [8]).

We find from theorem 4:

Corollary 7. *Assume that N is an n-dimensional connected submanifold of $R^m(k)$ ($n \geqq 2$) with constant dimensional first normal space. If we have through each point p of N an $(n-1)$-dimensional totally geodesic submanifold B_p of $R^m(k)$ and an l-dimensional ($l \geqq 1$) totally geodesic submanifold C_p of $R^m(k)$, which lie entirely on N and such that C_p is not contained in B_p, then N is a submanifold of an $R^{2n-l}(k)$. Especially if N is doubly foliated by $(n-1)$-dimensional totally geodesic submanifolds (which are at each point different) of $R^m(k)$, then N is a hypersurface in an $R^{n+1}(k)$.*

In fact the proof of theorem 3 remains valid for $n=2$. But we prefer to treat the case $n=2$ separately, because then the surface N has flat normal connection and this is not always the case for n-dimensional submanifolds ($n>2$) under the assumption of theorem 3. In this connection we have:

Theorem 5. *An n-dimensional submanifold N of $R^m(k)$ ($n \geqq 2$) with an $(n-1)$-dimensional asymptotic distribution and with flat normal connection has* $\dim N_1 \leqq 1$ *at each point. If moreover N is connected and has nowhere constant curvature k, then N is a hypersurface in an $R^{n+1}(k)$.*

Proof. Take an orthonormal base field $e_1, \cdots, e_n$ of N, such that $e_1, \cdots, e_{n-1}$ determine the $(n-1)$-dimensional asymptotic distribution, and an orthonormal base field $\xi_1, \cdots, \xi_{m-n}$ of $N^\perp$. Only the elements on the nth column and nth row of the matrices of A_{ξ_i} $i=1, \cdots, m-n$ with respect to $e_1, \cdots, e_n$ can be different from zero. Since N has flat normal connection, all these matrices must pairwise commute. This is only possible if we have one of the following situations at each point of N: all the matrices (of) A_{ξ_i} are zero except one or $V(e_1, e_n) = \cdots = V(e_{n-1}, e_n) = 0$ i.e. all the matrices (of) A_{ξ_i} are diagonal. In both cases we find that $\dim N_1 \leqq 1$ at each point. If we have at each point of N the second possbility, then it is easy to see that N has constant curvature k.

If N has at each point not constant curvature k, then we must have at each point of N the first possibility and moreover, not all the fields $V(e_1, e_n)$, $\cdots$, $V(e_{n-1}, e_n)$ are zero. So, in this case we have that the first normal space N_1 (which is everywhere one-dimensional) is spanned by $V(e_1, e_n), \cdots, V(e_{n-1}, e_n)$ and thus we find in the same way as in the proof of theorem 3 that N_1 is parallel in the normal bundle. Ths completes the proof.

Remark. From corollary 6 and theorem 5 we have: a minimal n-dimensional connected submanifold N of $R^m(k)$ with an $(n-1)$-dimensional asymptotic distribution and with flat normal connection is either totally geodesic or a minimal hypersuface in an $R^{n+1}(k)$.

Bibliography

[1] B. Y. Chen: *Geometry of submanifolds*. Marcel Dekker. New York 1973.

[2] J. Erbacher: *Reduction of the codimension of an isometric immersion*. Journal of Differential Geometry, vol. 5, nrs. 3 and 4, 1971.

[3] M. Spivak: *A comprehensive introduction to differential geometry*. Publish or perish inc. Boston 1970.

[4] C. Thas: *Properties of ruled surfaces in the euclidean space E^n*. Academia Sinica, vol. 6, nr. 1, 1978.

[5] C. Thas: *Minimal monosystems*. Yokohama Mathematical Journal. Vol. 26, 1978.

[6] C. Thas: *On minimal n-dimensional submanifolds of a space form $R^m(k)$, which are foliated* by *$(n-1)$-dimensional totally geodesic submanifolds of $R^m(k)$*. Kodai Mathematical Journal, Vol. 2, nr. 3, 1979.

[7] C. Thas: *A note on a class of submanifolds of a space form $R^m(k)$*. Soochow Journal of Mathematical and Natural Sciences, Vol. 4, 1978.

[8] Ü. Lumiste: *Die n-dimensionalen Minimalflachen mit einer $(n-1)$-dimensionalen Asymptotischen Richtung in jedem Punkte*. Mathematika-Loodusteaduskonna Töid, nr. 62, Tartu 1958.

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