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FURTHER RESULTS ON COMMON RIGHT FACTORS

By

FRED GROSS and CHUNG-CHUN YANG

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Introduction

As in [1] and [2], a meromorphic function $h(z)=f(g(z))$ is said to have $f(z)$ and $g(z)$ as left and right factors respectively, provided that $f(z)$ is nonlinear and meromorphic and $g(z)$ is nonlinear and entire (g may be meromorphic when $f(z)$ is a rational function). The first author showed in [2] that a meromorphic function F and its derivative F' cannot have a common right factor other than one of the form $e^{cz+b}+d$, where c, b and d are constants.

It is natural to ask what one can say about common right factors of a meromorphic function and its second derivative. This question leads to the rather difficult problem of fiinding meromorphic solutions f, g and l of the equation

$$(g')^2 f''(g)+g''f'(g)=l(g) .$$

Though the authors do have some partial results on this question the general problem remains open.

Thus, the problem of finding common meromorphic factors for F and F'' seems to be quite difficult. It is natural, therefore to look at the simpler problem of finding common right meromorphic factors of F, F'' and $F^{(n)}$, where $n>2$. For $n=3$, the answer follows immediately from the result stated in the first paragraph for F and F'. Simply replace F by F'' so that F'' and F''' have a common right meromorphic factor g. Then $g=e^{cz+b}+d$, where b, c, and d are constants.

Our main result in this paper is the solution of this problem for $n=4$. Before proceeding further, we shall define two functions g_1 and g_2 to be equivalent if and only if there exists a linear transformation $L(z)$ such that $g_1(z)=L(g_2(z))$ or $g_2(z)=L^{-1}(g_1(z))$.

We prove the following

Theorem 1. *Let F be a meromorphic function. If F, F'' and $F^{(4)}$ have the same right meromorphic factor g, then g is equivalent to one of the functions*

(i) az^2+bz+c. *For instance*: $F(z)=f(z^2+c)$ (f: *an entire function*),

(ii) $(az^2+bz+c)^{-1}$. *For instance*: $F(z)=f(1/z^2)$,

(iii) $ae^{bz}+c$. *For instance*: $F(z)=f(e^{cz})$,

(iv) $(ae^{bz}+c)^{-1}$. *For instance*: $F(z)=f(1/(e^{az}+c))$,

(v) $(a\cos(bz+c)+d)$. *For instance*: $F(z)=f(\cos z)$,

(vi) $(a\cos(bz+c)+d)^{-1}$. *For instance*: $F(z)=R(1/(\cos z+d))$
(R *is a rational function*),

or

(vii) *An elliptic functnoi of order two. For instance*: $F(z)=R(h)$
(h *is Wierstrass elliptic function*).

Here a, b, c, d are constants.

Before proceeding with the proof of the theorem, we shall prove the following

Lemma. *Let G and H be two nonlinear meromorphic functions with $G\neq aH+b$, where a and b are constants. If G, G'', H and H'' have a common right meromorphic factor g, then g has one of the forms* (i)-(vii) *in Theorem* 1.

Proof. Suppose that

$$G=p_1(g) \quad \text{and} \quad G''=p_2(g) \tag{1}$$

and

$$H=q_1(g) \quad \text{and} \quad H''=q_2(g)\,, \tag{2}$$

where p_i and q_i are meromorphic for $i=1$ and 2 and g is entire or p_i and q_i are rational and g is meromorphic.

From equations (1) and (2) we have

$$p_2(g)=p_1''(g)(g')^2+p_1'(g)g'' \tag{3}$$

and

$$q_2(g)=q_1''(g)(g')^2+q_1'(g)g''\,. \tag{4}$$

Since G and H are linearly dependent one can eliminate g'' from equations (3) and (4) and obtain the equation

$$(g')^2=R(g) \tag{5}$$

where

$$R(g)=\frac{p_2(g)q_1'(g)-p_1'(g)q_2(g)}{p_1''(g)q_1'(g)-q_1''(g)p_1'(g)}\,.$$

If $R(g)$ is transcendental, then it follows using a result of Clunie ([5], p. 54) and an argument of Briot and Bouquet ([3], p. 286) that $R(z)$ must be a rational function.

We now consider several cases.

If g is a polynomial, one readily verifies from equation (5) that g is of second degree. Suppose now that $g(z)$ is a rational function, i.e., $g(z)=g_1(z)/g_2(z)$, where $g_1(z)$ and $g_2(z)$ are polynomials and $g_2(z)$ is not a constant, then either $R(z)$ is a polynomial or $R(z)=S_1(z)/S_2(z)$, where S_i are polynomials for $i=1, 2$ and $S_2(z)=(z-a)^n$ (n a positive integer) and $g(z)-a$ different from zero. In other words, either $R(z)$ is a polynomial or $g(z)=a+1/t(z)$, where $t(z)$ is a polynomial. If $R(z)$ is a polynomial, we count the poles of g and g' and conclude from this count that $R(z)$ is either of degree 3 or 4. Hence we may write

$$R(z)=(z-a_1)^{k_1}(z-a_2)^{k_2}(z-a_3)^{k_3}(z-a_4)^{k_4}$$

with $k_1+k_2+k_3+k_4 \leqq 4$.

We may assume at this point without any loss of generality, that the degree of g_1 is not equal to the degree of g_2. The orders of the zeros of $g(z)-a_i$ for $i=1, 2, 3, 4$ indicate that either $k_1=1$ or $g(z)-a_i \neq 0$ for $i=1, 2, 3, 4$. If we now replace g in equation (5) by g_1/g_2 we find by a simple degree argument that $k_i=1$ for at most one i and in fact $g(z)-a_i \neq 0$. Thus, in any case $g(z)$ has the form $g(z)=a+1/t(z)$.

Using equation (5) once more, we readily conclude that $t(z)$ is a polynomial of degree two.

This completes our discussion for the case when $g(z)$ is a rational function.

We now assume that $g(z)$ is transcendental. Then we have

$$R(g)=\frac{Q_1(g)}{Q_{.}(g)},$$

where Q_1 and Q_2 are relatively prime polynomials. If Q_2 is nonconstant, then g must omit some value, say a, so that $1/(g-a)$ is entire. Thus, we may assume without any loss of generality that g is an entire function. Thus, we have

$$(g')^2 Q_2(g)=Q_1(g).$$

Hence, g is entire and has at least one finite Picard exceptional value and at least one other completely ramified value. Thus, g must be a constant, which of course is not the case. It follows that Q_2 must be a constant and $R(z)$ must be a polynomial. Since g cannot have more than four completely ramified values, it follows further that the degree of R is at most 4. Also, if g has a pole, one easily verifies that the degree of R is greater than two, and if it is entire, it follows from a result of Wittich [4] that the degree of $R(z)=2$. When R is of degree 3 or 4, one concludes from (5) that g is an elliptic function of second order. When R is of degree 2, we have

$$(g')^2=a(g-b)^2+c$$

and consequently that

$$g=a'\cos(b'z+c')+d$$

or

$$g=a'e^{c'z}+d\ ,$$

where a, b, c, a', b', c' and d are constants. This completes the proof of the lemma.

We now proceed with the proof of Theorem 1.

Proof. We set $G=F$ and $H=F''$. Suppose that $G=aH+b$, where a and b are constants and $a\neq 0$, then we have $F=aF''+b$, so F is entire and has the form

$$a_0+b_1e^{a_1z}+b_2e^{a_2z}\ , \qquad a_1=\pm a_2 \quad (a_i\neq 0,\ i=1,2)\ .$$

Hence,

$$F''=b_3e^{a_1z}+b_4e^{a_2z} \quad \text{and} \quad F^{(4)}=b_5e^{a_1z}+b_6e^{a_2z}\ ,$$

where a's and b's are constants.

By virtue of an earlier result of the first author [2], F, F'' and $F^{(4)}$ are pseudo-prime. Thus, either the left factors are rational functions or the right factors are polynomials. In the former case we have

$$F''=R_1(g)=b_3e^{a_1z}+b_4e^{a_2z}\ , \tag{6}$$

and

$$F^{(4)}=R_2(g)=b_5e^{a_1z}+b_6e^{a_2z}\ , \tag{7}$$

where R_1 and R_2 are rational functions. Solving the system (6), (7) for e^{a_1z} we get

$$e^{a_1z}=R_3(g)\ , \tag{8}$$

where R_3 is a rational function. Since F is entire, one may assume without any loss of generality that g is also entire. Thus, it is easy to conclude that

$$g=a+be^{cz}\ ,$$

where a, b and c are constants.

Suppose now that the right factors are polynomials. Then it follows from [6] that the polynomials must be of degree two. This completes the discussion of the case $F=aF''+b$ The case where this equality does not hold follows from the Lemma and the proof of our Theorem is complete.

For $n\geqq 5$, the problem remains unsolved. It is worth remarking that while the general problem of finding common right factors of F and F'' is a difficult one, one can obtain several interesting results when certain assumptions are made about the corresponding left factors. For example, by an argument somewhat more cumbersome than the one used to prove Theorem 1, one can prove

Theorem 2. *Let F be a nonlinear meromorphic function. There do not exist meromorphic functions p and g with g not identically constant such that $F=p(g)$ and $F''=p'(g)$ where p and p' are nonlinear.*

A similar result can be proven with p'' replacing p in Theorem 2.

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Naval Research Laboratory
Washington, D.C.
University of Maryland
Baltimore County
and
Naval Research Laboratory
Washington, D. C.
U.S.A.