YOKOHAMA MATHEMATICAL
JOURNAL VOL. 28, 1980

SOME LIMIT THEOREMS FOR MAXIMUM PARTIAL SUMS OF RANDOM VARIABLES

By

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(Received May 31, 1980)
(Revised June 16, 1980)

1. Introduction. Let $\{X_n, -\infty<n<\infty\}$ be a strictly stationary process. Denote by S_n the sum $X_1+X_2+\cdots+X_n$ $(S_0=0)$, and by S_n^* the maximum partial sum $\max_{0\leq j\leq n} S_j$. If $EX_1=0$, then, as is well-known, asymptotic behaviors of S_n and S_n^* are different. But, if $EX_1=\mu>0$, then asymptotic behaviors of S_n^* are very similar to the ones of S_n.

In the case where $\{X_n\}$ is an independent sequence with $EX_1=\mu>0$, Rogozin [5] and Ahmad [1] proved the speed of convergence of the distribution of $(S_n^*-n\mu)/\sqrt{n}$ to the normal distribution. In this note, we shall prove three theorems for S_n^* under the condition $EX_1=\mu>0$ without the assumption of independence. The first theorem is related to the speed of convergence of the distribution of $(S_n^*-n\mu)/\sqrt{n}$ to the normal distribution (Theorem 1). The second is the weak convergence to the Wiener process on [0, 1] (Theorem 2). And the last is a functional law of the iterated logarithm (Theorem 3).

Remark. In the first manuscript of this note, the author obtained the same conclusions as ones of Theorem 2 and 3 under more restrictive conditions by a different method from the present one. The present method proving Theorem 2 and 3 was suggested by Professor T. Mori.

Acknowledgement. The author would like to thank Professor T. Mori for his useful comments on the first manuscript.

2. Preliminaries and results. Let $\{X_n, -\infty<n<\infty\}$ be a strictly stationary process with $EX_1=\mu>0$ and a finite variance $E(X_1-\mu)^2<\infty$. In what follows, σ denotes a positive constant.

Now introduce the following notation:

$$\Delta_n=\sup_x \left| P\left(\frac{S_n-n\mu}{\sigma\sqrt{n}}\leq x\right)-\Phi(x)\right|$$

and

$$\Delta_n^* = \sup_x \left| P\left(\frac{S_n^* - n\mu}{\sigma\sqrt{n}} \leqq x\right) - \Phi(x) \right|$$

where $S_n^* = \max_{0 \leqq j \leqq n} S_j$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$.

Theorem 1. *Suppose that for some $\delta > 0$ and $K > 0$,*

$$E|S_n - n\mu|^{2+\delta} \leqq K n^{(2+\delta)/2} \tag{2.1}$$

for all n. Then we have

$$\Delta_n^* = O(\max\{\Delta_n, n^{-\delta/(4+2\delta)}\}). \tag{2.2}$$

Define two processes $X_n(t)$ and $X_n^*(t)$, $0 \leqq t \leqq 1$ as follows

$$X_n(t) = \frac{1}{\sigma\sqrt{n}} (S_{[nt]} - [nt]\mu), \quad 0 \leqq t \leqq 1 \tag{2.3}$$

and

$$X_n^*(t) = \frac{1}{n\sqrt{\sigma}} (S_{[nt]}^* - [nt]\mu), \quad 0 \leqq t \leqq 1$$

where $[x]$ denotes the integer part of a nonnegative number x.

Theorem 2. *If $X_n(\cdot)$ converges in distribution to the Wiener process W on $[0, 1]$ in the Skorohod J_1-topology, then $X_n^*(\cdot)$ converges in distribution to W in the Skorohod J_1-topology.*

For each $\omega \in \Omega$, define the functions $h_n(t, \omega)$ and $h_n^*(t, \omega)$ $(n \geqq 3)$ in $C = C[0, 1]$ as follows

$$h_n(t, \omega) = \begin{cases} (S_k(\omega) - k\mu)/\chi(n) & \text{for} \quad t = k/n,\ k = 0, 1, \dots, n \\ \text{linearly interpolated for } t \in [k/n, (k+1)/n] & k = 0, 1, \dots, n-1 \end{cases} \tag{2.4}$$

and

$$h_n^*(t, \omega) = \begin{cases} (S_k^*(\omega) - k\mu)/\chi(n) & \text{for} \quad t = k/n,\ k = 0, 1, \dots, n \\ \text{linearly interpolated for } t \in [k/n, (k+1)/n] & k = 0, 1, \dots, n-1 \end{cases}$$

where $\chi(n) = (2\sigma^2 n \log\log n)^{1/2}$.

We denote by K the subset of C consisting of all functions $h(t)$ absolutely continuous with respect to Lebesgue measure such that $\int_0^1 \dot{h}(t)^2 dt \leqq 1$, where $\dot{h}(t)$ stands for the Radon-Nikodym derivative of h. We say that a process $g_n(t, \omega)$, $0 \leqq t \leqq 1$, $n=1, 2, \ldots$ in C obeys Strassen's law of the iterated logarithm (SLIL) if, for almost every $\omega \in \Omega$, the sequence of functions $g_n(t, \omega)$ is precompact in C with supremum norm and its derived set is the set K.

Theorem 3. *If $h_n(t, \omega)$ obeys SLIL, then $h_n^*(t, \omega)$ also obeys SLIL.*

3. Proofs.

Lemma 3.1. *Suppose that, for some $\delta > 0$ and $K > 0$,*

$$E|S_n - n\mu|^{2+\delta} \leqq K n^{(2+\delta)/2} \tag{3.1}$$

for all n. Then for $\varepsilon > 0$, we have

$$P(S_n^* - S_n \geqq \varepsilon\sigma\sqrt{n}) = O((\varepsilon\sqrt{n})^{-\delta/2}). \tag{3.2}$$

Proof. It is easily seen that, if $S_n^* - S_n > 0$, then

$$S_n^* - S_n = \max_{0 \leqq j \leqq n-1} \{(-X_{j+1}) + (-X_{j+2}) + \cdots + (-X_n)\}.$$

So we have

$$\begin{aligned}
P(S_n^* - S_n \geqq \varepsilon\sigma\sqrt{n}) &\leqq \sum_{j=0}^{n-1} P((-X_{j+1}) + \cdots + (-X_n) \geqq \varepsilon\sigma\sqrt{n}) \\
&\leqq \sum_{j=0}^{n-1} P((-X_{j+1}) + \cdots + (-X_n) + (n-j)\mu \geqq \varepsilon\sigma\sqrt{n} + (n-j)\mu) \\
&\leqq \sum_{j=0}^{n-1} \frac{E|X_{j+1} + \cdots + X_n - (n-j)\mu|^{2+\delta}}{(\varepsilon\sigma\sqrt{n} + (n-j)\mu)^{2+\delta}} \qquad \text{by the Markov inequality} \\
&= \sum_{j=1}^{n} \frac{E|S_j - j\mu|^{2+\delta}}{(\varepsilon\sigma\sqrt{n} + j\mu)^{2+\delta}} \qquad \text{by stationarity of } \{X_n\} \\
&\leqq K \sum_{j=1}^{n} \frac{j^{(2+\delta)/2}}{(\varepsilon\sigma\sqrt{n} + j\mu)^{2+\delta}} \qquad \text{by the assumption} \\
&\leqq K \max_{1 \leqq j \leqq n} \frac{j^{1+\delta/2}}{(\varepsilon\sigma\sqrt{n} + j\mu)^{1+\delta/2}} \sum_{j=1}^{n} \frac{1}{(\varepsilon\sigma\sqrt{n} + j\mu)^{1+\delta/2}} \\
&\leqq K' \sum_{j=1}^{\infty} \frac{1}{(\varepsilon\sigma\sqrt{n} + j\mu)^{1+\delta/2}} = O((\varepsilon\sqrt{n})^{-\delta/2}).
\end{aligned}$$

Proof of Theorem 1. Since, for any two random variables X and Y and any $\varepsilon > 0$,

$$\sup_x |P(X+Y \leqq x) - \Phi(x)|$$
$$\leqq \sup_x |P(X \leqq x) - \Phi(x)| + \frac{1}{\sqrt{2\pi}} \varepsilon + P(|Y| \geqq \varepsilon),$$

putting $X=(S_n-n\mu)/\sigma\sqrt{n}$ and $Y=(S_n^*-S_n)/\sigma\sqrt{n}$, we have

$$\begin{aligned}\Delta_n^* &= \sup_x |P(S_n^*-n\mu \leqq \sigma\sqrt{n}\,x) - \Phi(x)| \\ &= \sup_x |P(S_n-n\mu+S_n^*-S_n \leqq \sigma\sqrt{n}\,x) - \Phi(x)| \\ &\leqq \sup_x |P(S_n-n\mu \leqq \sigma\sqrt{n}\,x) - \Phi(x)| + \frac{1}{\sqrt{2\pi}} \varepsilon + P(S_n^*-S_n \geqq \varepsilon\sigma\sqrt{n}).\end{aligned}$$

If we set $\varepsilon = n^{-\delta/(4+2\delta)}$, we have, by (3.2),

$$\Delta_n^* = O(\Delta_n) + O(n^{-\delta/(4+2\delta)}) + O(n^{-\delta/(4+2\delta)}).$$

This completes the proof of Theorem 1.

Theorem 2 and 3 can be proved as immediate consequences of a series of lemmas that follows.

Introduce two sets $\{f_n\}$ and $\{F_n\}$ of mappings of $C=C[0, 1]$ with uniform metric to itself as follows: for an $x \in C$,

$$f_n(x)(t) = \sup_{0 \leqq s \leqq t} (x(s) - a(n)(t-s)) \quad 0 \leqq t \leqq 1,\ n=1, 2,\ldots$$

and

$$F_n(x)(t) = \sup_{0 \leqq s \leqq t} (x(s) - b(n)(t-s)) \quad 0 \leqq t \leqq 1,\ n=3, 4,\ldots$$

where $a(n)=\sqrt{n}\mu/\sigma$ and $b(n)=\sqrt{n}\mu/\sqrt{2\sigma^2 \log\log n}$. Let us denote by d (,) the metric on C.

Lemma 3.2. *For all x, $y \in C$,*

(3.3) $$d(f_n(x), f_n(y)) \leqq d(x, y) \quad \textit{and} \quad d(F_n(x), F_n(y)) \leqq d(x, y)$$

for all n.

Proof. We prove only the former inequality.

$$\begin{aligned}d(f_n(x), f_n(y)) &= f_n(x)(t_0) - f_n(y)(t_0) \quad 0 \leqq t_0 \leqq 1 \quad \text{(say)} \\ &= x(s_1) - a(n)(t_0-s_1) - (y(s_2) - a(n)(t_0-s_2))\end{aligned}$$

where $f_n(x)(t_0) = x(s_1) - a(n)(t_0-s_1)$ and $f_n(y)(t_0) = y(s_2) - a(n)(t_0-s_2)$ $(0 \leqq s_1, s_2 \leqq t_0)$. Hence we have

$$d(f_n(x), f_n(y)) \leqq x(s_1) - a(n)(t_0 - s_1) - (y(s_1) - a(n)(t_0 - s_1))$$
$$= x(s_1) - y(s_1) \leqq d(x, y).$$

Lemma 3.3. *Let K be a precompact set in C. Then, for any $\varepsilon > 0$, there exists a positive integer N_0 such that, for all $n \geqq N_0$,*

$$\sup_{x \in K} d(f_n(x), x) \leqq \varepsilon \quad \textit{and} \quad \sup_{x \in K} d(F_n(x), x) \leqq \varepsilon. \tag{3.4}$$

Proof. We give the proof for f_n. The other can be proved by the same way. Since K is precompact, there exists a positive number M such that

$$\sup_{x \in K} \sup_{0 \leqq t \leqq 1} |x(t)| \leqq M,$$

and for given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in K$

$$|x(s) - x(t)| < \varepsilon \quad \text{if} \quad |s - t| < \delta.$$

We choose an integer $N_0 (\geqq 0)$ such that $a(N_0)\delta \geqq 2M$. Then by an easy consideration we have

$$f_n(x)(t) = \sup_{0 \vee (t-\delta) \leqq s \leqq t} (x(s) - a(n)(t-s)) \qquad 0 \leqq t \leqq 1$$

and

$$x(t) + \varepsilon \geqq f_n(x)(t) \geqq x(t) \qquad 0 \leqq t \leqq 1$$

for all $n \geqq N_0$, where $0 \vee (t - \delta)$ denotes $\max\{0, (t-\delta)\}$. Thus if $n \geqq N_0$, we have $d(f_n(x), x) \leqq \varepsilon$ for all $x \in K$.

For each integer n, denote by $\bar{X}_n$ (resp. $\bar{X}_n^*$) the polygonal function that is linear on each of the subintervals $[(i-1)/n, i/n]$, $i = 1, 2, \ldots, n$, and has the value $(S_i - i\mu)/\sigma\sqrt{n}$ (resp. $(S_i^* - i\mu)/\sigma\sqrt{n}$) at the point i/n.

Lemma 3.4. *For any $\varepsilon > 0$, as $n \to \infty$,*

$$P(\sup_{0 \leqq t \leqq 1} |\bar{X}_n(t) - X_n(t)| \geqq \varepsilon) \longrightarrow 0. \tag{3.5}$$

The same result remains valid for $\bar{X}_n^*$ and X_n^*.

Proof. We show (3.5) only.

$$P(\sup_{0 \leqq t \leqq 1} |\bar{X}_n(t) - X_n(t)| \geqq \varepsilon) \leqq \sum_{i=1}^{n} P(|X_i - \mu| \geqq \varepsilon\sigma\sqrt{n})$$
$$= nP(|X_1 - \mu| \geqq \varepsilon\sigma\sqrt{n}) \quad \text{by stationarity of } \{X_n\}$$
$$\leqq nE\left\{\frac{|X_1 - \mu|^2}{\varepsilon^2\sigma^2 n}\,;\ |X_1 - \mu| \geqq \varepsilon\sigma\sqrt{n}\right\}$$
$$= \frac{1}{\varepsilon^2\sigma^2} E\{|X_1 - \mu|^2;\ |X_1 - \mu| \geqq \varepsilon\sigma\sqrt{n}\} \longrightarrow 0 \qquad (n \to \infty).$$

Proof of Theorem 2. By Lemma 3.2, f_n's are continuous mapping from C into C, by Lemma 3.3, for any $x \in C$, $d(f_n(x), x) \to 0$ as $n \to \infty$, and by direct calculations, we have $f_n(\overline{X}_n) = \overline{X}_n^*$ for all n. Hence, by Theorem 5.5 [2] and the fact that, by Lemma 3.4, $X_n \xrightarrow{D} W$ in D with Skorohod J_1-topology implies $\overline{X}_n \xrightarrow{D} W$ in C, we have $\overline{X}_n^* \xrightarrow{D} W$ in C. On the other hand, by Lemma 3.4, $\overline{X}_n^* \xrightarrow{D} W$ in C implies $X_n^* \xrightarrow{D} W$ in D. Hence we have the conclusion of Theorem 2.

Proof of Theorem 3. Theorem 3 follows immediately from Lemma 3.3 and the equality $F_n(h_n)(t, \omega) = h_n^*(t, \omega)$.

4. Remarks. The moment condition (2.1) in Theorem 1 is satisfied by a wide class of weakly dependent strictly stationary processes, for examples, Markov processes satisfying Doeblin's condition [3], φ-mixing or strong mixing sequences [6] etc.

From Theorem 2, we have the asymptotic normality of S_n^* for a class of stationary processes $\{X_n\}$ assuming only $E|X_1 - \mu|^2 < \infty$. For example, see [2].

Sufficient conditions which assure SLIL for $h_n(t, \omega)$ are found, for example, in [4].

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