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ON THE JACOBSON RADICALS OF LIE TRIPLE SYSTEMS

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1. Introduction. The purpose of this paper is to study the Jacobson radical of a finite dimensional Lie triple system. Let T be a Lie triple system (L.t.s.) over a field of characteristic 0. For an ideal A of T, we put $A^{(1)}=[T\,A\,A]$ and $A^{(k)}=[T\,A^{(k-1)}\,A^{(k-1)}]$ $(k\geqq 2)$. An ideal A is to be called solvable if there is a positive integer k such that $A^{(k)}=0$. If T is finite dimensional, then it contains the unique maximal solvable ideal $R(T)$, which is called the radical of T. On the other hand, the Jacobson radical $J_R(T)$ of T is defined by intersection of all maximal ideals of T. Then we have the following theorem.

Theorem A. $J_R(T)=[T\,T\,T]\cap R(T)$.

Next, let L be the standard imbedding Lie algebra of T, i.e. $L=L(T, T)\oplus T$, and let Rad L (resp. $J_R(L)$) be the solvable radical (resp. Jacobson radical) of L. Then we obtain the following theorem.

Theorem B. $J_R(L)=L(T, R(T))\oplus J_R(T)$.

2. Proof of Theorems.

Lemma. $J_R(T)\subseteqq[T\,T\,T]$.

Proof. In the case $T=[T\,T\,T]$, this is trivial. So we may assume $T\neq[T\,T\,T]$. If $x\notin[T\,T\,T]$, then there is a subspace M of T which is complementary to the subspace $\langle x\rangle$ spanned by x and contains $[T\,T\,T]$. Then M is a maximal ideal of T. Since $J_R(T)\subseteqq M$, $x\notin J_R(T)$. Therefore $J_R(T)\subseteqq[T\,T\,T]$.

Proof of Theorem A. If I is a maximal ideal of T, then the factor triple system T/I is simple or $(T/I)^{(1)}=0$. In the former case, since T/I is simple, I must contain $R(T)$. From [1. Theorem 2.21], T is decomposed to $T=B_0\oplus R(T)$ (B_0 is a semisimple subtriple system of T). Hence I is of the form $M+R(T)$, where M is a maximal ideal of B_0. Since the semisimple subtriple system B_0 can be expressed as the direct sum of simple ideals [1. Theorem 2.9], the Jacobson radical of B_0 is 0. Hence the intersection of all such maximal ideals of T equals to $R(T)$. In the latter case, since $(T/I)^{(1)}=0$, I must contain $T^{(1)}$. Hence the intersection of all such maximal ideals of T contains $T^{(1)}$. Considering two case, we have

$$T^{(1)} \cap R(T) \subseteqq J_R(T) \subseteqq R(T).$$

Since $J_R(T) \subseteqq T^{(1)}$ by Lemma, we obtain

$$J_R(T)=[T\,T\,T] \cap R(T).$$

Corollary. *If T is a perfect* (i.e. $T=[T\,T\,T]$) *L.t.s., then $J_R(T)=R(T)$. In particular, if T is a semisimple, then $J_R(T)=0$.*

Corollary. *If T is a solvable L.t.s., then $J_R(T)=[T\,T\,T]$.*

Proof of Theorem B. In [2] and [3], it is proved that

$$\mathrm{Rad}\, L=L(T, R(T)) \oplus R(T), \tag{1}$$

$$J_R(L)=[L, \mathrm{Rad}\, L]=[L, L] \cap \mathrm{Rad}\, L. \tag{2}$$

Hence, we have

$$\begin{aligned} J_R(L) &= [L, \mathrm{Rad}\, L] \\ &= [L(T, T) \oplus T, L(T, R(T)) \oplus R(T)] \\ &= ([L(T, T), L(T, R(T))]+L(T, R(T))) \\ &\quad \oplus (L(T, T)R(T)-L(T, R(T))T) \\ &\subseteqq L(T, R(T)) \oplus ([T\,T\,T] \cap R(T)) \\ &= L(T, R(T)) \oplus J_R(T). \end{aligned} \tag{3}$$

On the other hand, from (1) and Theorem A, we have

$$(R(T) \cap [T\,T\,T]) \oplus L(T, R(T)) \subseteqq \mathrm{Rad}\, L$$

and

$$J_R(T) \oplus L(T, R(T)) \subseteqq \mathrm{Rad}\, L. \tag{4}$$

Since $L=L(T, T) \oplus T$,

$$\begin{aligned} [L, L] &= [L(T, T) \oplus T, L(T, T) \oplus T] \\ &= L(T, T) \oplus L(T, T)T \\ &\supseteqq L(T, R(T)) \oplus (R(T) \cap [T\,T\,T]) \\ &= L(T, R(T)) \oplus J_R(T). \end{aligned} \tag{5}$$

Therefore by (2), (4) and (5), we have

$$\begin{aligned} J_R(L) &= \mathrm{Rad}\, L \cap [L, L] \\ &\supseteqq L(T, R(T)) \oplus J_R(T). \end{aligned} \tag{6}$$

From (3) and (6), the theorem is proved.

Corollary. *If T is a perfect L.t.s. and $J_R(T)=0$, then $J_R(L)=0$.*

Corollary. *If T is a perfect L.t.s., then $J_R(L)=\mathrm{Rad}\, L$.*

References

[1] W. G. Lister: *A structure theory of Lie triple systems.* Trans. Amer. Math. Soc., **72**, 217–242 (1952).

[2] K. Meyberg: *Lecture on Algebras and triple systems.* The University of Virginia, (1972).

[3] N. Kamiya: *On the Jacobson Radicals of Infinite Dimensional Lie Algebras.* Hiroshima Math. Journal., **9**, 37–40 (1979).

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