

## A NOTE ON THE NONUNIFORM RATE OF CONVERGENCE TO NORMALITY

By

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**ABSTRACT:** Some results on the nonuniform bound for the distance between the distribution of normalized partial sums of i.i.d. random variables and a Chebyshev series, are given.

### 1. Introduction and results

Let  $\{X_k, k=1, 2, \dots\}$  be a sequence of independent and identically distributed random variables with  $EX_1=0$ ,  $EX_1^2=1$  and distribution function  $F(x)$ .  $F_n(x)$  denotes the distribution function of the normalized partial sum  $n^{-1/2}S_n$ , where  $S_n=X_1+X_2+\dots+X_n$ , and  $\Phi(x)$  denotes the standard normal distribution function. Let  $p \geq 0$  be an integer. Write

$$R_{np}(x) = |F_n(x) - G_{np}(x)|,$$

where  $G_{n0}(x) = \Phi(x)$  and for  $p \geq 1$

$$G_{np}(x) = \Phi(x) + (2\pi)^{-1/2} e^{-x^2/2} \sum_{j=1}^p n^{-j/2} Q_j(x).$$

Here  $Q_j(x)$  are polynomials of degree  $3j-1$  defined by Ibragimov [9] without presupposing the existence of moments of higher order than the second. That is, on the basis of a given arbitrary numerical sequence  $\beta_1=0, \beta_2=1, \beta_3, \beta_4, \dots$ , we form polynomials  $Q_j(x)$  in such a way that their coefficients are expressed in terms of  $\beta_1, \beta_2, \dots, \beta_{j+2}$  in the same way as the coefficients of the classical polynomials  $Q_j(x)$  in the Chebyshev series corresponding to the  $n^{-1/2}S_n$  are expressed in terms of the cumulants  $\kappa_1, \kappa_2, \dots, \kappa_{j+2}$  of the  $X_1$  (see [5] Section 38). Let  $\alpha_1=0, \alpha_2=1, \alpha_3, \alpha_4, \dots$  be the "moment" sequence corresponding to the "cumulant" sequence  $\beta_1=0, \beta_2=1, \beta_3, \beta_4, \dots$  (See also [8]).

When we consider the case  $p \geq 1$ , we further suppose Cramér condition

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(C)  $\limsup_{|t| \rightarrow \infty} |f(t)| < 1$   
to be satisfied, where  $f(t) = Ee^{itX_1}$ .

It has been shown that in order that for  $0 \leq \delta < 1$ ,

$$(1.1) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \sup_x (1+|x|)^{p+2} R_{np}(x) < \infty,$$

it is sufficient that

$$(1.2) \quad \begin{cases} E|X_1|^{p+2+\delta} < \infty & \text{for } 0 < \delta < 1, \\ E|X_1|^{p+2} \log(1+|X_1|) < \infty & \text{for } \delta = 0, \\ \alpha_j = EX_1^j, \quad j = 1, 2, \dots, p+2, \end{cases}$$

under Cramér condition (C) for  $p \geq 1$ . (The case  $p=0$  is due to Heyde [7] and the case  $p \geq 1$  due to the author [11].) (1.2) is also necessary for (1.1) to hold, when  $p=0$  ([5]) or when  $p \geq 1$  and  $0 < \delta < 1$  ([8]).

On the other hand, the conditions for

$$(1.3) \quad \sup_x (1+|x|)^{p+2+\delta} R_{np}(x) = O(n^{-(p+\delta)/2})$$

are known. (See [2], [3] and [10].) But it is easily shown that the nonuniformity of (1.3) cannot be improved in the sense that the power of  $(1+|x|)$  cannot be replaced by a higher one, under  $E|X_1|^{p+2+\delta} < \infty$ . (See [11].)

Now, in connection with (1.1) and (1.3), it seems natural to ask whether

$$(1.4) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \sup_x (1+|x|)^{p+2+\delta} R_{np}(x)$$

converges for  $0 < \delta < 1$  under  $E|X_1|^{p+2+\delta} < \infty$ , or not. So far as the author knows, this is not yet solved. This paper is concerned with this convergence problem. We recall here that, if  $\delta=0$ , then (1.4) converges under  $E|X_1|^{p+2} \log(1+|X_1|) < \infty$ , as we have seen in (1.1).

Let  $L(x)$  be a slowly varying function which is continuous and eventually non-decreasing.

We state our theorems.

**Theorem 1.** *Let  $p$  be a nonnegative integer and  $0 < \delta < 1$ . If  $E|X_1|^{p+2+\delta} < \infty$ ,  $\alpha_j = EX_1^j$ ,  $j = 1, 2, \dots, p+2$ , and, for  $p \geq 1$  if Cramér condition (C) is satisfied, then*

$$(1.5) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (L(n))^{1+(p+\delta)/2} R_{np}(c(L(n))^{1/2}) < \infty,$$

for all  $c > 0$ .

This theorem says that if we put  $x = c(L(n))^{1/2}$  instead of taking supremum of  $x$  in (1.4), then this series converges. This result is related to Theorem 1 of Michel [12], which is concerned with probabilities of moderate deviations.

Furthermore we have the following.

**Theorem 2.** *Let  $p$  be a nonnegative integer and  $0 < \delta < 1$ . In order that*

$$(1.6) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (\log n)^{1+(p+\delta)/2} R_{np}(c(\log n)^{1/2}) < \infty,$$

*for all  $c > 0$ , it is necessary and for  $p=0$  or for distributions satisfying (C) also sufficient that*

$$E|X_1|^{p+2+\delta} < \infty$$

*and*

$$\alpha_j = EX_1^j, \quad j = 1, 2, \dots, p+2.$$

Instead of (1.4), we can easily prove the following.

**Theorem 3.** *Let  $p$  be a nonnegative integer and  $0 < \delta < 1$ . In order that*

$$(1.7) \quad \sup_x (1 + |x|)^{p+2+\delta} \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} R_{np}(x) < \infty,$$

*it is necessary and for  $p=0$  or for distributions satisfying (C) also sufficient that*

$$E|X_1|^{p+2+\delta} < \infty$$

*and*

$$\alpha_j = EX_1^j, \quad j = 1, 2, \dots, p+2.$$

Finally, we shall show a simple result related to Theorem 2.

**Theorem 4.** *Let  $p$  be a nonnegative integer and  $0 < \delta < 1$ . In order that*

$$(1.8) \quad (\log n)^{1+(p+\delta)/2} R_{np}(c(\log n)^{1/2}) = O(n^{-(p+\delta)/2}),$$

*for each  $c > 0$ , it is necessary and for  $p=0$  or for distributions satisfying (C) also sufficient that*

$$(1.9) \quad P(|X_1| > z) = O(z^{-(p+2+\delta)}), \quad \text{as } z \rightarrow \infty,$$

*which is equivalent to*

$$(1.10) \quad \sup_x (1 + |x|)^{p+2+\delta} R_{np}(x) = O(n^{-(p+\delta)/2}).$$

## 2. Proof of Theorem 1

In the following, we shall use  $C$  as a positive constant which may depend on  $p$  and differ from one expression to another.

First, we show some properties concerning the eventually nondecreasing slowly varying function  $L(x)$ .

We fix  $c > 0$  in (1.5) in Theorem 1, and define  $a(m)$  by

$$(2.1) \quad m = c^2 a(m) L(a(m)),$$

namely,  $n = a(m)$  is an inverse function of  $m = c^2 n L(n)$ . Without loss of generality, we suppose  $L(x) \geq 1$ , then

$$nL(n) \geq n$$

so that

$$m = c^2 a(m) L(a(m)) \geq c^2 a(m).$$

By the monotonicity of  $L(x)$  for large  $x$ ,

$$L(m) \geq L(c^2 a(m)) \geq \frac{1}{2} L(a(m)) \quad \text{for all large } m.$$

Therefore

$$(2.2) \quad a(m) = m / (c^2 L(a(m))) \geq Cm / L(m) \quad \text{for all large } m.$$

By Theorem 1.5 of Seneta [15],

$$L(m) \leq CL(m/L(m)) \leq CL(Ca(m)) \leq CL(a(m)),$$

because of (2.2). Hence

$$(2.3) \quad a(m) = m / (c^2 L(a(m))) \leq Cm / L(m) \quad \text{for all large } m.$$

Now we proceed to the proof of Theorem 1. Without loss of generality, we assume  $c = 1$ . We use here the following nonuniform estimates in the central limit theorem:

$$(2.4) \quad \begin{aligned} R_{np}(x) \leq & C \left\{ \frac{n}{(n^{1/2}(1+|x|))^{p+2}} \int_{|u| > n^{1/2}(1+|x|)} |u|^{p+2} dF(u) \right. \\ & + \frac{n}{(n^{1/2}(1+|x|))^{p+3}} \int_{|u| \leq n^{1/2}(1+|x|)} |u|^{p+3} dF(u) \\ & \left. + A_p \left( \sup_{|t| \geq r} |f(t)| + \frac{1}{2n} \right) \frac{n^{(p+2)(p+3)/2}}{(1+|x|)^{p+3}} \right\}, \end{aligned}$$

under  $E|X_1|^{p+2} < \infty$  for some integer  $p \geq 0$ , where  $A_p = 0$  or 1 according as  $p = 0$  or  $p \geq 1$ , and  $r = (15E|X_1|^3)^{-1}$ . The case  $p = 0$  is given by Bikelis [1] and the case  $p \geq 1$  due to Osipov [14].

By (2.4),

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (L(n))^{1+(p+\delta)/2} R_{np}((L(n))^{1/2}) \\ & \leq C \sum_{n=1}^{\infty} \left\{ n^{-1+\delta/2} (L(n))^{\delta/2} \int_{|u| > n^{1/2}(1+(L(n))^{1/2})} |u|^{p+2} dF(u) \right. \\ & \quad \left. + n^{-3/2+\delta/2} (L(n))^{-1/2+\delta/2} \int_{|u| \leq n^{1/2}(1+(L(n))^{1/2})} |u|^{p+3} dF(u) \right\} \\ & \quad + CA_p \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (L(n))^{-1/2+\delta/2} \left( \sup_{|t| \geq r} |f(t)| + \frac{1}{2n} \right)^n n^{(p+2)(p+3)/2} \\ & \equiv I_1 + I_2 + I_3, \end{aligned}$$

say. Under Cramér condition (C),  $I_3 < \infty$  is trivial. Now,

$$\begin{aligned} I_1 &= C \sum_{n=1}^{\infty} n^{-1+\delta/2} (L(n))^{\delta/2} \int_{|u| > n^{1/2}(1+(L(n))^{1/2})} |u|^{p+2} dF(u) \\ &\leq C \sum_{n=1}^{\infty} n^{-1+\delta/2} (L(n))^{\delta/2} \sum_{m=[nL(n)]}^{\infty} \int_{m < u^2 \leq (m+1)} |u|^{p+2} dF(u) \\ &\leq C \sum_{m=1}^{\infty} m^{(p+2)/2} P(m < X_1^2 \leq (m+1)) \sum_{n=1}^{[a(m)+1]} n^{-1+\delta/2} (L(n))^{\delta/2}, \end{aligned}$$

where  $a(m)$  is the one defined in (2.1) and  $[t]$  is the integer part of  $t$ . From (2.3),

$$\begin{aligned} I_1 &\leq C \sum_{m=1}^{\infty} m^{(p+2)/2} P(m < X_1^2 \leq (m+1)) \sum_{n=1}^{[Cm/L(m)]+1} n^{-1+\delta/2} (L(n))^{\delta/2} \\ &\leq C \sum_{m=1}^{\infty} m^{(p+2)/2} P(m < X_1^2 \leq (m+1)) (L(Cm/L(m)))^{\delta/2} (Cm/L(m))^{\delta/2} \\ &\leq C \sum_{m=1}^{\infty} m^{(p+2+\delta)/2} P(m < X_1^2 \leq (m+1)), \end{aligned}$$

where we have used Theorem 1.5 of Seneta [15] again, so that

$$I_1 \leq CE|X_1|^{p+2+\delta} < \infty.$$

As to  $I_2$ ,

$$\begin{aligned} I_2 &= C \sum_{n=1}^{\infty} n^{-3/2+\delta/2} (L(n))^{-1/2+\delta/2} \int_{|u| \leq n^{1/2}(1+(L(n))^{1/2})} |u|^{p+3} dF(u) \\ &\leq C \sum_{m=1}^{\infty} m^{(p+3)/2} P((m-1) < X_1^2 \leq m) \sum_{n=[a(m)]}^{\infty} n^{-3/2+\delta/2} (L(n))^{-1/2+\delta/2} \end{aligned}$$

and by (2.2),

$$\begin{aligned}
I_2 &\leq C \sum_{m=1}^{\infty} m^{(p+3)/2} P((m-1) < X_1^2 \leq m) \sum_{n=[Cm/L(m)]}^{\infty} n^{-3/2+\delta/2} (L(n))^{-1/2+\delta/2} \\
&\leq C \sum_{m=1}^{\infty} m^{(p+3)/2} P((m-1) < X_1^2 \leq m) \\
&\quad \times (L(Cm/L(m)))^{-1/2+\delta/2} (Cm/L(m))^{-1/2+\delta/2} \\
&\leq C \sum_{m=1}^{\infty} m^{(p+2+\delta)/2} P((m-1) < X_1^2 \leq m) \\
&\leq CE|X_1|^{p+2+\delta} < \infty,
\end{aligned}$$

which completes the proof of the theorem.

### 3. Proof of Theorem 2

The sufficiency part is a special case of Theorem 1, so we need only show the necessity part. However, it is easily seen that

$$(3.1) \quad 1 - G_{np}(c(\log n)^{1/2}) + G_{np}(-c(\log n)^{1/2}) = O(n^{-c^2/2} Q((\log n)^{1/2})),$$

where  $Q(x)$  is a polynomial of at most degree  $3p-1$ . Therefore, if  $c > (p+\delta)^{1/2}$ , then

$$\sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (\log n)^{1+(p+\delta)/2} \{1 - G_{np}(c(\log n)^{1/2}) + G_{np}(-c(\log n)^{1/2})\} < \infty,$$

so that from (1.6),

$$(3.2) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} (\log n)^{1+(p+\delta)/2} P(|S_n| > c(n \log n)^{1/2}) < \infty$$

for all  $c > (p+\delta)^{1/2}$ . Hence, we can see  $E|X_1|^{p+2+\delta} < \infty$  by Theorem 2 of Davis [4].

### 4. Proof of Theorem 3

We first show the sufficiency part. From (2.4), we have

$$\begin{aligned}
&(1+|x|)^{p+2+\delta} \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} R_{np}(x) \\
&\leq C \left\{ (1+|x|)^{\delta} \sum_{n=1}^{\infty} n^{-1+\delta/2} \int_{|u| < n^{1/2}(1+|x|)} |u|^{p+2} dF(u) \right. \\
&\quad + \frac{1}{(1+|x|)^{1-\delta}} \sum_{n=1}^{\infty} n^{-3/2+\delta/2} \int_{|u| \leq n^{1/2}(1+|x|)} |u|^{p+3} dF(u) \\
&\quad \left. + \frac{A_p}{(1+|x|)^{1-\delta}} \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \left( \sup_{|t| \geq r} |f(t)| + \frac{1}{2n} \right)^n n^{(p+2)(p+3)/2} \right\} \\
&\equiv J_1 + J_2 + J_3,
\end{aligned}$$

say.  $J_3 < \infty$  is also trivial under Cramér condition (C). Furthermore,

$$\begin{aligned} J_1 &\leq C(1+|x|)^\delta \sum_{n=1}^{\infty} n^{-1+\delta/2} \sum_{m=[n(1+|x|)^2]}^{\infty} \int_{m < u^2 \leq (m+1)} |u|^{p+2} dF(u) \\ &\leq C(1+|x|)^\delta \sum_{m=1}^{\infty} m^{(p+2)/2} P(m < X_1^2 \leq (m+1)) \sum_{n=1}^{[m/(1+|x|)^2]+1} n^{-1+\delta/2} \\ &\leq C(1+|x|)^\delta \sum_{m=1}^{\infty} m^{(p+2)/2} P(m < X_1^2 \leq (m+1)) \left( \frac{m}{(1+|x|)^2} \right)^{\delta/2} \\ &\leq CE|X_1|^{p+2+\delta} < \infty, \end{aligned}$$

and

$$\begin{aligned} J_2 &\leq \frac{C}{(1+|x|)^{1-\delta}} \sum_{n=1}^{\infty} n^{-3/2+\delta/2} \sum_{m=1}^{[n(1+|x|)^2]+1} \int_{m < u^2 \leq (m+1)} |u|^{p+3} dF(u) \\ &\leq \frac{C}{(1+|x|)^{1-\delta}} \sum_{m=1}^{\infty} m^{(p+3)/2} P(m < X_1^2 \leq (m+1)) \sum_{n=[m/(1+|x|)^2]}^{\infty} n^{-3/2+\delta/2} \\ &\leq \frac{C}{(1+|x|)^{1-\delta}} \sum_{m=1}^{\infty} m^{(p+3)/2} P(m < X_1^2 \leq (m+1)) \left( \frac{m}{(1+|x|)^2} \right)^{-1/2+\delta/2} \\ &\leq CE|X_1|^{p+2+\delta} < \infty. \end{aligned}$$

We now proceed to prove the necessity part. According to Heyde-Leslie [8], we write

$$F_n^*(x) = F_n(x) * (1 - F_n(-x - 0))$$

and

$$G_{np}^*(x) = G_{np}(x) * (1 - G_{np}(-x)),$$

where  $*$  means a convolution. Then (1.7) implies

$$(4.1) \quad \sup_x \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} |F_n^*(x) - G_{np}^*(x)| < \infty.$$

Here we show that (1.7) implies  $E|X_1|^{p+2} < \infty$ . It follows from (1.7) that

$$\begin{aligned} &\int_{-\infty}^{\infty} (1+|x|)^{p+1} \left\{ \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} R_{np}(x) \right\} dx \\ &\leq C \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^{1+\delta}} < \infty, \end{aligned}$$

since  $\delta > 0$ . Therefore,

$$\sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \int_{-\infty}^{\infty} (1+|x|)^{p+1} R_{np}(x) dx < \infty$$

and so

$$\int_{-\infty}^{\infty} (1+|x|)^{p+1} R_{np}(x) dx < \infty \quad \text{for some } n=m, m+1.$$

Hence, keeping in mind the form of  $G_{np}(x)$ , we have

$$\int_0^{\infty} (1+|x|)^{p+1} \{1 - F_n(x) + F_n(-x)\} dx < \infty \quad \text{for } n=m, m+1,$$

which implies  $E|X_1|^{p+2} < \infty$ . Therefore, we can use the same argument as in Heyde-Leslie [8].

Letting  $f_n(t) = \int e^{itx} dF_n(x)$  and  $g_{np}(t) = \int e^{itx} dG_{np}(x)$ , we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \left| \int_{-\infty}^{\infty} (|f_n(t)|^2 - |g_{np}(t)|^2) e^{-t^2/2} dt \right| \\ &= (2\pi)^{1/2} \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \left| \int_{-\infty}^{\infty} (F_n^*(x) - G_{np}^*(x)) x e^{-x^2/2} dx \right| \\ &\leq (2\pi)^{1/2} \int_{-\infty}^{\infty} \left( \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} |F_n^*(x) - G_{np}^*(x)| \right) |x| e^{-x^2/2} dx \\ &\leq (2\pi)^{1/2} \sup_x \left( \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} |F_n^*(x) - G_{np}^*(x)| \right) < \infty \end{aligned}$$

because of (4.1). Hence, we have also in our case

$$(4.2) \quad \sum_{n=1}^{\infty} n^{-1+(p+\delta)/2} \left| \int_{-\infty}^{\infty} (|f_n(t)|^2 - |g_{np}(t)|^2) e^{-t^2/2} dt \right| < \infty,$$

and it is shown in Heyde-Leslie [8] that (4.2) implies  $E|X_1|^{p+2+\delta} < \infty$ .

## 5. Proof of Theorem 4

Note that (1.10) is equivalent to

$$(5.1) \quad \int_{|u|>z} |u|^{p+2} dF(u) = O(z^{-\delta}),$$

which was shown by Bikelis [2] for  $p=0$  and by Karoblis [10] for  $p \geq 1$ , respectively. Also, it is shown in Michel [13] that (1.9) and (5.1) are equivalent. Therefore, the equivalency of (1.9) and (1.10) is given. Furthermore, (1.8) follows from (1.10), so that the sufficiency part is shown.

It remains to prove the necessity part. First we suppose that  $X_1$  is symmetric. From (1.8) and (3.1),

$$P(|S_n| > c(n \log n)^{1/2}) = O(n^{-(p+\delta)/2} (\log n)^{-1-(p+\delta)/2}),$$

if  $c > (p+\delta)^{1/2}$ . For symmetric random variables,



$$P(|S_n| > a_n) \geq \frac{1}{2} P(\max_{1 \leq k \leq n} |X_k| > a_n) \sim \frac{1}{2} nP(|X_1| > a_n).$$

Therefore,

$$P(|X_1| > (n \log n)^{1/2}) = O((n \log n)^{-(p+2+\delta)/2})$$

as  $n \rightarrow \infty$ , which is

$$P(|X_1| > z) = O(z^{-(p+2+\delta)})$$

as  $z \rightarrow \infty$ .

Relaxing the restriction of the symmetricity of  $X_1$  will be handled by the ordinary argument such as, for example, in Heyde-Leslie [8].

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