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SIMPLE CROSSED PRODUCTS OF C^*-ALGEBRAS BY LOCALLY COMPACT ABELIAN GROUPS

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Abstract: We introduce a new invariant $\tilde{\Gamma}(\alpha)$, a closed subsemigroup of the dual of G, of a C^*-dynamical system $(\mathfrak{a}, G, \alpha)$ where $\mathfrak{a}$ is a C^*-algebra, and G is a locally compact abelian group with an action α on $\mathfrak{a}$. We show that the crossed product $\mathfrak{a}\times_\alpha G$ is simple if and only if $\mathfrak{a}$ is α-simple (i.e. $\mathfrak{a}$ does not have any non-trivial α-invariant ideals) and $\tilde{\Gamma}(\alpha)$ equals the dual of G. We discuss some cases where $\tilde{\Gamma}(\alpha)$ coincides with the Connes spectrum $\Gamma(\alpha)$. Finally we give examples of simple crossed products of Cuntz algebras by locally compact abelian groups.

§1. Introduction

In a paper [7] by D. Olesen and G. K. Pedersen the Connes spectrum $\Gamma(\alpha)$ [6] of a C^*-dynamical system $(\mathfrak{a}, G, \alpha)$, with a locally compact abelian group G, plays an important role in characterizing primeness of the crossed product $\mathfrak{a}\times_\alpha G$. We introduce a new invariant $\tilde{\Gamma}(\alpha)$, which is a closed subsemigroup of $\Gamma(\alpha)$, and show that $\tilde{\Gamma}(\alpha)$ is relevant in characterizing simplicity of $\mathfrak{a}\times_\alpha G$. After $\tilde{\Gamma}(\alpha)$ being introduced, our results and methods are quite similar to the prime case above-mentioned.

By using a characterization of $\tilde{\Gamma}(\alpha)$ in terms of ideals of the crossed product and the dual action on it, we show that $\tilde{\Gamma}(\alpha)$ coincides with $\Gamma(\alpha)$ in some cases, in particular, when G is discrete and $\mathfrak{a}$ is α-simple.

Unfortunately $\tilde{\Gamma}(\alpha)$ seems to be hard to compute, at least, directly from its definition. Hence our examples of simple crossed products could be given independently of the above-mentioned general theory. We show that the crossed product of a Cuntz algebra O_n [3] by a so-called quasi-free automorphism group [4] corresponding to a unitary representation u, on the n-dimensional Hilbert space, of a locally compact abelian group G, is simple if and only if the closed subsemigroup generated by $\mathrm{Sp}\, u$ and $-p$ equals the dual of G, for any $p\in \mathrm{Sp}\, u$ if $n<\infty$, and for $p=0$ if $n=\infty$. In some of those C^*-dynamical systems we can compute $\tilde{\Gamma}(\alpha)$.

We refer the reader to [7] for our terminology, definition and notation. But we denote the crossed product by $\mathfrak{a}\times_\alpha G$ rather than $G\times_\alpha \mathfrak{a}$.

§2. $\tilde{\Gamma}(\alpha)$

Let G be a locally compact abelian group with its dual Γ and let $(\mathfrak{a}, G, \alpha)$ be a C^*-dynamical system, i.e. α is a homomorphism of G into the automorphism group of $\mathfrak{a}$ such that $t \to \alpha_t(x)$ is norm-continuous for any $x \in \mathfrak{a}$. For $f \in L^1(G, dt)$ with a Haar measure dt, the map α_f on $\mathfrak{a}$ is defined by

$$\alpha_f(x) = \int f(t)\alpha_t(x)dt .$$

The α-spectrum, $\mathrm{Sp}_\alpha(x)$, of $x \in \mathfrak{a}$ is defined as $\cap \{z(f) : \alpha_f(x) = 0\}$ where

$$z(f) = \{p \in \Gamma : \hat{f}(p) = 0\}.$$

The spectral subspace $\mathfrak{a}^\alpha(\Omega)$ corresponding to a closed subset Ω of Γ is $\{x \in \mathfrak{a} : \mathrm{Sp}_\alpha(x) \subset \Omega\}$. The spectrum of α, $\mathrm{Sp}\,\alpha$, is the set of $p \in \Gamma$ such that for any closed neighbourhood Ω of p, $\mathfrak{a}^\alpha(\Omega)$ is non-zero. See, for detail, e.g. [2].

We define the *strong* spectrum of α, $\tilde{\mathrm{Sp}}(\alpha)$, as the set of p such that for any closed neighbourhood Ω of p, $\mathfrak{a}(\Omega) = \mathfrak{a}$, where $\mathfrak{a}(\Omega)$ is the closed linear span of $\mathfrak{a}^\alpha(\Omega)^* \cdot \mathfrak{a} \cdot \mathfrak{a}^\alpha(\Omega)$, which is, in general, a hereditary C^*-subalgebra of $\mathfrak{a}$.

We denote by $\mathscr{H}^\alpha(\mathfrak{a})$ the set of non-zero, α-invariant hereditary C^*-subalgebras of $\mathfrak{a}$. We define the *strong* Connes spectrum of α, $\tilde{\Gamma}(\alpha)$, by

$$\tilde{\Gamma}(\alpha) = \cap \tilde{\mathrm{Sp}}(\alpha \mid B), \quad B \in \mathscr{H}^\alpha(\mathfrak{a}).$$

The Connes spectrum $\Gamma(\alpha)$ is just defined without tildes in the above formula [6]. Our first proposition is quite obvious:

Proposition 2.1. (i) $0 \in \tilde{\mathrm{Sp}}(\alpha) \subset \mathrm{Sp}(\alpha)$ *and* $\tilde{\mathrm{Sp}}(\alpha)$ *is a closed subsemigroup of* Γ,

(ii) $0 \in \tilde{\Gamma}(\alpha) \subset \Gamma(\alpha)$ *and* $\tilde{\Gamma}(\alpha)$ *is a closed subsemigroup of* Γ.

For a subset H of Γ we denote by $S(H)$ the largest closed subsemigroup satisfying $H + S(H) \subset H$, i.e.

$$S(H) = \bigcap_{p \in H} H - p$$

when H is closed (c.f. [6]). We characterize $\tilde{\Gamma}(\alpha)$ by using covariant representations (π, u) of $(\mathfrak{a}, G, \alpha)$:

Proposition 2.2.

$$\tilde{\Gamma}(\alpha) = \bigcap_{(\pi, u)} \bigcap_{B \in \mathscr{H}^\alpha(\mathfrak{a}), \pi(B) \neq (0)} S(\mathrm{Sp}\, u e_{\pi(B)})$$

where $e_{\pi(B)}$ is the identity of the weak closure $\overline{\pi(B)}^w$ of $\pi(B)$.

Proof. A hereditary C^*-subalgebra B of $\mathfrak{a}$ is of the form $e\mathfrak{a}e \cap \mathfrak{a}$ by using an open projection e in $\mathfrak{a}^{**}$ and satisfies $B^{**} = e\mathfrak{a}^{**}e$. $e_{\pi(B)}$ is the image of e under the extension $\bar{\pi}$ of π to $\mathfrak{a}^{**}$, and commutes with $u(G)$ if B is α-invariant. From now on we denote $e_{\pi(B)}$ by e.

If $p \in \tilde{\mathrm{S}}\mathrm{p}\,(\alpha \mid B)$, then $\pi(B^\alpha(p+\Omega))\psi \neq (0)$ for any non-zero $\psi \in e\mathscr{H}_\pi$ and for any compact neighbourhood Ω of 0. This implies that $\mathrm{Sp}\, ue + p \subset \mathrm{Sp}\, ue$, i.e. $\mathrm{Sp}\, ue + \tilde{\mathrm{S}}\mathrm{p}\,(\alpha \mid B) \subset \mathrm{Sp}\, ue$.

Conversely let $p \notin \tilde{\mathrm{S}}\mathrm{p}\,(\alpha \mid B)$ with $B \in \mathscr{H}^\alpha(\mathfrak{a})$ and let Ω be a compact neighbourhood of p such that $B(\Omega) \neq B$. Since B and $B(\Omega)$ are α-invariant, we regard $B \times_\alpha G$ and $B(\Omega) \times_\alpha G$ as C^*-subalgebras of $\mathfrak{a} \times_\alpha G$ and we have that $B(\Omega) \times_\alpha G \subsetneqq B \times_\alpha G \subseteq \mathfrak{a} \times_\alpha G$ (Lemma 2.10 in [7]). Let ϕ be a pure state of $B \times_\alpha G$ such that $\phi \mid B(\Omega) \times_\alpha G = 0$ and let $\bar{\phi}$ be its (unique) extension to a state of $\mathfrak{a} \times_\alpha G$. We now have the covariant representation (π, u) of $(\mathfrak{a}, G, \alpha)$ associated with $\bar{\phi}$, e.g., $u(t) = \pi_{\bar{\phi}}(\lambda(t))$ $t \in G$ where $\lambda(\cdot)$ is the natural unitary group which implements α in the multiplier algebra $M(\mathfrak{a} \times_\alpha G)$ of $\mathfrak{a} \times_\alpha G$. Let e (resp. e_1) be the identity of $\overline{\pi(B)}$, the same as the one of $\overline{\pi_{\bar{\phi}}(B \times_\alpha G)}^w$, (resp. the identity of $\overline{\pi(B(\Omega))}^w$). Then $e, e_1 \in u(G)'$ and $e \gneqq e_1$. Let $q \in \mathrm{Sp}\, u(e-e_1)$ and let Ω_1 be a compact neighbourhood of q. With $e(\Omega_1)$ being the spectral projection of $u(e-e_1)$ corresponding to Ω_1, we have

$$[\pi(\mathfrak{a})e(\Omega_1)\mathscr{H}_\pi] = \mathscr{H}_\pi$$

since the projection onto the left hand side is in $\pi(\mathfrak{a})' \cap u(G)'$. Since

$$e[\pi(\mathfrak{a})e(\Omega_1)\mathscr{H}_\pi] = [\pi(B)e(\Omega_1)\mathscr{H}_\pi]$$

$$[\pi(B^\alpha(\Omega))e(\Omega_1)\mathscr{H}_\pi] = (0)$$

for any compact neighbourhood Ω_1 of q, we can easily conclude that $\mathrm{Sp}\, ue \not\ni p+q$.

q.e.d.

Remark 2.3. The intersection over the covariant representations can be restricted to the irreducible covariant representations (π, u), i.e. the ones with $\pi(\mathfrak{a})' \cap u(G)' = \mathbf{C} \cdot 1$, in the above proposition.

Remark 2.4. If $\tilde{\Gamma}(\alpha) \neq \Gamma$, it follows easily that $\mathfrak{a} \times_\alpha G$ is not simple (c.f. [9]).

We may consider the invariant in von Neumann algebra case corresponding to $\tilde{\Gamma}$, but it turns out to be the Connes spectrum. Hence $\tilde{\Gamma}$ can be considered as another version of the Connes spectrum (originally defined for von Neumann algebras).

§3. Simple crossed products

Let $(\mathfrak{a}, G, \alpha)$ be a C^*-dynamical system with a locally compact abelian group G. Let $K \equiv C(L^2(G))$ be the algebra of all compact operators on $L^2(G)$ and λ the regular representation of G on $L^2(G)$. Let $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes K$ and $\tilde{\alpha} = \alpha \otimes \mathrm{Ad}\,\lambda$ an action of G on $\tilde{\mathfrak{a}}$.

Lemma 3.1. $\tilde{\Gamma}(\tilde{\alpha}) = \tilde{\Gamma}(\alpha)$.

Proof. Suppose $p \not\in \tilde{\Gamma}(\alpha)$. Then there are a covariant representation (π, u) and $B \in \mathscr{H}^{\alpha}(\mathfrak{a})$ such that $\mathrm{Sp}\, ue \ni 0$ and $\mathrm{Sp}\, ue \not\ni p$ where e is the identity of $\overline{\pi(B)}^w$ (the assumption $\mathrm{Sp}\, ue \ni 0$ is always achieved by multiplying u by some character of G).

We construct a covariant representation $(\tilde{\pi}, \tilde{u})$ of $(\tilde{\mathfrak{a}}, G, \tilde{\alpha})$ by simply tensoring the identity representation of $(K, G, \mathrm{Ad}\,\lambda)$, e.g.

$$\tilde{u}(t) = u(t) \otimes \lambda(t), \quad t \in G.$$

Let Ω be a compact neighbourhood of 0 such that $p - \Omega \subset (\mathrm{Sp}\, ue)^c$ and let e_1 be the spectral projection of λ corresponding to Ω. Set $D = e_1 K e_1$. Then $D \in \mathscr{H}^{\mathrm{Ad}\lambda}(K)$ and $B \otimes D \in \mathscr{H}^{\tilde{\alpha}}(\tilde{\mathfrak{a}})$ and $e \otimes e_1$ is the identity of $\overline{\tilde{\pi}(B \otimes D)}^w$. We have that $\mathrm{Sp}\, \tilde{u} \cdot (e \otimes e_1) \not\ni p$ since

$$\mathrm{Sp}\, \tilde{u} \cdot (e \otimes e_1) \subset \mathrm{Sp}\, ue + \mathrm{Sp}\, \lambda e_1 = \mathrm{Sp}\, ue + \Omega.$$

Since $\mathrm{Sp}\, \tilde{u} \cdot (e \otimes e_1) \ni 0$, we can conclude that $\tilde{\Gamma}(\tilde{\alpha}) \not\ni p$.

Suppose $p \not\in \tilde{\Gamma}(\tilde{\alpha})$. Then there are a covariant representation (π, u) of $(\tilde{\mathfrak{a}}, G, \tilde{\alpha})$ and $B \in \mathscr{H}^{\tilde{\alpha}}(\tilde{\mathfrak{a}})$ such that $\mathrm{Sp}\, ue \ni 0$ and $\mathrm{Sp}\, ue \not\ni p$ where e is the identity of $\overline{\pi(B)}^w$.

First we assert that B can be chosen to satisfy that $B \supset C^*(G) B\, C^*(G)$, where $C^*(G)$ is identified with $1 \otimes C^*(G)$ in $M(\tilde{\mathfrak{a}})$. Set

$$L = \{x \in \tilde{\mathfrak{a}} : \pi(x) u(f) = 0;\ \forall f \in L^1(G),\ \mathrm{supp}\,\hat{f} \cap \mathrm{Sp}\, ue = \phi\},$$

where $u(f) = \int f(t) u(t) dt$. Then L is an $\tilde{\alpha}$-invariant closed left ideal of $\tilde{\mathfrak{a}}$ containing B. Since $\tilde{\pi}(C^*(G)) \subset u(G)'$, we have that $LC^*(G) \subset L$. Set $B_1 = L \cap L^*$. Then $B_1 \in \mathscr{H}^{\tilde{\alpha}}(\tilde{\mathfrak{a}})$ and B_1 satisfies the above assertion.

Now we assume that $B \supset C^*(G) B C^*(G)$. Hence $e \in \tilde{\pi}(C^*(G))'$. Let Ω be a compact neighbourhood of 0 such that $p + \Omega - \Omega \subset (\mathrm{Sp}\, ue)^c$. Let $e(q)$ be the spectral projection of λ corresponding to $q - \Omega$, in particular $e(q) \in M(\tilde{\mathfrak{a}})$. Since the supremum of $e(q)$, $q \in \Gamma$, in $M(\tilde{\mathfrak{a}})$ is 1, there is $q_0 \in \Gamma$ satisfying $\mathrm{Sp}\, ue\, \tilde{\pi}(e(q_0)) \ni 0$. Let $D = e(q_0) K e(q_0) \in \mathscr{H}^{\mathrm{Ad}\lambda}(K)$ and now we restrict the representation to $e(q_0) \tilde{\mathfrak{a}} e(q_0) \simeq \mathfrak{a} \otimes D \in \mathscr{H}^{\tilde{\alpha}}(\tilde{\mathfrak{a}})$ with the representation space $\mathscr{K} \equiv \tilde{\pi}(e(q_0)) \mathscr{H}_{\pi}$.

We define the following unitary representation $\hat{u}$ on $\mathscr{K}$:

$$\hat{u}(t)=u(t)\bar{\pi}(\lambda(-t))\langle t, q_0\rangle, \quad t\in G.$$

Then $\hat{u}$ commutes with $\bar{\pi}(D)$, $\mathrm{Ad}\,\hat{u}(t)(\bar{\pi}(a))=\bar{\pi}\circ\alpha_t(a)$ for $a\in\mathfrak{a}$, and

$$\mathrm{Sp}\,\hat{u}e\cap\Omega\neq\phi,$$

$$\mathrm{Sp}\,\hat{u}e\subset\mathrm{Sp}\,ue-\mathrm{Sp}\,\lambda e(q_0)+q_0\subset\mathrm{Sp}\,ue+\Omega\subset(p+\Omega)^c.$$

Set

$$L=\{x\in\mathfrak{a}\otimes D\colon \pi(x)\hat{u}(f)=0, \quad \forall f\in L^1, \quad \mathrm{supp}\hat{f}\subset p+\Omega\}$$

Then L is an $\alpha\otimes$id.-invariant closed left ideal of $\mathfrak{a}\otimes D$ containing $e(q_0)Be(q_0)$. Further $LD\subset L$. Hence L is of the form $L_1\otimes D$ where L_1 is an α-invariant closed left ideal of $\mathfrak{a}$. Set $B_1=L_1\cap L_1^*$, and let e_1 be the identity of $\bar{\pi}(B_1)$. Then $\mathrm{Sp}\,\hat{u}e_1\subset(p+\Omega)^c$ and $\mathrm{Sp}\,\hat{u}e_1\cap\Omega\neq\phi$. Thus $p\notin\mathrm{Sp}\,\hat{u}e_1-q$ for $q\in\mathrm{Sp}\,\hat{u}e_1\cap\Omega$, i.e. $p\notin\tilde{\Gamma}(\alpha)$.

Now we consider the dual system $(\mathfrak{a}\times_\alpha G, \Gamma, \hat{\alpha})$ and characterize $\tilde{G}(\hat{\alpha})$, similarly to Lemma 3.2 in [7];

Lemma 3.2.

$$\tilde{G}(\hat{\alpha})=\{t\in G\colon \alpha_t(I)\subset I \quad \text{for any ideal } I \text{ of } \mathfrak{a}\}.$$

Proof. Let I be an ideal of $\mathfrak{a}$ and let $t\in G$. Suppose that $\alpha_t(I)\not\subset I$.

Let (Ω_ι) be a net of compact neighbourhoods of 0 such that $\cap\,\Omega_\iota=(0)$. Let

$$I_\Omega=\bigcap_{s\in\Omega}\alpha_s(I).$$

We assert that $\cup_\iota I_{\Omega_\iota}$ is dense in I. For let $x\in I$ be positive and find positive elements e_n and x_n in the C^*-subalgebra generated by x such that

$$e_nx_n=x_n, \quad \|x-x_n\|\leq 1/n.$$

Then by Lemma 3.2 in [7] there is Ω_ι such that $\alpha_s(x_n)\in I$ for $s\in-\Omega_\iota$. Thus $x_n\in I_{\Omega_\iota}$. Hence the closure of $\cup_\iota I_{\Omega_\iota}$ contains x.

Suppose that $I_{\Omega_\iota-\Omega_\iota}\subset\alpha_{-t}(I_{\Omega_\iota})$ for any ι. Since $\cup\,I_{\Omega_\iota-\Omega_\iota}$ is also dense in I, this would imply that $I\subset\alpha_{-t}(I)$, a contradiction. Thus there is ι such that $I_{\Omega_\iota-\Omega_\iota}\not\subset\alpha_{-t}(I_{\Omega_\iota})$.

Let $J=I_{\Omega_\iota-\Omega_\iota}$. Then since the ideal $\mathrm{V}_{s\in\Omega_\iota}\alpha_s(J)$ generated by $\alpha_s(J)$, $s\in\Omega_\iota$ is contained in I_{Ω_ι}, we have that

$$J\not\subset\bigvee_{s\in\Omega_\iota}\alpha_{s-t}(J).$$

Let $B=\overline{J\cdot \mathfrak{a}\times_\alpha G\cdot J}\in\mathscr{H}^{\hat\alpha}(\mathfrak{a}\times_\alpha G)$ and let $\Omega=\Omega_\iota$. Then $B^{\hat\alpha}(t-\Omega)$ is the closed linear span of

$$x\lambda(f)y,\quad x,\, y\in J,\quad f\in L^1\cap L^2\quad\text{with}\quad \operatorname{supp} f\subset t-\Omega.$$

Hence the hereditary $*$-algebra generated by elements of the form $y^*\lambda(f)^*x^*x\lambda(f)y$ is dense in $B(t-\Omega)$.

$a\equiv y^*\lambda(f)^*x^*x\lambda(f)y$ is $\hat\alpha$-integrable, i.e. there is a positive $I(a)$ in $M(\mathfrak{a}\times_\alpha G)$ (in fact in $\mathfrak{a}\subset M(\mathfrak{a}\times_\alpha G)$) such that

$$\phi(I(a))=\int_\Gamma \phi(\hat\alpha_p(a))dp$$

for every $\phi\in(\mathfrak{a}\times_\alpha G)^*$. Explicitly

$$I(a)=\int |f(s)|^2 y^*\alpha_{-s}(x^*x)yds\in J\cdot\bigvee_{s\in\Omega}\alpha_{s-t}(J)\equiv J_1\,.$$

Since $B(t-\Omega)$ is $\hat\alpha$-invariant, it follows that $I(a)\in B(t-\Omega)^{**}$. Hence the hereditary $*$-algebra generated by elements of the form $I(a)$ is contained in $B(t-\Omega)$ and of course is dense in $B(t-\Omega)$. Hence

$$B(t-\Omega)\subset\overline{J_1\cdot\mathfrak{a}\times_\alpha G\cdot J_1}\,.$$

Since $J_1\subsetneqq J$, $B(t-\Omega)\neq B$, i.e. $t\not\in\tilde G(\hat\alpha)$.

Suppose that $t\not\in\tilde G(\hat\alpha)$. Then there are a covariant representation (π, u) of $(\mathfrak{a}\times_\alpha G, \Gamma, \hat\alpha)$ and $B\in\mathscr{H}^{\hat\alpha}(\mathfrak{a}\times_\alpha G)$ such that $\operatorname{Sp} ue\ni 0$ and $\operatorname{Sp} ue\not\ni t$ where e is the identity of $\overline{\pi(B)}^w$. Let Ω be a compact neighbourhood of $0\in G$ such that $t+\Omega-\Omega\subset(\operatorname{Sp} ue)^c$. Then for any $x\in B$ and $s\in\Omega$,

$$\pi(x)\bar\pi(\lambda(s))u(f)=0,\quad f\in L^1(\Gamma)\quad\text{with}\quad \operatorname{supp}\hat f\subset t+\Omega.$$

Let L be the left ideal of $\mathfrak{a}\times_\alpha G$ with $B=L\cap L^*$. Then from the above calculation, the left ideal L_1 generated by

$$\cup\, L\lambda(f),\quad f\in L^1(G)\quad\text{with}\quad \operatorname{supp} f\subset\Omega,$$

satisfies that for $x\in L_1$,

$$\pi(x)u(f)=0,\quad f\in L^1(\Gamma)\quad\text{with}\quad \operatorname{supp}\hat f\subset t+\Omega. \tag{$*$}$$

Set $B_1=L_1\cap L_1^*\in\mathscr{H}^{\hat\alpha}(\mathfrak{a}\times_\alpha G)$ and let e_1 be the identity of $\overline{\pi(B_1)}^w$. Then $\operatorname{Sp} ue_1\ni 0$ and $\operatorname{Sp} ue_1\not\ni t$.

The positive cone of B_1 has a total set of $\hat\alpha$-integrable elements of the form $a=\lambda(f)^*x^*x\lambda(f)$, $x\in B$, $f\in L^1\cap L^2$ with $\operatorname{supp} f\subset\Omega$. Let J be the ideal of $\mathfrak{a}$ generated by elements $I(a)$ with all such a. Then $B_1\subset\overline{J\cdot\mathfrak{a}\times_\alpha G\cdot J}$. Since $\overline{J\cdot\mathfrak{a}\times_\alpha G\cdot J}$

is generated by elements of the form $x_1 I(a_1) y I(a_2) x_2$ with $x_i \in \mathfrak{a}$, $y \in \mathfrak{a} \times_\alpha G$ and since $I(a_1) y I(a_2) \in B_1$, we have that $\overline{J \cdot \mathfrak{a} \times_\alpha G \cdot J} = \overline{J B_1 J}$.

Set $B_2 = \overline{J \cdot \mathfrak{a} \times_\alpha G \cdot J} \in \mathscr{H}^{\hat{\alpha}}(\mathfrak{a} \times_\alpha G)$. Then $x \in B_2$ satisfies (*) since $\bar{\pi}(\mathfrak{a})$ commutes with $u(\Gamma)$. Hence there is a compact neighbourhood Ω_1 of t such that $B_2(\Omega_1) \neq B_2$.

Since $B_1^{\hat{\alpha}}(\Omega_1)$ is the closed linear span of

$$x\lambda(f)y, \; x, \; y \in J, \; f \in L^1 \cap L^2 \quad \text{with} \quad \operatorname{supp} f \subset \Omega_1,$$

similarly to the first part of the proof, $B_2(\Omega_1)$ is the hereditary C^*-subalgebra generated by elements of the form

$$I(y^*\lambda(f)^* x^* x \lambda(f) y) = \int |f(s)|^2 y^* \alpha_{-s}(x^* x) y ds .$$

Hence $B_2(\Omega_1) \supset \alpha_{-t}(J) J \cdot \mathfrak{a} \times_\alpha G \cdot \alpha_{-t}(J) J$, which implies that $J \not\perp \alpha_{-t}(J)$. q.e.d.

Here we give a comment. Our reference on $\hat{\alpha}$-integrability [2.4, 7] contains an error in the definition of I. The correct form should be the one given in the above proof (otherwise Lemma 2.6 in [7] would fail), i.e. $a \in M(B)_+$ is β-integrable if there is a (necessarily unique) $I(a) \in M(B)$ such that

$$\phi(I(a)) = \int_\Gamma \phi \circ \beta_p(a) dp$$

for every $\phi \in B^*$, where $B = \mathfrak{a} \times_\alpha G$ and $\beta = \hat{\alpha}$ in this case. Here $p \to \phi \circ \beta_p(a)$ is continuous.

To prove that the β-integrable elements are hereditary, we adopt an argument similar to the one in [2.4, 7] by using, e.g., Lemma 2.1 in [1], although this fact is not quite necessary in the above proof, because we have considered only elements of the form $\lambda(f)^* x^* x \lambda(f)$ (or $y^* \lambda(f)^* x^* x \lambda(f) y$ if $\hat{\alpha}_p(y) = y$, $p \in \Gamma$) as integrable elements, which is justified by Proposition 2.8 in [7].

Lemma 3.3 [10]. *The C^*-dynamical system* $(\mathfrak{a} \times_\alpha G \times_{\hat{\alpha}} \Gamma, G, \hat{\hat{\alpha}})$ *is covariantly isomorphic to* $(\tilde{\mathfrak{a}}, G, \tilde{\alpha})$.

Lemma 3.4.

$$\tilde{\Gamma}(\alpha) = \{p \in \Gamma : \hat{\alpha}_p(I) \subset I \text{ for any ideal } I \text{ of } \mathfrak{a} \times_\alpha G\}$$

Proof. It follows from Lemmas 3.2 and 3.3.

By the above lemma and Lemma 3.1 in [7] we have

Theorem 3.5. *Let* $(\mathfrak{a}, G, \alpha)$ *be as above. The following conditions are equivalent*:

(i) $\mathfrak{a}\times_\alpha G$ *is simple*;

(ii) $\mathfrak{a}$ *is α-simple and* $\tilde{\Gamma}(\alpha)=\Gamma$.

As a corollary to Lemma 3.4 we give

Proposition 3.6. *Let* $(\mathfrak{a}, G, \alpha)$ *be as above. Suppose that there is another C^*-dynamical system* (B, G, β) *which is exteriorly equivalent to* $(\mathfrak{a}, G, \alpha)$. *Then* $\tilde{\Gamma}(\alpha)=\tilde{\Gamma}(\beta)$.

Proof. $(\mathfrak{a}\times_\alpha G, \Gamma, \hat{\alpha})$ is covariantly isomorphic to $(B\times_\beta G, \Gamma, \hat{\beta})$. q.e.d.

It seems more difficult to compute $\tilde{\Gamma}(\alpha)$ than $\Gamma(\alpha)$ in most of cases. But sometimes $\tilde{\Gamma}(\alpha)$ coincides with $\Gamma(\alpha)$. We shall show some of these cases.

The following lemma can be found, e.g. in [11, Lemma 22]:

Lemma 3.7. *Suppose that* $\mathfrak{a}$ *is α-simple and that G/G_I is compact for any primitive ideal I of* $\mathfrak{a}$, *where*

$$G_I=\{t\in G\colon \alpha_t(I)=I\}\,.$$

Then the primitive ideal space of $\mathfrak{a}$ *with the transposed action of G is isomorphic to G/G_0 with the action of G by translations, where $G_0=G_I$ for any primitive ideal I.*

Proof. Let I be a primitive ideal, and let (Ω_ι) be a net of compact neighbourhoods of $0\in G/G_I$ such that $\cap\,\Omega_\iota=(0)$. Since $\alpha_t(I)=\alpha_s(I)$ if t and s in G have the same image $\dot{t}=\dot{s}$ in G/G_I, we can define

$$I(\dot{t}+\Omega_\iota)=\bigcap_{\dot{s}\in\Omega_\iota}\alpha_{t+s}(I)\,.$$

There is a finite set S_ι of G/G_I such that

$$\cup_{\dot{s}\in S_\iota}(\dot{s}+\Omega_\iota)=G/G_I\,.$$

Since $\mathfrak{a}$ is α-simple, we have

$$\bigcap\nolimits_{\dot{s}\in S_\iota}I(\dot{s}+\Omega_\iota)=(0)\,.$$

Let J be a primitive ideal. Since $\bigcap_{\dot{s}\in S_\iota}I(\dot{s}+\Omega_\iota)\subset J$, there is an $\dot{s}_\iota\in S_\iota$ such that

$$I(\dot{s}_\iota+\Omega_\iota)\subset J\,.$$

Since G/G_I is compact, we may suppose that $\dot{s}_\iota$ converges, say to $\dot{s}$. Then $\dot{s}+\Omega_\iota\ni\dot{s}_j$ for sufficiently large $j\geq\iota$. Hence $\dot{s}+\Omega_\iota+\Omega_\iota\supset\dot{s}_j+\Omega_j$ which implies

$$I(\dot{s}+\Omega_\iota+\Omega_\iota)\subset J\,.$$

We have that $\alpha_s(I) \subset J$, since $\cup_\iota I(\dot{s}+\Omega_\iota+\Omega_\iota)$ is dense in $\alpha_s(I)$, as shown from the first part of the proof of Lemma 3.2 and the fact that the quotient map $G \to G/G_I$ is open. Similarly we get that $\alpha_t(J) \subset I$ for some $t \in G$, i.e.

$$\alpha_{s+t}(J) \subset \alpha_s(I) \subset J .$$

Since G/G_J is compact, we can conclude that $\alpha_{s+t}(J)=J$. Thus $\alpha_s(I)=J$.

Hence the set of primitive ideals is $\{\alpha_t(I): \dot{t} \in G/G_I\}$, i.e. there is a one-one correspondence between the primitive ideal space and G/G_I, by choosing one primitive ideal I of $\mathfrak{a}$, which obviously preserves the actions of G. For any subset S of G/G_I, we have

$$\bigcap_{\dot{s} \in S} \alpha_s(I) = \bigcap_{\dot{s} \in S} \alpha_s(I) .$$

By the same argument as above,

$$\bigcap_{\dot{s} \in S} \alpha_s(I) \subset \alpha_t(I)$$

implies that $\dot{t} \in \bar{S}$. Hence the closure operations coincide through the correspondence. q.e.d.

Proposition 3.8. *Let $(\mathfrak{a}, G, \alpha)$ be a C^*-dynamical system where G is a discrete abelian group and $\mathfrak{a}$ is α-simple. Then $\Gamma(\alpha)=\tilde{\Gamma}(\alpha)$.*

Proof. We apply Lemma 3.7 to the dual system $(\mathfrak{a} \times_\alpha G, \Gamma, \hat{\alpha})$, where now Γ is compact, and use the formula for $\tilde{\Gamma}(\alpha)$ in Lemma 3.4 and the one for $\Gamma(\alpha)$ in Corollary 5.4 in [7].

Proposition 3.9. *Let $(\mathfrak{a}, G, \alpha)$ be a separable C^*-dynamical system (i.e. both $\mathfrak{a}$ and G are separable). Suppose that G_I is discrete for any primitive ideal I of $\mathfrak{a}$ where*

$$G_I = \{t \in G: \alpha_t(I) = I\} .$$

If $\mathfrak{a}$ is α-simple, then the primitive ideal space of $\mathfrak{a} \times_\alpha G$ with the transposed action of $\hat{\alpha}$ of Γ is isomorphic to Γ/H (with action of Γ) for some closed subgroup H of $\hat{G}$ such that Γ/H is compact, in particular $\tilde{\Gamma}(\alpha)=\Gamma(\alpha)(=H)$.

Proof. (c.f. [8, Theorem 3.1]) By [5, Corollary 3.2], for any primitive ideal I, Γ/Γ_I is compact. Apply Lemma 3.7. For the last statement, see the proof of Proposition 3.8.

For applications of the above proposition we refer the reader to [8].

§4. Crossed products of O_n

Let $\mathscr{H}_n$ be an n-dimensional Hilbert space ($2 \leq n \leq \infty$), and let $F(\mathscr{H}_n)$ be the Fock Hilbert space over $\mathscr{H}_n$. For each $f \in \mathscr{H}_n$, $O(f)$ is a bounded operator defined by

$$O(f) g_1 \otimes \cdots \otimes g_n = f \otimes g_1 \otimes \cdots \otimes g_n$$
$$O(f)\Omega = f$$

where Ω is the vacuum vector in $F(\mathscr{H}_n)$. If $\|f\| = 1$, $O(f)$ is an isometry. We denote by $O(\mathscr{H}_n)$ the C^*-algebra generated by $O(f)$, $f \in \mathscr{H}_n$.

If n is finite, the Cuntz algebra O_n [3] is isomorphic to the quotient of $O(\mathscr{H}_n)$ by the compact operator algebra on $F(\mathscr{H}_n)$ and if $n = \infty$, O_n is isomorphic to $O(\mathscr{H}_n)$. Each unitary u on $\mathscr{H}_n$ induces an automorphism of O_n through that of $O(\mathscr{H}_n)$ defined by

$$O(f) \longrightarrow O(uf), \quad f \in \mathscr{H}_n .$$

We call *quasi-free* those automorphisms of O_n obtained in this way. See, for the detail, Evance [4].

From now on we assume that n is finite.

Let G be a locally compact abelian group with its dual Γ, as before, and let u be a continuous unitary representation of G on $\mathscr{H}_n$. By thinking of elements of the form

$$O(f_1) \cdots O(f_n) O(g_1)^* \cdots O(g_m)^* ,$$

it is clear that $\mathrm{Sp}\,\alpha$ is the closed subgroup of Γ generated by $\mathrm{Sp}\,u$.

Lemma 4.1. *Let (O_n, G, α) be as above. Then $\Gamma(\alpha) = \mathrm{Sp}\,\alpha$.*

Proof. Let $(\phi_i)_{i=1}^n$ be an orthonormal system in $\mathscr{H}_n$ such that $u_t \phi_i = \langle t, p_i \rangle \phi_i$ with $p_i \in \Gamma$. Let S_i be the image of $O(\phi_i)$ into O_n.

Set for each $k = 1, 2, \dots$

$$S_i^{(k)} = \sum_{\mu : l(\mu) = k} S_\mu S_i S_\mu^* ,$$

where the summation is taken over all the words μ of $\{1, \dots, n\}$ with length $l(\mu) = k$, and $S_{\{i_1, \dots, i_k\}} = S_{i_1} \cdots S_{i_k}$ (c.f. [3]). All $S_i^{(k)}$ are isometries and satisfy that $\lim \|[S_i^{(k)}, x]\| = 0$ for any $x \in F^n \subset O_n$ where F^n is the algebra of fixed points under the gauge automorphism group γ, i.e. the quasi-free automorphism group induced by $\{z \cdot 1;\ |z| = 1\}$ on $\mathscr{H}_n$.

Let ν be an infinite aperiodic sequence of letters $\{1,\dots, n\}$ (c.f. Lemma 1.8 in [3]) and let ν_m be the restriction of ν to the first m letters. Set

$$Q_m = \sum_{\mu, l(\mu)=m} S_\mu S_{\nu_m} S_{\nu_m}^* S_\mu^* .$$

Then $\{Q_m\}$ are γ- and α-invariant projections and satisfy

$$\lim \|Q_m \varepsilon(x) Q_m - Q_m x Q_m\| = 0$$
$$\lim \|Q_m x Q_m\| = \|\varepsilon(x)\|$$

where $\varepsilon(x) = \int \gamma_t(x)dt$ is the projection of O_n onto F^n (c.f. Proposition 1.7 in [3]). Then for any positive $x \in O_n$,

$$\lim_m \lim_k \|Q_m x Q_m S_i^{(k)} x\| = \lim \lim \|S_i^{(k)} Q_m \varepsilon(x) Q_m x\|$$
$$= \lim_m \|Q_m \varepsilon(x) Q_m x\| \geq \|\varepsilon(x)\|^2 .$$

Let $B \in \mathscr{H}^\alpha(O_n)$ and Ω a compact neighbourhood of 0. Then there is a non-zero positive $x \in B$ with $\mathrm{Sp}_\alpha(x) \subset \Omega$. It follows from the above calculation that there are m and k such that $xQ_m S_i^{(k)} x \neq 0$. This implies that

$$\mathrm{Sp}\,\alpha \,|\, B \cap (p_i + \Omega + \Omega) \neq \phi$$

Since Ω and B are arbitrary, we can conclude that $\Gamma(\alpha) \ni p_i$. Thus $\Gamma(\alpha) = \mathrm{Sp}\,\alpha$ since $\Gamma(\alpha)$ is a closed subgroup [6].

We denote by $\bar{\gamma}$ the extension of the gauge action γ to an action on $O_n \times_\alpha G$. This is possible because γ commutes with α. In the following we denote by H the intersection of the closed subsemigroups of Γ generated by $\mathrm{Sp}\,u$ and $-p$, with $p \in \mathrm{Sp}\,u$.

Lemma 4.2. *Let (O_n, G, α) be as above. Then $H \supset \tilde{\Gamma}(\alpha)$.*

Proof. We construct certain α-invariant states of O_n. For $i = 1,\dots, n$ and $k = 1, 2,\dots$, set

$$P_i^{(k)} = S_i^k S_i^{*k} .$$

Then $\{P_i^{(k)}\}_{k=1,2,\dots}$ is a decreasing sequence of γ-invariant projections. Let ϕ_i be a γ-invariant state satisfying

$$\phi_i(x) = \phi_i(P_i^{(k)} x P_i^{(k)}), \quad x \in O_n, \quad k = 1, 2, \dots .$$

Then it is shown that ϕ_i is α-invariant (and in fact unique) and that the continuous functions on $G \ni t \to \phi(x\alpha_t(y))$ for $x, y \in O_n$ are contained in the closed algebra generated by characters $p_1, \ldots, p_n$ and $-p_i$, e.g.

$$\phi_i(x\alpha_t(S_\mu S_\nu^*)) = \langle t, p_{j_1} + \cdots + p_{j_m} - p_{k_1} - \cdots - p_{k_n} \rangle \phi_i(xS_\mu S_\nu^*)$$

is non-zero only if $\nu \equiv \{k_1, \ldots, k_n\} = \{i, \ldots, i\}$, where $\mu = \{j_1, \ldots, j_m\}$. This implies that in the GNS representation associated with ϕ_i, the canonical representation U of G defined by

$$U_t \pi_{\phi_i}(x)\Omega_{\phi_i} = \pi_{\phi_i} \circ \alpha_t(x)\Omega_{\phi_i}, \quad x \in O_n$$

has spectrum in the closed subsemigroup H_i generated by $p_1, \ldots, p_m$ and $-p_i$. By Proposition 2.2 we have that $\tilde{\Gamma}(\alpha) \subset H_i$.

Lemma 4.3. *Let (O_n, G, α) be as above. Then for any $\bar{\gamma}$-invariant ideal I of $O_n \times_\alpha G$, it follows that $\hat{\alpha}_p(I) \subset I$ for $p \in H$.*

Proof. Let ρ be a representation of $O_n \times_\alpha G$ whose kernel is $\bar{\gamma}$-invariant. Let $x \in O_n^\gamma \times_\alpha G \equiv (O_n \times_\alpha G)^{\bar{\gamma}}$ or be of the form $\sum a_i \otimes f_i$ with $a_i \in O_n^\gamma$, $f_i \in C^*(G)$. Then since $\lim \|S_i^{(k)*} x S_i^{(k)} - \hat{\alpha}_{p_i}(x)\| = 0$, we have

$$\lim_k \|\rho(S_i^{(k)*} x S_i^{(k)})\| = \|\rho \circ \hat{\alpha}_{p_i}(x)\| .$$

The left hand side equals

$$\lim_k \|\rho(x)\rho(S_i^{(k)} S_i^{(k)*})\|$$

since $S_i^{(k)} S_i^{(k)*}$ asymptotically commutes with x. Further since $S_i^{(k)} S_i^{(k)*}$ are projections and $\sum_i S_i^{(k)} S_i^{(k)*} = 1$, we have

$$\|\rho(x)\| \geq \|\rho \circ \hat{\alpha}_{p_i}(x)\|$$

$$\|\rho(x)\| = \max_i \|\rho \circ \hat{\alpha}_{p_i}(x)\| \tag{*}$$

For a fixed $x \in O_n^\gamma \times_\alpha G$ we can find an infinite sequence $\{i_k\}$ of $\{1, \ldots, n\}$ such that

$$\|\rho(x)\| = \|\rho \circ \hat{\alpha}_{p_{i_1} + \cdots + p_{i_k}}(x)\| \tag{**}$$

for all $k = 1, 2, \ldots$. There is an $i \in \{1, 2, \ldots, n\}$ which infinitely often appears in $\{i_k\}$. For such an i we have

$$\|\rho(x)\| = \|\rho \circ \hat{\alpha}_{np_i}(x)\|$$

for $n = 1, 2, \ldots$, since for any subset J of $\{1, \ldots, k\}$, (**) is less than or equal to

$$\|\rho\circ\hat{\alpha}_{(\sum_{j\in J} p_{ij})}(x)\| \leq \|\rho(x)\| .$$

Let $p\in H$. By the assumption there is a sequence in the subsemigroup generated by $\{p_1,\dots, p_n, -p_i\}$ which converges to p, i.e. there is a sequence

$$q_l=\sum_k n_k^{(l)}p_k-m^{(l)}p_i-p,\ n_k^{(l)}\geq 0,\ m^{(l)}\geq 0$$

which converges to zero in Γ. Then

$$\begin{aligned}\|x-\hat{\alpha}_{q_l}(x)\| &\geq \|\rho\circ\hat{\alpha}_{m^{(l)}p_i}(x)-\rho\circ\alpha_{(\Sigma n_k^{(l)}p_k-p)}(x)\| \\ &\geq \|\rho(x)\|-\|\rho\circ\hat{\alpha}_{-p}(x)\|\end{aligned}$$

which implies that $\|\rho\circ\hat{\alpha}_{-p}(x)\|\geq\|\rho(x)\|$. Hence

$$\hat{\alpha}_p(I)\cap O_n^\gamma\times_\alpha G\subseteq I\cap O_n^\gamma\times_\alpha G$$

where I is the kernel of ρ. Since I is generated by $I\cap O_n^\gamma\times_\alpha G$, we can conclude that $\hat{\alpha}_p(I)\subset I$. q.e.d.

Theorem 4.4. *Let (O_n, G, α) be as above. The crossed product $O_n\times_\alpha G$ is simple if and only if the closed subsemigroup of Γ generated by* Sp u *and* $-p$ *is Γ itself for any $p\in$* Sp u.

Proof. Since $H\supset\tilde{\Gamma}(\alpha)$ by Lemma 4.2, if $H\neq\Gamma$, $O_n\times_\alpha G$ is not simple by Theorem 3.5.

Suppose $H=\Gamma$. Then Lemma 4.3 implies that any $\bar{\gamma}$-invariant ideal is $\hat{\alpha}$-invariant. Since O_n is simple [3], there are not any non-trivial $\hat{\alpha}$-invariant ideals of $O_n\times_\alpha G$. Thus $O_n\times_\alpha G$ is $\bar{\gamma}$-simple. Since $O_n\times_\alpha G$ is prime by Lemma 4.1 and [7, Theorem 5.8], it follows from [7, Lemma 6.4] (or Lemma 3.7) that $O_n\times_\alpha G$ is simple.

Proposition 4.5. *Let (O_n, G, α) be as above and suppose that $\alpha(G)$ contains the gauge automorphism group γ. Then $\tilde{\Gamma}(\alpha)$ is the intersection of the closed subsemigroups of Γ generated by* Sp u *and* $-p$, *with* $p\in$ Sp u.

Proof. Since $\bar{\gamma}$ is inner under the above assumption, any ideal of $O_n\times_\alpha G$ is $\bar{\gamma}$-invariant. The rest of the proof follows from Lemmas 4.2, 4.3 and 3.4. q.e.d.

§5. Crossed products of O_∞

In this section we consider the case $n=\infty$. As in Section 4, let u be a weakly continuous unitary representation of a locally compact group G on a separable infinite-dimensional Hilbert space $\mathscr{H}$ and let α be the corresponding quasi-free action on $O_\infty = O(\mathscr{H})$. It is immediate that $\mathrm{Sp}\,\alpha$ is the closed subgroup generated by $\mathrm{Sp}\,u$.

Let $F = F(\mathscr{H}) = \sum_0^\infty \mathscr{H}^{\otimes n}$ be the Fock space and π_F the Fock representation of O_∞ on F [4]. Let U_F be the canonical representation of G on the Fock space, i.e.

$$U_F(t) \mid \mathscr{H}^{\otimes n} = u_t \otimes \cdots \otimes u_t \ (n\text{-tuples}) \equiv u_t^{\otimes n}.$$

It is clear that $\mathrm{Sp}\,U_F$ is the closed subsemigroup H generated by $\mathrm{Sp}\,u$. The pair (π_F, U_F) gives a representation $\pi_F \times U_F$ of $O_\infty \times_\alpha G$ in an obvious way. If $H \neq \Gamma$, $\pi_F \times U_F$ is not faithful, in particular $O_\infty \times_\alpha G$ is not simple (c.f. [9]).

Theorem 5.1. *Let (O_∞, G, α) be as above. The crossed product $O_\infty \times_\alpha G$ is simple if and only if the closed subsemigroup H generated by* $\mathrm{Sp}\,u$ *is Γ.*

Proof. We have shown that if $H \neq \Gamma$, then $O_\infty \times_\alpha G$ is not simple. Hence we now assume that $H = \Gamma$.

First we want to show that $\pi_F \times U_F$ is faithful. Since $\pi_F \times U_F$ is irreducible, this in particular implies that $O_\infty \times_\alpha G$ is prime.

For each $n = 1, 2, \ldots$, there is a natural unitary map W_n from $F \otimes \mathscr{H}^{\otimes n}$ onto $\sum_n^\infty \mathscr{H}^{\otimes k} \subset F$, such that $W_n(\psi \otimes \phi) = \sum_0^\infty \psi_k \otimes \phi$ where $\psi = \sum_0^\infty \psi_k$ with $\psi_k \in \mathscr{H}^{\otimes k}$ and $\psi_k \otimes \phi \in \mathscr{H}^{\otimes k+n}$. Note that

$$W_n \cdot U_F(t) \otimes u_t^{\otimes n} = U_F(t) W_n.$$

In the following, however, we omit W_n.

Let (Ω_ι) be a decreasing sequence of compact neighbourhoods of 0 in Γ such that $\cap\, \Omega_\iota = (0)$. There are $p_\iota \in \mathrm{Sp}\,u$ and $\phi_\iota \in \mathscr{H}$ with $\|\phi_\iota\| = 1$ such that $\mathrm{Sp}_u \phi_\iota \subset p_\iota + \Omega_\iota$ and ϕ_ι tends to zero weakly.

Let $p \in \Gamma$. Then, since $\mathrm{Sp}\,U_F = \Gamma$ due to the assumption, there are positive integers m_ι and $\psi_\iota \in \mathscr{H}^{\otimes m_\iota}$ with $\|\psi_\iota\| = 1$ such that $\mathrm{Sp}_{\,U_F}\,\psi_\iota \subset p - p_\iota + \Omega_\iota$.

Let $x = \sum a_k \otimes f_k \in O_\infty \times_\alpha G$ where a_k are monomials (i.e. of the type $O(f_1) \cdots O(f_i) O^*(g_1) \cdots O^*(g_j)$) and $f_k \in L^1(G)$.

Note that for any $\psi \in F$,

$$\lim \left\| \int f_k(t) U_F(t) (\psi \otimes \phi_\iota \otimes \psi_\iota) dt - \int f_k(t) \langle t, p \rangle (U_F(t)\psi) \otimes \phi_\iota \otimes \psi_\iota \, dt \right\| = 0$$

$$\lim \|\pi_F(a_k)(\psi \otimes \phi_\iota \otimes \psi_\iota) - (\pi_F(a_k)\psi) \otimes \phi_\iota \otimes \psi_\iota\| = 0.$$

Hence we have that

$$\lim \|(\pi_F \times U_F)(x)(\psi \otimes \phi_\iota \otimes \psi_\iota) - ((\pi_F \times U_F) \circ (\hat{\alpha}_p(x)\psi)) \otimes \phi_\iota \otimes \psi_\iota\| = 0.$$

Since $\psi \in F$ is arbitrary, we have that $\|\pi_F \times U_F(x)\| \geqq \|(\pi_F \times U_F) \circ \hat{\alpha}_p(x)\|$, which in turn implies, since p is arbitrary,

$$\|(\pi_F \times U_F)\circ\hat{\alpha}_p(x)\| = \|(\pi_F \times U_F)(x)\|, \quad p \in \Gamma.$$

Since O_∞ is simple [3], we can conclude that $\pi_F \times U_F$ is faithful.

Now let $\pi \times U$ be any irreducible representation of $O_\infty \times_\alpha G$. By a similar reason as given in the proof of Theorem 4.4 it suffices to show that $\ker(\pi \times U) \cap O_\infty^\gamma \times_\alpha G = (0)$.

Let (f_i) be a complete orthonormal system of $\mathscr{H}$. Then $\pi(O(f_i)O^*(f_i))$ are mutually orthogonal projections. Let

$$P = \sum_{i=1}^{\infty} \pi(O(f_i)O^*(f_i)).$$

P is a projection in $U(G)'$ which is independent from choice of (f_i).

First suppose that $P \neq 1$. Let e be a projection in $U(G)'$ such that $0 \neq e \leqq 1-P$. Then by the irreducibility, we have that $[\pi(O_\infty)e\mathscr{H}_\pi] = \mathscr{H}_\pi$. But $(1-P)[\pi(O_\infty)e\mathscr{H}_\pi] = e\mathscr{H}_\pi$, i.e. $1-P=e$. Hence $1-P$ is one-dimensional. Now it is easily shown that π is equivalent to π_F and that $U = pU_F$ with some $p \in \Gamma$. Hence the faithfulness of $\pi \times U \simeq \pi_F \times U_F\circ\hat{\alpha}_p$ follows from the above.

Suppose that $P=1$. Let $p \in \mathrm{Sp}\, u + \cdots + \mathrm{Sp}\, u$ (k terms). There is a sequence of unit vectors $\psi_{1\iota} \otimes \cdots \otimes \psi_{k\iota} \in \mathscr{H}^{\otimes k}$ such that $\mathrm{Sp}_{\,U_F}(\psi_{1\iota} \otimes \cdots \otimes \psi_{k\iota}) \subset p + \Omega_\iota$ with Ω_ι given before. Now we define a family of isometries:

$$V_\iota^{(0)} = \pi(O(\psi_{1\iota})\cdots O(\psi_{k\iota}))$$
$$V_\iota^{(m)} = \sum \pi(O(f_i))V_\iota^{(m-1)}\pi(O^*(f_i)).$$

Note that $V_\iota^{(m)}$ does not depend on (f_i).

Let $x = \sum a_k \otimes f_k \in O_\infty^\gamma \times_\alpha G$ where a_k are monomials, say, $a_k = O(g_1)\cdots O(g_{m_k})\cdot O^*(h_1)\cdots O^*(h_{m_k})$. For $m \geqq m_k$ we have that

$$\pi(a_k)V_\iota^{(m)} = V_\iota^{(m)}\pi(a_k)$$
$$\lim \|U_t V_\iota^{(m)} - \langle t, p\rangle V_\iota^{(m)} U_t\| = 0.$$

Thus, for $m \geq \max(m_k)$

$$\lim \|(\pi \times U)(x)V_\iota^{(m)} - V_\iota^{(m)}(\pi \times U)\circ\hat{\alpha}_p(x)\| = 0.$$

Since $V_\iota^{(m)}$ are isometries, we have that $\|(\pi \times U)(x)\| \geq \|(\pi \times U)\circ\hat{\alpha}_p(x)\|$. Since such p is dense in Γ, we have

$$\|(\pi \times U)(x)\| = \|(\pi \times U)\circ\hat{\alpha}_p(x)\|, \quad \forall\, p \in \Gamma.$$

Since O_∞ is simple, we know that $(\pi \times U) \mid O_\infty^\gamma \times_\alpha G$ is faithful.

Remark 5.2. If the set of elements of Sp U_F added by those of the essential spectrum of u is equal to Sp U_F itself and if $\alpha(G)$ contains the gauge automorphism group γ, then it follows from the above proof that $\tilde{\Gamma}(\alpha)$ equals Sp U_F. For example $\tilde{\Gamma}(\gamma) = Z_+$.

In passing we give a remark on a quasi-free automorphism α_u of O_∞ which is induced by a unitary u on $\mathscr{H}$ such that u^n tends to zero weakly as $n \to \infty$. The following proposition implies, in particular, that the Fock state is the only α_u-invariant state of O_∞.

Proposition 5.3. *Let α_u be as above. Then for any $x \in O_\infty$,*

$$M_N(x) \equiv (2N+1)^{-1} \sum_{n=-N}^{N} \alpha_u^n(x)$$

converges in norm to a multiple of the identity.

Proof. Suppose $x = O(f_1) \cdots O(f_n) O^*(g_1) \cdots O(g_m)^*$ with $n + m \geq 1$, and $\|f_1\| = \cdots = \|g_m\| = 1$. Then if $n \geq 1$,

$$\begin{aligned} \|M_N(x)\|^2 &= \|M_N(x)^* M_N(x)\| \\ &\leq (2N+1)^{-2} \sum_{n,m=-N}^{N} |\langle u^n f_1\, u^m f_1 \rangle| \end{aligned}$$

which implies that $\lim \|M_N(x)\|^2 = 0$. Similarly we have the same in case $m \geqq 1$. The linear span of 1 and elements of the form $O(f_1) \cdots O^*(g_m)$ is dense in O_∞, which completes the proof.

With a little more care we can conclude that the system $(O_\infty, \mathbf{Z}, \alpha)$ with $\alpha_n \equiv \alpha_{u^n}$ is weakly asymptotically abelian.

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