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VON NEUMANN REGULARITY AND WEAK P-INJECTIVITY

By

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Introduction. In [25], rings whose cyclic left modules are either projective or p-injective (called left CPP rings) are considered. In this note, we study WP-rings (left weak p-injective rings) which are defined as follows: A is called a WP-ring if every left ideal of A not isomorphic to ${}_AA$ is p-injective. Conditions for WP-rings to be von Neumann regular are given. A WP-ring with maximum condition on left annihilators is proved to be either semi-simple Artinian or a left principal ideal domain. Left perfect WP-rings are semi-simple Artinian. Left Noetherian left V-rings are characterized in terms of finitely generated intersections of maximal left ideals. Certain questions raised by Fisher [4] are considered and results in [25] are generalized.

Throughout, A represents an associative ring with identity and A-modules are unitary. Z, S and J will denote respectively the left singular ideal, the left socle and the Jacobson radical of A. Recall that a left A-module M is p-injective (resp. f-injective) if, for any principal (resp. finitely generated) left ideal I of A and any left A-homomorphism $g: I \to M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. Then A is regular iff every left A-module is p-injective (f-injective) [19]. The following connection between p-injectivity and flatness may be noted: For any p-injective left ideal I of A, A/I is a cyclic flat left A-module. A result of Ikeda and Nakayama [8, Theorem 1] states that A is a left p-injective ring iff every principal right ideal of A is a right annihilator. Following [15], A is called a left p-V-ring if every simple left A-module is p-injective. Left f-V-rings [25] are similarly defined. As usual, an ideal of A means a two-sided ideal of A and A is called left duo if every left ideal is an ideal of A. Call A an ELT ring if every essential left ideal of A is an ideal of A. It is well-known [9, Theorem 2.3] that A is an ELT left self-injective ring iff every left ideal of A is quasi-injective (such rings are called left q-rings). Rings whose left ideals not isomorphic to ${}_AA$ are quasi-injective are studied in [12] and [13].

§ 1. *WP*-rings

WP-rings (defined above) generalize von Neumann regular rings and left principal ideal domains.

Lemma 1.1. *If A is a WP-ring, then A is a semi-prime left semi-hereditary ring whose finitely generated left ideals are principal and such that the left socle S is p-injective.*

Proof. Since any finitely generated p-injective left ideal is a direct summand of ${}_AA$ [24, Lemma 1.2], then A is left semi-hereditary and every finitely generated left ideal is principal. For any $0 \neq a \in A$, if Aa is p-injective, then there exists a non-zero idempotent e such that $Aa = Ae$ and hence $(Aa)^2 \neq 0$. If $Aa \approx {}_AA$, then $1(c) = 0$ for some $c \in Aa$ which implies $0 \neq Aac \subseteq (Aa)^2$. This proves that A is semi-prime. Finally, in case $S \neq 0$, if there exists a minimal left ideal isomorphic to ${}_AA$, then A is a division ring while otherwise, S being a direct sum of minimal p-injective left ideals is p-injective.

Rings whose cyclic right modules not isomorphic to A_A are injective (called right *PCI* rings) are studied in [3]. Since an integral domain containing a non-zero p-injective left or right ideal is a division ring, then [7, Corollary 9] and Lemma 1.1 yield.

Theorem 1.2. *A is a left CPP, WP-ring iff A is either von Neumann regular or a simple left principal ideal, left PCI domain.*

Lemma 1.3. *If A is a WP-ring containing a non-trivial central idempotent, then A is von Neumann regular.*

Proof. If e is a non-trivial central idempotent in A, then neither Ae nor $A(1-e)$ is isomorphic to ${}_AA$. Therefore $A = Ae \oplus A(1-e)$ is a left p-injective ring and every principal left ideal, being p-injective, is a direct summand of ${}_AA$ which proves A regular.

As usual, A is called reduced if A contains no non-zero nilpotent element. It is well-known that every idempotent in a reduced ring is central. The next proposition then follows immediately.

Proposition 1.4. *A reduced WP-ring with non-zero socle is strongly regular.*

Recall that A is directly finite (or Dedekind finite) if $xy = 1$ implies $yx = 1$ for any $x, y \in A$. Then A is directly finite iff ${}_AA \oplus {}_AB \approx {}_AA$ implies $B = 0$.

Theorem 1.5. *If A is a directly finite WP-ring, then A is either regular or a left principal ideal domain.*

Proof. Suppose that A is not a domain and let $0 \neq b \in A$ such that $1(b) \neq 0$. Since Ab is projective, then $1(b) = Ae$ for some non-trivial idempotent e. Then A directly finite and $A = Ae \oplus A(1-e)$ imply that both Ae and $A(1-e)$ must be p-injective. Therefore A is left p-injective which yields A regular.

Cozzens [2, Theorem 1.4] proved that a simple principal left and right ideal domain which is right PCI needs not be Artinian. Applying [22, Lemma 1] and Lemma 1.1 to Theorem 1.5, we get

Corollary 1.6. *The following conditions are equivalent*:

(1) *A is either strongly regular or a left principal ideal domain*;

(2) *A is a WP-ring whose complement left ideals are ideals of A*;

(3) *A is a reduced WP-ring.*

We now consider WP-rings with chain conditions.

Corollary 1.7. *The following conditions are equivalent*:

(1) *A is either semi-simple Artinian or a left principal ideal domain*;

(2) *A is a WP-ring with finite left Goldie dimension*;

(3) *A is a WP-ring with maximum condition on left* (*or right*) *annihilators.*

Lemma 1.8. *Let A be a WP-ring. Then*

(1) *For any idempotent e, either Ae or $A(1-e)$ is p-injective*;

(2) *For any $a \in A$, either Aa is p-injective or $1(a)$ is principal p-injective.*

Proof. Since a direct sum of left A-modules is p-injective iff each direct summand is p-injective, the proof of [12, Proposition 1.6] then yields (1).

(2) For any $a \in A$, $1(a)$ is a direct summand of ${}_AA$ (Lemma 1.1) and by (1), either Aa or $1(a)$ is p-injective.

Proposition 1.9. *Let A be a WP-ring. Then A is von Neumann regular if anyone of the following conditions is satisfied*:

(1) *A has a finitely generated non-zero left socle*;

(2) *A contains a central zero divisor*;

(3) *There exists a proper finitely generated left ideal F which is an ideal of A such that A/F is a regular ring*;

(4) *A is a direct sum of two left ideals A_1, A_2 which are of infinite left Goldie dimension.*

Proof. (1) follows from [5, Lemma 2.33], [24, Lemma 1.2], Lemmas 1.1 and 1.3.

(2) If c is a central zero divisor, then by Lemma 1.8 (2), either Ac or $1(c)$ is

generated by a non-zero central idempotent whence A is regular by Lemma 1.3.

(3) Let F be a non-zero proper finitely generated left ideal which is an ideal such that A/F is regular. Then F is principal (Lemma 1.1) and A/F is a p-injective left A-module. If ${}_AF \approx {}_AA$, there exists $c \in F$ such that $1(c)=0$. Define the left A-homomorphism $g: Ac \to A/F$ by $g(ac)=a+F$ for all $a \in A$. Then there exists $d \in A$ such that $1+F=g(c)=cd+F$. Since $cd \in F$ (an ideal of A), then $1 \in F$ which contradicts $F \neq A$. Thus ${}_AF$ is p-injective which implies A contains a non-trivial central idempotent by [5, Lemma 2.33] and [24, Lemma 1.2], whence A is regular by Lemma 1.3.

(4) Let A_1 contain infinitely many independent non-zero left ideals $\{L_j\}_{j \in J}$. Since $L = A_2 \oplus (\bigoplus_{j \in J} L_j)$ is not isomorphic to ${}_AA$, then L is p-injective which implies A_2 p-injective. Similarly, A_1 is p-injective which proves $A = A_1 \oplus A_2$ left p-injective. Thus A is regular.

Obviously, left duo WP-rings are not necessarily regular. WP, left or right V-rings need not be regular either [2]. Following [6], A is called a left π'-regular ring if, for each $a \in A$, there exist a positive integer n and $b, c \in A$ such that $a^n = ba^n ca^n$. We now introduce a class of rings which generalize von Neumann regular rings, left $\wp$-rings [9] and left duo rings.

Definition. A is called an *EPT* ring if, for any essential left ideal L of A, either ${}_AL$ is p-injective or L is an ideal of A.

Theorem 1.10. *The following conditions are equivalent:*

(1) *A is von Neumann regular;*

(2) *A is a left CPP, WP-ring containing a non-zero p-injective left or right ideal;*

(3) *A is an EPT, right p-V-ring;*

(4) *A is an EPT fully right idempotent ring;*

(5) *A is an EPT, WP-ring whose simple right modules are flat;*

(6) *A is an EPT fully left idempotent WP-ring;*

(7) *A is an EPT left π'-regular WP-ring.*

Proof. (1) implies (2) through (7) obviously.

(2) implies (1) by Theorem 1.2.

(3) implies (4) by [20, Lemma 1].

Since a direct summand of a p-injective left A-module is p-injective, then the proof of [25, Proposition 9] and [24, Lemma 1.2] show that (4) implies (1).

Assume (5). Suppose there exists $b \in A$ such that Ab is not a direct summand of ${}_AA$. Let K be a left ideal such that $L=Ab\oplus K$ is essential in ${}_AA$. Then $L\neq A$ and ${}_AL$ is not p-injective which implies that L is an ideal of A and ${}_AL\approx{}_AA$. Therefore L contains an element c such that $1(c)=0$. If R is a maximal right ideal of A containing L, then A/R is a flat right A-module which implies $c=dc$ for some $d\in R$ [1, Proposition 2.1], whence $1=d\in R$, a contradiction. Thus (5) implies (1).

Assume either (6) or (7). Again suppose $b\in A$ such that Ab is not a direct summand of ${}_AA$. Let K be a left ideal such that $L=Ab\oplus K$ is essential in ${}_AA$. Then L is a proper ideal of A and ${}_AL\approx{}_AA$ which implies $1(c)=0$ for some $c\in L$. Since A is either fully left idempotent or left π'-regular, then there exist a positive integer n and $d\in L$ such that $c^n=dc^n$. Therefore $1(c)=0$ implies $1=d\in L$, a contradiction. This proves that either (6) or (7) implies (1).

The next corollary is related to [4, Query (c)] and [25, Question].

Corollary 1.11. *If A is an EPT, WP, left V-ring, then A is regular.*

[21, Proposition 6] and Theorem 1.10 (4) imply the next result (cf. [4, Problem 1]).

Proposition 1.12. *Let A be an EPT ring whose essential right ideals are ideals of A. Then A is regular iff every factor ring of A is semi-primitive.*

Reduced WP, left V-rings need not be regular [2]. However we have

Theorem 1.13. *The following conditions are equivalent:*

(1) *A is strongly regular;*

(2) *A is a reduced EPT left V-ring;*

(3) *A is a reduced WP-ring whose principal left ideals are complement left ideals;*

(4) *The left ideal generated by any finite subset B of A is a relative complement of* $1(B)$.

Proof. The equivalence of (1) and (2) follows from [14, Proposition 6.1] and the proof of [19, Proposition 3].

(1) implies (3) obviously.

Assume (3). By Corollary 1.6, A is either strongly regular or a left principal ideal domain. In the latter case, A is a left Ore domain which implies that A is the only non-zero principal left ideal, when A is a division ring. Thus (3) implies (1).

If A is strongly regular, I the left ideal generated by a finite subset B, then $A=I\oplus 1(B)$ which shows that (1) implies (4).

Assume (4). For any $b\in A$, $Ab\cap 1(b)=0$ implies A reduced. Then any finitely generated left ideal I of A is a relative complement of $1(I)$. Suppose there exists

$y \in 1(r(I))$, $y \notin I$. Then $K = I + Ay \subseteq 1(r(I))$ implies $1(I) = r(I) = r(1(r(I))) \subseteq r(K) = 1(K) \subseteq 1(I)$ whence $1(I) = 1(K)$. Therefore $K \cap 1(I) = K \cap 1(K) = 0$ which contradicts I a relative complement of $1(I)$. Thus every finitely generated left ideal of A is a left annihilator and (4) implies (1) by [22, Theorem 1].

§ 2. Left V-rings

Since several years, von Neumann regular rings and V-rings are studied by many authors (cf. for example, [2] to [18]). Recall that left p-V-rings are fully left idempotent and generalize both regular rings and left V-rings since regular rings need not be left V-rings (C. Faith) and the converse is not true either (J. H. Cozzens). For any left ideal I of A, write I^* for the intersection of all maximal left ideals of A containing I. A well-known theorem of Villamayor [11, Theorem 2.1] asserts that A is a left V-ring iff $I = I^*$ for every left ideal I of A. The next result improves [25, Theorem 8].

Theorem 2.1. *The following conditions are equivalent:*

(1) *A is a left V-ring;*

(2) *A is a left p-V-ring such that any complement or essential left ideal I of A is an ideal of I^*.*

Proof. (1) implies (2) by [11, Theorem 2.1].

Assume (2). The proof of "(2) implies (1)" in [25, Theorem 8] shows that $I = I^*$ for any complement or essential left ideal I of A. Let M be a simple left A-module, L a proper essential left ideal and $f: L \to M$ a non-zero left A-homomorphism. Then $K = \ker f$ is a maximal left subideal of L and $LK \approx M$. If $L = K \oplus U$ for some minimal left ideal U of A, the set of left ideals containing K and having zero intersection with U has a maximal member C, which is a complement left ideal of A. Since $C = C^*$, there exists a maximal left ideal B such that $C \subseteq B$ but $C \oplus U \nsubseteq B$. Then $B \cap U = 0$ which implies $A = B \oplus U$, whence f may be extended to $g: A \to M$. If ${}_AK$ is essential in ${}_AL$, then $K = K^*$ and the proof of [25, Theorem 8] shows that f may again be extended to $h: A \to M$. This proves that (2) implies (1).

Left self-injective regular rings need not be left V-rings [4, p. 107]. Since a complement left ideal in a left continuous ring [18] is a direct summand, the next corollary then follows from [25, Corollary 7] and Theorem 2.1.

Corollary 2.2. *If A is a left continuous regular ring whose essential left ideals L are ideals of L^*, then A is a left V-ring.*

Proposition 2.3. *If A is a semi-prime ELT left continuous ring, then A is a* *regular left V-ring. In that case, the following conditions are equivalent:* (a) *Every left ideal of A is quasi-injective*; (b) *The injective hull of ${}_AA$ is a cyclic left A-module.*

Proof. If $z \in Z$ such that $z^2=0$, then $(Az)^2 \subseteq 1(z)Az=1(z)z=0$ implies $z=0$. By [24, Lemma 2.1], $Z=0$ whence A is regular [18, Lemma 4.1] and A is a left V-ring by Corollary 2.2. Assuming (a), we have A left self-injective which implies (b) evidently. Conversely, assume (b). Then [23, Theorem 2] implies that the injective hull of ${}_AA$ is projective whence A is left self-injective by the following result of I. Kaplansky: Any finitely generated submodule of a projective left module over a regular ring is a direct summand. Therefore every left ideal of A is quasi-injective by [9, Theorem 2.3].

We now characterize left Noetherian left V-rings in terms of finitely generated intersections of maximal left ideals.

Theorem 2.4. *The following conditions are equivalent:*

(1) *A is a left Noetherian left V-ring;*

(2) *A is a left f-V-ring such that L^* is finitely generated for every essential left ideal L of A.*

Proof. (1) implies (2) obviously.

Assume (2). Suppose there exists a proper essential left ideal L such that $L \neq L^*$. If $L^* = \sum_{i=1}^{n} Ay_i$, there exists an integer m $(1 \leqslant m \leqslant n)$ such that $y_m \notin I = L + \sum_{j=1}^{m-1} Ay_j$ and $I + Ay_m = L^*$. By Zorn's Lemma, there exists a left ideal K containing I which is a maximal left subideal of L^*. Then $L \subseteq I$ implies $L^* \subseteq I^* \subseteq K^*$ and since $K^* \subseteq (L^*)^* = L^*$, then $K^* = L^*$ which implies $K \neq K^*$. But this contradicts [25, Theorem 6]. Thus every essential left ideal of A is finitely generated which implies A left Noetherian and therefore (2) implies (1).

Corollary 2.5. *A is a left hereditary left Noetherian left V-ring iff A is a left f-V-ring such that L^* is finitely generated projective for every essential left ideal L of A.*

The next two propositions are related to [4, Problem 3].

Proposition 2.6. *Let A be a prime left f-V-ring with a left ideal I containing a maximal left subideal K such that K^* is finitely generated p-injective. Then A is primitive with non-zero socle.*

Proof. If we suppose that $K \neq K^*$, then as in the proof of Theorem 2.4, there exists a maximal left subideal M of K^* with $K \subseteq M$. Then $K^* = M^*$ which contradicts [25, Theorem 6]. Thus $K=K^*$ is a direct summand of $_AA$ [24, Lemma 2.1] which implies $I=K\oplus U$, where U is a minimal left ideal of A.

Fully left idempotent rings play an important role in the study of regular rings and V-rings (cf. [4]). It is now known that prime regular rings need not be primitive (O. I. Domanov (1977)).

Proposition 2.7. *Let A be a prime ELT fully left idempotent ring. Then A is primitive with non-zero socle.*

Proof. Since A is fully left idempotent, the Jacobson radical $J=0$. Suppose A has zero socle. If $b\in A$ such that $b^2=0$, for any maximal left ideal M of A, either $1(b)\subseteq M$ or $M+1(b)=A$. In the latter case, $1=c+d$, $c\in M$, $d\in 1(b)$ which implies $b=cb\in M$ (an ideal of A). Thus $b\in M$ in any case which implies $b\in J=0$. This proves that A is reduced. Therefore A is fully right idempotent which implies A strongly regular by [25, Proposition 9]. Thus A is a division ring which is a contradiction.

The next corollary is related to [4, Problem 2].

Corollary 2.8. *Let A be an ELT fully left idempotent ring such that every primitive factor ring is either WP or fully right idempotent. Then A is regular.*

Proof. Every factor ring of A is an ELT fully left idempotent ring. Then every prime factor ring is primitive by Proposition 2.7 and therefore regular by either Theorem 1.10 (6) or [25, Proposition 9]. This proves that A is regular [4, p. 114].

[18, Lemma 4.1], [4, Theorem 14], Corollaries 2.2 and 2.8 together imply.

Corollary 2.9. *Let A be an ELT fully left idempotent ring whose primitive factor rings are left continuous. Then A is a regular left V-ring.*

Recall that A is an I-ring if every non-nil left ideal contains a non-zero idempotent [4, p. 104]. If we apply [4, Theorem 4] and [24, Theorem 2.4] to Corollary 2.9, we get

Corollary 2.10. *Let A be an ELT ring such that each factor ring is a semi-primitive I-ring and each primitive factor ring is left continuous. Then A is a regular left V-ring.*

We return to WP-rings which we use to characterize semi-simple Artinian rings.

Theorem 2.11. *The following conditions are equivalent:*

(1) *A is semi-simple Artinian;*

(2) *A is a left perfect WP-ring;*

(3) *A is a right p-injective WP-ring with maximum condition on left an-*

nihilators;

(4) *A is a left p-injective left f-V-ring such that L^* is finitely generated for every essential left ideal L of A.*

Proof. (1) implies (2) through (4) obviously.

Assume (2). If I is a p-injective left ideal, then A/I is a flat left A-module and since A is left perfect, then A/I is projective [1, Theorem 3.2] which implies I a direct summand of ${}_AA$. Since A is WP, then every left ideal is principal whence A is left Artinian. The left socle S is therefore essential in ${}_AA$. Since ${}_AS$ is injective by Lemma 1.1, then $S=A$ which proves that (2) implies (1).

Assume (3). Every finitely generated left ideal is a left annihilator by [8, Theorem 1] and Lemma 1.1. Then A is left Noetherian which implies A is a left non-singular ring (Lemma 1.1) whose left ideals are left annihilators whence (3) implies (1).

(4) implies (1) by [20, Theorem 7] and Theorem 2.4.

Remarks. (1) A left π'-regular, WP-ring is right non-singular;
(2) The following conditions are equivalent: (a) A is a simple left hereditary left Noetherian left V-ring whose singular left modules are injective; (b) A is a prime ring with maximum condition on left annihilators and such that the singular submodule of every semi-simple left A-module is injective. (Such rings need not be Artinian [2, Theorem 1.4]).

Luedeman, McMorris and Sim introduced p-injectivity and proved that certain results on regular rings and V-rings have their analogues in the theory of semi-groups [10].

Question: Do there exists semi-group analogues of Theorems 1.2, 1.10 and 1.13?

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