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M-HYPONORMAL OPERATORS

By

S. C. ARORA* and RAMESH KUMAR

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Let H be a complex Hilbert space. An operator (bounded linear) T on H is said to be hyponormal if

$$\|T^{*}x\| \leq \|Tx\|$$

for all $x \in H$ [4]. T is said to be M-hyponormal if there exists $M>0$ such that

$$\|(T-zI)^{*}x\| \leq M\|(T-zI)x\|$$

for all $x \in H$ and for all complex numbers z. If in addition to this T satisfies

$$\|(T-zI)^{n}x\|^{2} \leq M\|(T-zI)^{2n}x\| \ \|x\|$$

for all complex numbers z, all integers n and all x in H, then T is said to be of M-power class (N) [2]. We write $\sigma(T)$ for the spectrum of T; $w(T)$ for the Weyl spectrum of T; and $\pi_{00}(T)$ for the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity. T satisfies Weyl's theorem if

$$\sigma(T) \sim w(T) = \pi_{00}(T).$$

Coburn [1] proved that Weyl's theorem holds for any hyponormal operator. The purpose of this paper is generalize this result for any M-hyponormal operator. V. Istratescu [2] has proved this result for an operator of M-power class (N). We also prove a conjecture of [2] in this paper.

It is easily seen that if T is an M-hyponormal operator then for each complex number z, $T-zI$ and zT are M-hyponormal. Also if T is M-hyponormal on H and N is a subspace of H invariant under T, then $T|_N$ is M-hyponormal. In our work we use the following lemmas.

Lemma 1. *If T is an M-hyponormal operator on H and if N is a subspace of H invariant under T such that $T|_N$ is normal, then N reduces T.*

Lemma 2. *Every M-hyponormal quasinilpotent operator is zero.*

For the proofs we refer to [5, Lemma 2] and [3, Corollary 5].

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Theorem 3. *Any isolated point in the spectrum of an M-hyponormal operator is its eigenvalue.*

Proof. Since $T-zI$ is M-hyponormal for each complex z, therefore we can assume the isolated point in the spectrum $\sigma(T)$ of T to be zero. Choose $R>0$ such that the only point of $\sigma(T)$ strictly within $\{z: |z|=R\}$ is zero and $\{z: |z|=R\} \cap \sigma(T) = \emptyset$. Set

$$E=\int_{|z|=R} \frac{I}{T-zI}\, dz\,.$$

Then E is a non-zero projection commuting with T and hence its range space N is invariant under T. This implies that $T|_N$ is M-hyponormal. Also then

$$\begin{aligned}\sigma(T|_N) &= \sigma(T) \cap \{z: |z|<R\}\\ &= \{0\}.\end{aligned}$$

Thus $T|_N$ is M-hyponormal quasinilpotent operator and hence by Lemma 2 is zero. Let $0 \neq x_0 \in N$. Then $Tx_0=0$. This proves the theorem.

Theorem 4. *If T is an M-hyponormal operator then*

$$\omega(T)=\sigma(T)\sim\pi_{00}(T)\,.$$

Proof. It suffices to establish that $0\in\sigma(T)\sim\omega(T)$ if and only if $0\in\pi_{00}(T)$. Since T is M-hyponormal, therefore

$$\|T^*x\| \leq M\|Tx\|$$

for each $x\in H$. Hence

$$N(T)\subset N(T^*)=(\overline{R(T)})^{\perp}\,.$$

Now, let $0\in\sigma(T)\sim\omega(T)$. Then T is a Fredholm operator of index zero. This gives

$$N(T)\subset(\overline{R(T)})^{\perp}=R(T)^{\perp}$$

and

$$\dim(N(T))=\dim(R(T)^{\perp})<\infty.$$

We obtain therefore $N(T)=R(T)^{\perp}$. Hence the decomposition $H=N(T)\oplus N(T)^{\perp}$ gives $T=0\oplus S$, where S is one-one and onto and hence invertible. This gives

$$\sigma(T)=\{0\}\cup\sigma(S)\,.$$

Since $0\notin\sigma(S)$, it is isolated in $\sigma(T)$ and is an eigenvalue of finite multiplicity.

Conversely, if $0 \in \pi_{00}(T)$, the decomposition $H = N(T) \oplus N(T)^{\perp}$ gives $T = 0 \oplus S$, where S is one-one and M-hyponormal. Once again

$$\sigma(T) = \{0\} \cup \sigma(S).$$

If S is not invertible, then 0 being an isolated point of $\sigma(S)$ is an eigenvalue of S by Theorem 3. This contradicts that S is one to one. Hence S is invertible and in particular surjective. This would imply that T is a Fredholm operator of index zero. The result follows.

Corollary 5. *Every M-hyponormal operator T can be written as*

$$T = A \oplus S$$

where A is normal and S is M-hyponormal with $\omega(S) = \sigma(S)$.

Proof. By theorem 4, $\sigma(T) \sim \omega(T) = \pi_{00}(T)$. Let N be the closed linear subspace of H generated by $\bigcup_{\lambda_i \in \pi_{00}(T)} N(T - \lambda_i)$. Then N is reduced by T. The decompositon $H = N \oplus N^{\perp}$ gives $T = A \oplus S$, where A is normal and S is M-hyponormal. One can see that $\omega(S) = \sigma(S)$.

Theorem 6. *If T is M-hyponormal operator with a single limit point of the spectrum, then T is normal.*

Proof. We can assume the limit point to be zero. By hypothesis, every non-zero point of the spectrum being isolated is an eigenvalue. M-hyponormality of T implies that each eigenspace of T is reducing and T is normal on that eigenspace. Let N be the closed linear span of H generated by $\cup N(T - \lambda_i)$, where λ_i runs over non-zero values in $\sigma(A)$. N is thus a closed linear subspace of H reducing T and $T|_N$ is normal. But then by the decomposition $H = N \oplus N^{\perp}$ we get $T|_{N^{\perp}}$ to be M-hyponormal quasinilpotent operator and hence is zero. Hence T is normal.

Istratescu [2, Remark 1.11] conjectured that if T is of M-power class (N) and $\sigma(T)$ has only finitely many limit points, then T is normal. However we prove the following

Theorem 7. *If T is M-hyponormal with only a finite number of limit points in its spectrum, then T is normal.*

Proof. Let z_1 be a limit point of $\sigma(T)$ and choose a simple closed curve G which does not intersect $\sigma(T)$ and contains the only one limit point z_1 in its interior.

$$E_1 = \int_G \frac{I}{T - zI} \, dz .$$

Then E_1 is a non-zero projection on H such that E_1H is invariant under T. Also then

$$\sigma(T|_{E_1H}) = \sigma(T) \cap [\text{interior of } G].$$

Hence $T|_{E_1H}$ can have only one limit point and therefore is normal by Theorem 6. Hence T is reduced by E_1H by Lemma 1. Now considering T on $(E_1H)^\perp$ and continuing the same process we conclude that T being direct sum of normal operators is normal.

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Department of Mathematics
Hans Raj College
University of Delhi
Delhi-110007 India
and
Department of Mathematics
S.G.T.B. Khalsa College
University of Delhi
Delhi-110007 India