YOKOHAMA MATHEMATICAL
JOURNAL VOL. 28, 1980

A NOTE ON WEAK CONVERGENCE OF EMPIRICAL PROCESSES FOR ϕ-MIXING RANDOM VARIABLES

By

RYOZO YOKOYAMA

(Received Aug. 1, 1979)

1. Introduction and result

Let $\{x_j, j\geqq 1\}$ be a strictly stationary sequence of random variables satisfying a ϕ-mixing condition

(1) $$\sup\{|P(B|A)-P(B)|: A\in\mathscr{M}_1^k,\quad B\in\mathscr{M}_{k+n}^{\infty}\}\leqq\phi(n)\downarrow 0\quad(n\to\infty),$$

here $\mathscr{M}_a^b$ denotes the σ-field generated by x_j $(a\leqq j\leqq b)$.

Denote by $F_n(t)$ the empirical distribution function of the sequence $\{x_j, j\geqq 1\}$ at stage n. Without loss of generality (see the proof of Theorem 22.1 of Billingsley [1]) we assume that x_j is uniformly distributed over [0, 1]. Let

(2) $$Y_n(t)=n^{1/2}[F_n(t)-t],\qquad 0\leqq t\leqq 1$$

be the corresponding empirical process. Further set for $0\leqq s$, $t\leqq 1$,

(3) $$\sigma(s, t)=E\{g_s(x_1)g_t(x_1)\}+\sum_{k=2}^{\infty}E\{g_s(x_1)g_t(x_k)\} + \sum_{k=2}^{\infty}E\{g_s(x_k)g_t(x_1)\},$$

where $g_t(x)=I_{[0,t]}(x)-t$ and $I_A(\cdot)$ is the indicator function of A, and whenever $|\sigma(s, t)|<\infty$ (it holds if $\Sigma\phi(n)<\infty$), define a tied down Gaussian random function $Y(t)$, $0\leqq t\leqq 1$, by

(4) $$E\{Y(t)\}=0,\quad E\{Y(s)Y(t)\}=\sigma(s, t).$$

The weak convergence of Y_n to Y has been established by Billingsley [1, Theorem 22.1] under the condition $\Sigma n^2\phi^{1/2}(n)<\infty$. After this, Sen [3] and Yoshihara [5, 6] obtained further developments on this line (in the last paper, the above condition is relaxed to $\phi(n)=O(n^{-1-\delta})$ for some $\delta>0$). We show here that the above result remains true under the less restrictive condition $\Sigma\phi(n)<\infty$ using the method introduced by Yoshihara [6].

Theorem. *Let* $\{x_j, j\geqq 1\}$ *be a strictly stationary sequence of random vari-*

ables satisfying a ϕ-mixing condition (1) *with*

(5) $$\sum_{n=1}^{\infty} \phi(n) < \infty .$$

Suppose that x_j is uniformly distributed over [0, 1]. *Then*

$$Y_n \xrightarrow{D} Y$$

where Y_n and Y are defined by (2) *and* (4) *respectively.*

2. Proof

Throughout this section, K_i denote constants not depending on n and l. For fixed s and t with $0 \leqq s < t \leqq 1$, write

$$z_j = g_t(x_j) - g_s(x_j) = I_{(s,t]}(x_j) - l, \quad l = t - s .$$

Then $\{z_j, j \geqq 1\}$ is ϕ-mixing with $Ez_j = 0$ and $P(|z_j| > 1) = 0$. Fix δ $(0 < \delta < 1)$. We now show that there exists a positive τ such that

(6) $$P(|n^{-1/2} S_n| \geqq \lambda) \leqq K_1 \lambda^{-(2+\delta)} [n^{-\tau} l + l^{(2+\delta)/2}]$$

for all positive λ and all n sufficiently large, where $S_n = z_1 + \cdots + z_n$ (cf. Yoshihara [6, Lemma]). Let r be the largest integer such that $2^{r+1} \leqq n$. Put $p = 2^{[r\beta]}$ $(\beta = (8 - 3\delta)/16)$ and $m = 2^{r-[r\beta]}$, where $[s]$ denotes the largest integer contained in s. We write

$$\xi_i = \sum_{j=ip+1}^{(i+1)p} z_j \qquad (0 \leqq i \leqq 2m-1)$$

and set

$$T_k = \sum_{i=0}^{k-1} \xi_{2i} \, (1 \leqq k \leqq m), \quad T'_m = \sum_{i=0}^{m-1} \xi_{2i+1}, \quad T''_m = S_n - T_m - T'_m .$$

Since x_1 is uniformly distributed over [0, 1], we have

(7) $$E|z_1| \leqq 2l, \quad Ez_1^2 \leqq l .$$

By the inequality (20.28) of Billingsley [1, p. 171] and (7),

(8) $$\begin{aligned} E\xi_0^2 &= pEz_1^2 + 2 \sum_{j=1}^{p-1} (p-j) E z_1 z_{1+j} \\ &\leqq pl[1 + 8 \sum_{j=1}^{\infty} \phi(j)] \leqq K_2 pl . \end{aligned}$$

Since $|\xi_0| \leqq p$ and $E|\xi_0| \leqq pE|z_1| \leqq 2pl$, using (20.28) of Billingsley [1] again,

$$|E\xi_0\xi_{2i}| \leqq 2pE|\xi_0|\phi(pi) \leqq 4p^2 l\phi(pi) . \tag{9}$$

Further since $\phi(n)$ is monotone,

$$\sum_{i=1}^{k-1} \phi(pi) \leqq \sum_{i=1}^{k-1} p^{-1} \sum_{s=(i-1)p+1}^{ip} \phi(s) \leqq p^{-1} \sum_{i=1}^{\infty} \phi(i) . \tag{10}$$

Combining (8)–(10) we obtain

$$ET_k^2 = kE\xi_0^2 + 2\sum_{i=1}^{k-1} (k-i)E\xi_0\xi_{2i} \leqq K_3 kpl. \tag{11}$$

Therefore for some positive γ if n is chosen so large that

$$[\phi(p)]^{1/(2+\delta)} < \gamma/8 ,$$

we have from the proof of Lemma 18.5.1 of Ibragimov-Linnik [2] and (11) that for $1 \leqq k \leqq m/2$

$$\begin{aligned} E|T_{2k}|^{2+\delta} &\leqq (2+\gamma)E|T_k|^{2+\delta} + 4(ET_k^2)^{(2+\delta)/2} \\ &\leqq (2+\gamma)E|T_k|^{2+\delta} + K_4(kpl)^{(2+\delta)/2} . \end{aligned} \tag{12}$$

Now (12) agrees with (2.12) of Yoshihara [6], and thus the inequality (6) follows directly from the proof of Lemma in [6].

The proof is then completed in the same way as that of Theorem 22.1 of Billingsley [1] by using our (6) instead of his Lemma 22.1.

Remark 1. The weak convergence of p-dimensional ($p \geqq 2$) empirical processes for ϕ-mixing random vectors to an appropriate Gaussian process can also be proved under the condition (5) (cf. Sen [4] and Yoshihara [6]).

Remark 2. The condition (5) is the best one expected at present time, because for general ϕ-mixing process there is no method to prove that $\sigma(s, t)$ defined by (3) converges under weaker condition than (5).

References

[1] P. Billingsley: *Convergence of probability measures.* New York, Wiley 1968.

[2] I. A. Ibragimov and Yu. V. Linnik: *Independent and stationary sequences of random variables.* Groningen, Wolters-Noordhoff 1971.

[3] P. K. Sen: *A note on weak convergence of empirical processes for sequences of ϕ-mixing random variables.* Ann. Math. Statist., **42** (1971), 2131–2133.

[4] P. K. Sen: *Weak convergence of multidimensional empirical processes for stationary ϕ-mixing processes.* Ann. Probability, **2** (1974), 147–154.

[5] K. Yoshihara: *Extensions of Billingsley's theorems on weak convergence of empirical processes.* Z. Wahrscheinlichkeitstheorie verw. Gebiete, **29** (1974), 87–92.

[6] K. Yoshihara: *Note on multidimensional empirical processes for ϕ-mixing random vectors.* J. Multivariate Anal., **8** (1978), 584–588.

Department of Mathematics
Osaka Kyoiku University
Ikeda, Osaka 563, Japan