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GAP SERIES AND α-BLOCH FUNCTIONS

By

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1. Introduction. Let f be a function holomorphic in $D=\{|z|<1\}$ with the gap series expansion

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k},\ z\in D, \tag{1.1}$$

where for a constant $q>1$ the natural numbers n_k, $k\geq 1$, satisfy

$$n_{k+1}/n_k\geq q,\ k\geq 1. \tag{1.2}$$

A function g in D is called α-Bloch ($\alpha>0$) if g is holomorphic in D and if

$$\sup_{z\in D}(1-|z|)^{\alpha}|g'(z)|<\infty;$$

the family of all α-Bloch functions is denoted by B^{α}. A Bloch function [6] is precisely a 1-Bloch function. Let $B_0^{\alpha}(\alpha>0)$ be the family of g holomorphic in D such that

$$\lim_{|z|\to 1}(1-|z|)^{\alpha}|g'(z)|=0.$$

It is easy to observe that $B_0^{\alpha}\subset B^{\alpha}$.

Our main result in the present paper is

Theorem 1. *Let f be a holomorphic function in D with* (1.1) *and* (1.2). *Then for $\alpha>0$, the following two propositions hold.*

(I) *$f\in B^{\alpha}$ if and only if*

$$\limsup_{k\to\infty}|a_k|n_k^{1-\alpha}<\infty. \tag{1.3}$$

(II) *$f\in B_0^{\alpha}$ if and only if*

$$\lim_{k\to\infty}|a_k|n_k^{1-\alpha}=0. \tag{1.4}$$

Theorem 1(I) in the case $\alpha=1$ is known; for the proof one should combine [5, Theorem 1] with [5, Theorem 2 (iii)]; note that the latter half of [8, Theorem] is identical with [5, Theorem 2 (iii)].

In Sections 3 and 4 we shall propose some applications of Theorem 1 (II) in the case $\alpha=1$.

The present work arises from the communications with Professor Peter A. Lappan to whom I wish to express my cordial thanks.

2. Proof of Theorem 1. We begin with

Lemma. *Let* $g(z)=\sum_{n=0}^{\infty} b_n z^n$ *be holomorphic in* D. *If* $g\in B^{\alpha}(g\in B_0^{\alpha}$, *respectively) for* $\alpha>0$, *then*

$$\limsup_{n\to\infty}|b_n|n^{1-\alpha}<\infty \quad (\lim_{n\to\infty}|b_n|n^{1-\alpha}=0,\ resp.).$$

As a special case we obtain [5, Theorem 1] that $\{b_n\}$ is bounded if $g\in B^1$.

For the proof of Lemma we first note that $(1-n^{-1})^{1-n}\to e$ as $n\to\infty$. Assume that $g\in B^{\alpha}$. By the Cauchy formula one obtains for $n\geq 1$,

$$|b_n|=|(2\pi in)^{-1}\int_0^{2\pi} g'(re^{i\theta})r^{1-n}e^{i(1-n)\theta}d\theta|\leq C_1 n^{-1}(1-r)^{-\alpha}r^{1-n}$$

for all $0<r<1$; hereafter $C_k(k=1,\dots, 7)$ denote positive constants. For $n>1$ and for $r=1-n^{-1}$ we thus obtain

$$|b_n|\leq C_1 n^{\alpha-1}(1-n^{-1})^{1-n},$$

whence

$$\limsup_{n\to\infty}|b_n|n^{1-\alpha}<\infty .$$

The proof for the case $g\in B_0^{\alpha}$ is similar to the above with a few modifications.

In view of Lemma the rest we should prove in Theorem 1 is the "if" parts in (I) and (II). We first consider the case (I). First of all we notice by

$$\frac{1}{(1-|z|)^{1+\alpha}}=\sum_{n=0}^{\infty}A_n|z|^n, \quad A_n\sim\Gamma(1+\alpha)^{-1}n^{\alpha},$$

that

$$\sum_{n=0}^{\infty}(n+1)^{\alpha}|z|^n\leq\frac{C_2}{(1-|z|)^{1+\alpha}}, \quad z\in D. \tag{2.1}$$

It then follows from (1.3) that

$$|zf'(z)|=|\sum_{k=1}^{\infty} a_k n_k z^{n_k}|\leq C_3\sum_{k=1}^{\infty} n_k^{\alpha}|z|^{n_k},$$

whence, on making use of the Cauchy product, one obtains

$$\frac{|zf'(z)|}{1-|z|} \leq C_3 \sum_{n=1}^{\infty} \Big(\sum_{n_k \leq n} n_k^{\alpha}\Big)|z|^n .$$

Let $K = \max\{k;\ n_k \leq n\}$. Then,

$$n^{-\alpha} \sum_{n_k \leq n} n_k^{\alpha} = \left(\frac{n_K}{n}\right)^{\alpha}\left[1+\left(\frac{n_{K-1}}{n_K}\right)^{\alpha}+\cdots+\left(\frac{n_1}{n_K}\right)^{\alpha}\right] \tag{2.2}$$

$$\leq 1+q^{-\alpha}+q^{-2\alpha}+\cdots = \frac{q^{\alpha}}{q^{\alpha}-1} = C_4 .$$

Therefore,

$$\frac{|zf'(z)|}{1-|z|} \leq C_5 \sum_{n=1}^{\infty} n^{\alpha}|z|^n = C_5|z| \sum_{n=0}^{\infty} (n+1)^{\alpha}|z|^n$$

$$\leq \frac{C_6|z|}{(1-|z|)^{1+\alpha}} \qquad \text{for} \quad z \in D ,$$

by (2.1), whence $f \in B^{\alpha}$. We prove next the "if" part of (II). Given $\varepsilon > 0$ we may find $k_0 \geq 2$ such that

$$|a_k| n_k^{1-\alpha} < \varepsilon \qquad \text{for all} \quad k \geq k_0 .$$

Set

$$P(z) = |z|^{-1} \sum_{k=1}^{k_0-1} |a_k| n_k |z|^{n_k} ,$$

so that P is bounded on D. Then, there exists $0 < r < 1$ such that

$$(1-|z|)^{\alpha} P(z) < \varepsilon \qquad \text{for} \quad r < |z| < 1 . \tag{2.3}$$

Now,

$$|zf'(z)| \leq \sum_{k=1}^{\infty} |a_k| n_k |z|^{n_k} \leq |z| P(z) + \varepsilon \sum_{k=k_0}^{\infty} n_k^{\alpha} |z|^{n_k} ,$$

so that

$$\frac{|zf'(z)|}{1-|z|} \leq \frac{|z|P(z)}{1-|z|} + \varepsilon \sum_{n=1}^{\infty} \Big(\sum_{\substack{n_k \leq n \\ k \geq k_0}} n_k^{\alpha}\Big)|z|^n .$$

It then follows from (2.2), together with (2.1), that

$$\frac{|zf'(z)|}{1-|z|} \leq \frac{|z|P(z)}{1-|z|} + \frac{\varepsilon C_7 |z|}{(1-|z|)^{1+\alpha}} . \tag{2.4}$$

Combining (2.4) with (2.3) one obtains

$$(1-|z|)^{\alpha}|f'(z)| \leq (1+C_7)\varepsilon, \quad r < |z| < 1,$$

which proves that $f \in B_0^{\alpha}$.

3. Fatou points. We begin with a corollary of Theorem 1.

Corollary. *Let f be holomorphic in D with* (1.1) *and* (1.2). *Assume that*

$$\sum_{k=1}^{\infty} |a_k|^2 = \infty \quad and \quad \lim_{k\to\infty} |a_k| = 0 .$$

Then, $f \in B_0^1$ and f has not finite radial limit at almost every point of the circle $\Gamma = \{|z| = 1\}$.

Lappan proved the special case $n_k = q^k$ and $a_k = k^{-1/2}$.

For the proof of Corollary we first note that (1.4) with $\alpha = 1$ holds because $|a_k| \to 0$. It follows from the theorems of G. H. Hardy and J. E. Littlewood and of A. Zygmund (see, for instance, [4, Theorem A and Theorem B]) that f has not finite radial limit at almost every point of Γ.

Let f be meromorphic in D, let $F(f)$ be the set of all Fatou points [3, p. 21] of f, and let $F^*(f)$ be the set of $\zeta \in F(f)$ where f has a finite angular limit. Then $F(f)$–$F^*(f)$ is of Lebesgue measure zero. Since f is pole-free in a terminal part of each angular domain at $\zeta \in F^*(f)$ it follows from the Cauchy formula for f' that $F^*(f) \subset F'(f)$, where $F'(f)$ is the set of $\zeta \in \Gamma$ where $(1-|z|)|f'(z)|$ has zero as the angular limit.

Consider now f in Corollary. Then $F'(f) = \Gamma$ and $F^*(f)$ is of measure zero, in other words, the set $F'(f)$–$F^*(f)$ is metrically very large.

4. Conformal and semiconformal points. Let S be the family of all functions holomorphic and univalent in D. Then $f \in S$ is called conformal at $\zeta \in \Gamma$ if f has the angular limit $f(\zeta) \neq \infty$ at ζ and if the function

$$\arg [(f(\zeta) - f(z))/(\zeta - z)]$$

of z has a finite angular limit at ζ[7, p. 303]. We call $f \in S$ semiconformal at $\zeta \in \Gamma$ if the radial limit $f^*(\zeta) \neq \infty$ (being also the angular limit) exists and if the function

$$(f^*(\zeta) - f(z))/((\zeta - z) f'(z))$$

of z has the angular limit one at ζ; see [2] and [10]. We denote by $\mathscr{C}(f)$ ($\mathscr{C}_s(f)$, resp.) the set of all conformal (semiconformal, resp.) points of $f \in S$. It is known that $\mathscr{C}(f) \subset \mathscr{C}_s(f)$. J. L. Walsh and D. Gaier [9, p. 85] essentially proved that there exists $f \in S$ such that $1 \in \mathscr{C}_s(f) - \mathscr{C}(f)$. A natural question, therefore, arises: *How large may be the set $\mathscr{C}_s(f) - \mathscr{C}(f)$ for $f \in S$?* We shall show

Theorem 2. *There exists $g \in S$ such that $\mathscr{C}_s(g) = \Gamma$ and $\mathscr{C}(g)$ is of measure zero.*

Proof. Consider $f \in B_0^1$ of Corollary in Section 3. We may assume, on dividing f by a suitable positive constant, that

$$\sup_{z \in D} (1 - |z|^2) |f'(z)| < 1 . \tag{4.1}$$

Set

$$g(z) = \int_0^z \exp [i(f(w) - f(0))] dw, \quad z \in D ,$$

so that $g(0) = g'(0) - 1 = 0$. First of all it follows from (4.1), together with [1, Corollary 4.1, p. 36] and $|g''/g'| = |f'|$, that $g \in S$. It is known [10, Lemma 1] that $\zeta \in \mathscr{C}_s(g)$ if and only if the angular limit of $(1 - |z|)|g''(z)/g'(z)|(= (1 - |z|)|f'(z)|)$ at ζ is zero. Therefore, $\Gamma = \mathscr{C}_s(g)$. On the other hand, it is known [11, Theorem 2, p. 121] that $\zeta \in \mathscr{C}(g)$ if and only if $\arg g' = \mathrm{Re}(f - f(0))$ has a finite angular limit at ζ. It then follows from Plessner's theorem [3, Theorem 8.2, p. 147], applied to $f - f(0)$, that

$$\mathscr{C}(g) - F^*(f)$$

is of measure zero. Since $F^*(f)$ is of measure zero, it follows that $\mathscr{C}(g)$ is of measure zero.

Remark. It is easy to see that g may be extended one-to-one quasiconformally to the whole extended plane.

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