YOKOHAMA MATHEMATICAL
JOURNAL VOL. 28, 1980

FIXED POINT THEOREMS FOR GENERALISED CONTRACTION MAPPINGS

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(Received Feb. 27, 1979)

In recent years a number of generalisations of the famous Banach's contraction principle have appeared. Of all these, the generalisation of Ciric [4] (see also Theorem 1) stands at the top. Of course, Benno Fuchssteiner [1] obtained a lattice theoretic generalisation of Ciric's result. While Ciric concentrated on a single operator, Chi Song Wong [2], [3] dealt with a pair of operators.

Ciric took the control function to be a constant, while Chi Song Wong [3] allowed the control functions to be upper semi-continuous.

Ciric's result, as it stands, is not valid if the control function is supposed to be upper semi-continuous (see Example 3). Now a natural question is, what further conditions are to be imposed, when the control functions are assumed to be upper semicontinuous? Section 1 of this paper deals with this question.

In Section 2, we try to obtain Ciric type results for a pair of mappings, from which certain results of Chi Song Wong follow as corollaries. The corresponding version of Ciric's theorem, for a pair of mappings, of course, is not true (see Example 5).

In Section 3, a number of examples are provided to give an insight into the results in Sections 1 and 2.

Throughout this paper, (X, d) stands for a complete metric space.

§1. Ciric [4] proved the following theorem.

Theorem 1. *Let* $0 \leq \alpha < 1$ *and* $f: X \to X$ *be such that* $d(fx, fy) \leq \alpha \max\{d(x, y), d(x, fx), d(y, fy), d(y, fx), d(x, fy)\}$ *for all* x, y *in* X. *Then* f *has a unique fixed point and for any* x *in* X, *the sequence* $\{f^n(x)\}$ *of iterates converges to the fixed point.*

From this, immediately, follows the

Corollary 2. (*Hardy and Rogers* [5], *Theorem* 1). *Let* $f: X \to X$ *and* a, b, c *be non-negative real numbers such that*

(i) $a+2b+2c<1$

and

(ii) $d(fx, fy) \le ad(x, y)+bd(x, fx)+bd(y, fy)+cd(x, fy)+cd(y, fx)$ *for all* x, y *in* X.

Then f *has a unique fixed point.*

Remarks. (1) Thus Theorem 1 is a generalisation of Hardy-Rogers Theorem. Example 1 shows that this is a proper generalisation. Ciric [4] also gave an example to this effect, but there X is an infinite set.

(2) It can also be shown that (see Example 2) there is a situation where Theorem 1 guarantees the existence of a fixed point while Chi Song Wong's Theorem ([3], Theorem 1) does not. See Corollary 5 also for a modified version of Wong's Theorem.

(3) Theorem 1 is not true if α is replaced by an upper semi-continuous non-negative real valued function on $(0, \infty)$ as is evident from Example 3.

However, we have

Theorem 3. *Let* $\alpha: (0, \infty)\to[0, 1)$ *be upper semi-continuous from the right and* $f: X\to X$ *be such that*

$$d(fx, fy) \le \alpha(t) \max \{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy)+d(y, fx)]\}$$

whenever $x, y \in X$ *and* $t=d(x, y)>0$. *Then, for any* x *in* X, *the sequence* $\{f^n(x)\}$ *of iterates is Cauchy. If further,* α *is upper semi-continuous, then there exists a point* z *in* X *such that* $\{f^n(x)\}$ *converges to* z *for all* x *in* X; *consequently, if* f *has a fixed point, then* z *is the unique fixed point of* f.

Proof. We follow the proof of Theorem 1 of Chi Song Wong [3]. Let $x_0 \in X$ and $x_n = f^n(x_0)$ for $n=1, 2, 3, \cdots$.

We first show that $\{x_n\}$ is Cauchy. For this, we may assume that $x_n \neq x_{n+1}$ for all n.

Let $t_n = d(x_{n-1}, x_n)$ for $n=1, 2, 3, \cdots$.

Now

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \le \alpha(t_n) \max \left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2} d(x_{n-1}, x_{n+1})\right\}$$

Since $\alpha(t_n)<1$, we must have

$$d(x_n, x_{n+1}) \leq \alpha(t_n) \max\left\{d(x_{n-1}, x_n), \frac{1}{2} d(x_{n-1}, x_{n+1})\right\}$$
$$\leq \alpha(t_n) \max\left\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right\}$$

so that $d(x_n, x_{n+1}) \leq \alpha(t_n) d(x_{n-1}, x_n)$ for $n \geq 1$. Thus

(1) $$t_{n+1} \leq \alpha(t_n) t_n \quad \text{for} \quad n \geq 1,$$

so that $\{t_n\}$ is a decreasing sequence of positive reals and hence converges, say to $\beta \geq 0$.

If $\beta > 0$, letting $n \to \infty$ in (1), we get

$$\beta \leq \limsup \alpha(t_n)\beta \leq \alpha(\beta)\beta,$$

since α is upper semi-continuous from the right, a contradiction. Thus $\beta = 0$.

Assume $\{x_n\}$ is not Cauchy. Then $\exists \varepsilon > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers, such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon \quad \text{and} \quad d(x_{m_k}, x_{n_k - 1}) < \varepsilon \ \forall k.$$

Thus,

$$\gamma_k \longrightarrow \varepsilon \text{ as } k \longrightarrow \infty, \quad \text{where} \quad \gamma_k = d(x_{m_k}, x_{n_k}).$$

So we can choose $\{\gamma_k\}$ such that $\gamma_k \downarrow \varepsilon$. Now

$$d(x_{m_k+1}, x_{n_k+1}) \leq \alpha(\gamma_k) \max\Big\{\gamma_k, d(x_{m_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}),$$
$$\frac{1}{2}[d(x_{m_k}, x_{n_k+1}) + d(x_{n_k}, x_{m_k+1})]\Big\}$$

so that, letting $k \to \infty$, $\varepsilon \leq \alpha(\varepsilon)\varepsilon$, a contradiction. Thus $\{x_n\}$ is Cauchy. Since X is complete, $\exists z \in X \ni f^n(x_0) \to z$.

Now suppose α is upper semi-continuous. Let $y_0 \in X$ and assume that the sequence $\{y_n\}$ of iterates converges to v, where $y_n = f^n(y_0)$. Let $\delta_n = d(x_n, y_n)$ so that $\delta_n \to d(z, v) = \delta$ (say). If $z \neq v$, then $x_n \neq y_n \ \forall\, n \geq 0$ so that

$$d(x_{n+1}, y_{n+1}) \leq \alpha(\delta_n) \max\Big\{d(x_n, y_n), d(x_n, x_{n+1}),$$
$$d(y_n, y_{n+1}), \frac{1}{2}[d(x_n, y_{n+1}) + d(y_n, x_{n+1})]\Big\}$$

Letting $n \to \infty$ and making use of the upper semi-continuity of α, we get $\delta \leq \limsup \alpha(\delta_n)\delta \leq \alpha(\delta)\delta$ a contradiction.

Remark 1. The upper semi-continuity of α in the second part of Theorem 3 can be equivalently replaced by continuity. For, if $\alpha\colon (0, \infty)\to[0, 1)$ is upper semi-continuous, then there exists a continuous function $\beta\colon (0, \infty)\to[0, 1)$ such that $\alpha(t)\le\beta(t)$ for all t in $(0, \infty)$.

Proof. First we observe that an upper semi-continuous function attains its maximum on a bounded closed interval. Let

$$a_n = \max\left\{\alpha(t)\,|\,\frac{1}{n+1}\le t\le\frac{1}{n}\right\}$$
$$b_n = \max\{\alpha(t)\,|\,n\le t\le n+1\}$$
$$c_n = \max\{a_k|1\le k\le n\}$$

and

$$d_n = \max\{a_1, b_k|1\le k\le n\}$$

for $n=1, 2,\cdots$.

Define $\beta\colon (0, \infty)\to[0, 1)$ by

$$\beta(n+t)=t\,d_{n+1}+(1-t)d_n \quad \text{if} \quad 0\le t\le 1,\ n=1, 2,\cdots,$$
$$\beta\left(\frac{t}{2}+1-t\right)=tc_2+(1-t)d_1 \quad \text{if} \quad 0\le t\le 1$$

and

$$\beta\left(\frac{t}{n}+\frac{1-t}{n+1}\right)=t\,c_n+(1-t)d_{n+1} \quad \text{if} \quad 0\le t\le 1,\ n=2, 3,\cdots.$$

Then β serves the purpose.

However, if $\alpha\colon (0, \infty)\to[0, 1)$ is upper semi-continuous from the right, then there may not exist a continuous function $\beta\colon (0, \infty)\to[0, 1)$ such that $\alpha\le\beta$. For example, take $\alpha(t)=t$ if $0<t<1$ and $\alpha(t)=\frac{1}{2}$ if $t\ge 1$.

Remark 2. The hypothesis of Theorem 3 does not guarantee the existence of a fixed point. In fact, Example 4 shows that f may not have a fixed point even if X is a compact convex subset of a Banach space and even if we take into consideration the maximum over the first three terms only.

However, a slight strengthening of the hypothesis of Theorem 3 ensures the existence of a fixed point. In fact, we have the following

Theorem 4. Let $\alpha\colon (0, \infty)\to[0, 1)$ be upper semi-continuous from the right,

a: $[0, \infty) \to (0, 1)$, b: $[0, \infty) \to (0, 1)$ be continuous at 0 and $a(t)+b(t)=1 \ \forall \ t \in [0, \infty)$ and $f: X \to X$ be such that

(I)
$$d(fx, fy) \le \alpha(t) \max \Big\{ d(x, y),\ a(t)d(x, fx)+b(t)d(y, fy),\ b(t)d(x, fx)+a(t)d(y, fy),\ \frac{1}{2}[d(x, fy)+d(y, fx)] \Big\}$$

whenever $x, y \in X$ and $t=d(x, y)>0$. Then f has a unique fixed point and for any x in X, the sequence $\{f^n(x)\}$ of iterates converges to the fixed point.

Proof. Let $x_0 \in X$. Then from Theorem 3, it follows that the sequence $\{x_n\}$, where $x_n = f^n(x_0)$, is Cauchy. Since X is complete, $\exists z \in X$ such that $\{x_n\}$ converges to z. Let $t_n = d(x_n, z)$ for $n=1, 2, 3, \cdots$.

If $x_n = z$ for infinitely many n, then $fz=z$. So we may assume that $\exists \ N$ such that $x_n \neq z$ for $n \ge N$.

For $n \to N$, consider

$$d(x_{n+1}, fz) = d(fx_n, fz) \le \alpha(t_n) \max \Big\{ d(x_n, z),\ [a(t_n)d(x_n, x_{n+1})+b(t_n)d(z, fz)],\ [b(t_n)d(x_n, x_{n+1})+a(t_n)d(z, fz)],\ \frac{1}{2}[d(x_n, fz)+d(z, x_{n+1})] \Big\}.$$

Since $\alpha(t_n)<1$, by taking limits as $n \to \infty$, we obtain $d(z, fz) \le \max \Big\{ a(0),\ b(0),\ \frac{1}{2} \Big\}$ $d(z, fz)$ so that $fz=z$.

The uniqueness part is obvious.

Remark. From the proof of Theorem 4, it follows that for f satisfying (I) with $\alpha(t)$ replaced by 1, if the sequence of iterates converges then the limit is a fixed point.

Corollary 5. (*Chi Song Wong* [3], *Theorem* 1) *Let* α_i: $(0, \infty) \to [0, 1)$ (*for* $i=1, 2, 3, 4, 5$) *be upper semi-continuous from the right and* $\sum_{i=1}^{5} \alpha_i(t) < 1$ *for any* t. *Let* $f: X \to X$ *be such that*

$$d(fx, fy) \le \alpha_1(t)d(x, y)+\alpha_2(t)d(x, fx)+\alpha_3(t)d(y, fy)+\alpha_4(t)d(x, fy)+\alpha_5(t)d(y, fx)$$

whenever $x, y \in X$ *and* $t=d(x, y)>0$.

Then f *has a unique fixed point.*

Note. In this corollary we may suppose that $\alpha_2=\alpha_3$ and $\alpha_4=\alpha_5$.

Remark. Thus Theorem 4 is a generalisation of Wong's Theorem. That it is a proper generalisation is evident from Example 2.

§ 2. In this section, we prove some theorems on the existence of fixed point for a pair of mappings.

For a pair of mappings, we have the following analogue of Theorem 4.

Theorem 6. *Let* $\alpha: (0, \infty)\to[0, 1)$ *be upper semi-continuous,* $a: [0, \infty)\to(0, 1)$, $b: [0, \infty)\to(0, 1)$ *be continuous at* 0 *and* $a(t)+b(t)=1\ \forall\ t\in[0, \infty)$ *and* $f: X\to X$, $g: X\to X$ *be such that*

$$\text{(I)}\qquad d(fx, gy)\leq\alpha(t)\max\Big\{d(x, y), [a(t)d(x, fx)+b(t)d(y, gy)], [b(t)d(x, fx)+a(t)d(y, gy)], \frac{1}{2}[d(x, gy)+d(y, fx)]\Big\}$$

whenever $x, y\in X$ *and* $t=d(x, y)>0$.

Then either f or g has a fixed point. If further,

$$d(fx, gx)\leq\max\Big\{[a(0)d(x, fx)+b(0)d(x, gx)], [b(0)d(x, fx)+a(0)d(x, gx)], \frac{1}{2}[d(x, fx)+d(x, gx)]\Big\}$$

for all x in X, then each of f, g has a unique fixed point and these points coincide.

Proof. Let $x_0\in X$, $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n=0, 1, 2,\cdots$. We may assume that $x_n\neq x_{n+1}$ for any n. Let $t_n=d(x_n, x_{n+1})$. Then $d(x_{2n}, x_{2n+1})=d(fx_{2n}, gx_{2n-1})$ and using (I), we get

$$d(x_{2n}, x_{2n+1})\leq\alpha(t_{2n-1})d(x_{2n-1}, x_{2n}).$$

Similarly

$$d(x_{2n+1}, x_{2n+2})\leq\alpha(t_{2n})d(x_{2n}, x_{2n+1})$$

so that

$$t_n\leq\alpha(t_{n-1})t_{n-1}\quad\text{for}\quad n=1, 2,\cdots.$$

Now, by the upper semi-continuity of α, as in Theorem 3, it follows that $t_n\downarrow 0$. Suppose $\{x_n\}$ is not Cauchy. Then there exist $\varepsilon>0$ and subsequences $\{m_k\}$ and $\{n_k\}$ of the sequence of positive integers such that $d(x_{m_k}, x_{n_k})\to\varepsilon$.

Let $A=\{k|m_k$ and n_k are even$\}$, $B=\{k|m_k$ and n_k are odd$\}$, $C=\{k|m_k$ is even and n_k is odd$\}$ and $D=\{k|m_k$ is odd and n_k is even$\}$. Then, at least one of A, B, C, D is infinite.

Assume A is infinite. Then, by considering $d(x_{m_k+1}, x_{n_k})=d(fx_{m_k}, gx_{n_k-1})$, making use of (I) and letting $k\to\infty$, we get $\varepsilon\leq\alpha(\varepsilon)\varepsilon$ a contradiction, so that A can not be infinite.

Similarly B, C, D can not be infinite, consequently $\{x_n\}$ is Cauchy. Let $x_n\to z$. Then $x_n\neq z$ for infinitely many n. If $x_{2n}\neq z$ for infinitely many n, then, by considering $d(x_{2n+1}, gz)=d(fx_{2n}, gz)$, making use of (I) and letting $n\to\infty$, we get, $gz=z$. Similarly, if $x_{2n+1}\neq z$ for infinitely many n, then $fz=z$.

The second part of the theorem is obvious.

Corollary 7. *(Chi Song Wong [3], Theorem 2). Let α_i: $(0, \infty)\to[0, 1)$ for $i=1, 2, 3$ be upper semi-continuous and $\alpha_1(t)+2\alpha_2(t)+2\alpha_3(t)<1$ $\forall$ t. Let $f: X\to X$, $g: X\to X$ be such that*

$$d(fx, gy)\leq\alpha_1(t)d(x, y)+\alpha_2(t)[d(x, fx)+d(y, gy)]+\alpha_3(t)[d(x, gy)+(y, fx)]$$

whenever $x, y\in X$ and $t=d(x, y)>0$. Then either f or g has a fixed point.

From Theorem 6, we have also the following

Corollary 8. *Let $0\leq\alpha<\frac{1}{2}$, $f: X\to X$, $g: X\to X$ be such that $d(fx, gy)\leq \alpha\max\{d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}$ for all x, y in X. Then either f or g has a fixed point.*

However, Corollary 8 is not true if $\alpha=\frac{1}{2}$ as is evident from Examples 5 and 6. But there are situations, where f or g has a fixed point if $0<\alpha<1$ and some contraction inequality is satisfied. In fact, we have the following

Theorem 9. *Let $0\leq\alpha<1$, $f: X\to X$, $g: X\to X$ be such that*

$$\text{(I)} \qquad d(fx, gy)\leq\alpha\max\left\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, gy)+d(y, fx)]\right\}$$

whenever $x, y\in X$ and $x\neq y$.

Then either f or g has a fixed point. If further,

$$d(fx, gx)\leq\alpha\max\{d(x, fx), d(x, gx)\} \quad \forall x\in X,$$

then each of f and g has a unique fixed point, these fixed points coincide and for any $x_0\in X$, the sequence $\{x_n\}$ (where $x_{2n+1}=f(x_{2n})$ and $x_{2n+2}=g(x_{2n+1})$ for $n=0, 1, 2,\cdots$) *of iterates converges to the common fixed point.*

In fact, Theorem 9 can be improved as follows:

Theorem 10. *Let $f: X \to X$, $g: X \to X$ be such that whenever x and y are distinct elements of X,*

(I) $$d(fx, gy) \leq \max\{\alpha d(x, y), \alpha d(x, fx), \alpha d(y, gy), ad(x, gy) + bd(y, fx)\}$$

whenever $x, y \in X$ and $x \neq y$, where $0 \leq \alpha < 1$, $a \geq 0$, $b \geq 0$, $a + b < 1$, and $\alpha \cdot \max\left\{\frac{a}{1-a}, \frac{b}{1-b}\right\} < 1$. Then at least one of f and g has a fixed point. If further, (I) *holds for all x, y in X, then each of f and g has a unique fixed point and these fixed points coincide and for any $x_0 \in X$, the sequence $\{x_n\}$ of iterates converges to the common fixed point.*

Proof. Let $x_0 \in X$, $x_{2n+1} = f(x_{2n})$ and $x_{2n+2} = g(x_{2n+1})$ for $n = 0, 1, 2, \cdots$. We may assume that $x_n \neq x_{n+1}$ for any n. Consider

$$d(x_{2n+1}, x_{2n}) = d(fx_{2n}, gx_{2n-1})$$
$$\leq \max\{\alpha d(x_{2n-1}, x_{2n}), \alpha d(x_{2n}, x_{2n+1}), bd(x_{2n-1}, x_{2n+1})\}.$$

Since $0 \leq \alpha < 1$, we must have

$$d(x_{2n+1}, x_{2n}) \leq \max\{\alpha d(x_{2n-1}, x_{2n}), bd(x_{2n-1}, x_{2n+1})\}$$

and hence

(II) $$d(x_{2n}, x_{2n+1}) \leq \max\left\{\alpha, \frac{b}{1-b}\right\} d(x_{2n-1}, x_{2n}).$$

Similarly, we can show that

(III) $$d(x_{2n+1}, x_{2n+2}) \leq \max\left\{\alpha, \frac{a}{1-a}\right\} d(x_{2n}, x_{2n+1}).$$

Let

$$c = \left(\max\left\{\alpha, \frac{a}{1-a}\right\}\right) \cdot \left(\max\left\{\alpha, \frac{b}{1-b}\right\}\right). \quad \text{Then } 0 \leq c < 1.$$

From (II) and (III)

$$d(x_{2n}, x_{2n+1}) \leq cd(x_{2n-2}, x_{2n-1})$$

and

$$d(x_{2n+1}, x_{2n+2}) \leq cd(x_{2n-1}, x_{2n}) \qquad \text{for } n = 1, 2, \cdots.$$

Hence

$$d(x_{2n}, x_{2n+1}) \leq c^n d(x_0, x_1) \quad \text{and} \quad d(x_{2n+1}, x_{2n+2}) \leq c^n d(x_1, x_2)$$

for $n=1, 2,\cdots$. Consequently $\{x_n\}$ is Cauchy. Since X is complete, $\{x_n\}$ converges, say, to z. Proceeding as in Theorem 6, it can be seen that either $fz=z$ or $gz=z$. The second part of the theorem is obvious.

From Theorem 9, we have the following

Corollary 11. *Let* $0\leq\alpha<\frac{1}{2}$ *and* $f: X\to X$, $g: X\to X$ *be such that*

$$d(fx, gy)\leq\alpha \max \{d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}$$

for distinct x, y *in* X. *Then either* f *or* g *has a fixed point. Further, if the inequality holds when* $x=y$ *also, then both* f *and* g *have unique fixed points and these fixed points coincide.*

Theorem 12. *Let* $0\leq\alpha<1$, a *and* b *be non-negative numbers such that* $a+b<1$,

(I) $$\alpha|a-b|<1-(a+b)$$

and $f: X\to X$, $g: X\to X$ *be such that whenever* x, y *are distinct elements in* X

(II) $$d(fx, gy)\leq\alpha \max \{d(x, y), d(x, fx), d(y, gy)\} +(1-\alpha)[ad(x, gy)+bd(y, fx)].$$

Then at least one of f *and* g *has a fixed point. If further,* (II) *holds for all* x, y *in* X, *then the sequence of iterates, for any* x_0 *in* X, *converges to the common fixed point of* f *and* g.

Proof. Let $x_0\in X$, $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n=0, 1, 2,\cdots$. We may assume that $x_n\neq x_{n+1}$ for any n. Now,

$$d(x_{2n+1}, x_{2n+2})=d(fx_{2n}, gx_{2n+1})\leq\alpha \max \{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} +(1-\alpha)ad(x_{2n}, x_{2n+2})$$

$$\Longrightarrow d(x_{2n+1}, x_{2n+2})\leq \max \{\alpha d(x_{2n}, x_{2n+1})+(1-\alpha)ad(x_{2n}, x_{2n+2}), ad(x_{2n}, x_{2n+2})\}$$

$$\Longrightarrow d(x_{2n+1}, x_{2n+2})\leq\beta d(x_{2n}, x_{2n+1}), \quad \text{where } \beta=\max \left\{\frac{\alpha+(1-\alpha)a}{1-(1-\alpha)a}, \frac{a}{1-a}\right\}.$$

Similarly

$$d(x_{2n}, x_{2n+1})\leq\gamma d(x_{2n-1}, x_{2n}), \quad \text{where } \gamma=\max \left\{\frac{\alpha+(1-\alpha)b}{1-(1-\alpha)b}, \frac{b}{1-b}\right\},$$

for $n=1, 2,\cdots$.

Let $c=\beta\gamma$.

If $a, b\in\left[0, \frac{1}{2}\right)$, then $\beta<1$ and $\gamma<1$ so that $0\le c<1$.

If $\max\{a, b\}\ge\frac{1}{2}$, then, since $\frac{\alpha+(1-\alpha)x}{1-(1-\alpha)x}\le\frac{x}{1-x}\Leftrightarrow\frac{1}{2}\le x\ \forall x\in[0, 1)$, it follows from (I) that $0\le c<1$.

The rest of the proof follows as in Theorem 10.

Corollary 13. *Let $f: X\to X$, $g: X\to X$ be such that for all x, y in X*

$$d(fx, gy)\le a_1d(x, y)+a_2d(x, fx)+a_3d(y, fy)+a_4d(x, gy)+a_5d(y, fx)$$

where $a_i\ge 0\ \forall i$, $\sum_{i=1}^{5} a_i<1$ and

$$\text{(I)}\qquad |a_4-a_5|(a_1+a_2+a_3)<1-\sum_{i=1}^{5} a_i.$$

Then f and g have a unique common fixed point.

Proof. Take $\alpha=a_1+a_2+a_3$, $a=\frac{a_4}{1-\alpha}$ and $b=\frac{a_5}{1-\alpha}$ in Theorem 12.

Remark. (i) When, $a_4=a_5$, (I) in Corollary 13 is trivially satisfied. Consequently, a part of Wong's Theorem ([2], Theorem 1) follows from the above.

(ii) When $a_1+a_2+a_3+2\max\{a_4, a_5\}=1$, (I) in Corollary 13 is satisfied.

Theorem 14. *Let $0\le\alpha<1$, $f: X\to X$ and $g: X\to X$ be such that*

$$\text{(I)}\qquad d(fx, gy)\le\alpha\max\{d(x, y), d(x, gy), d(y, fx)\}\quad\forall\ x, y\ \text{ in }\ X$$

and let one of f and g have a fixed point. Then each of f and g has a fixed point and these fixed points coincide. Moreover, for any x_0 in X, the sequence $\{x_n\}$ of iterates where $x_{2n+1}=fx_{2n}$ and $x_{2n+2}=gx_{2n+1}$ for $n=1, 2,\cdots$, converges to the common fixed point.

Proof. Suppose $fx=x$. Then

$$d(x, gx)=d(fx, gx)\le\alpha\max\{d(x, gx), d(x, fx)\}=\alpha d(x, gx)$$

so that $gx=x$.

If $fx=x$ and $gy=y$, then

$$d(x, y)=d(fx, gy)\le\alpha\max\{d(x, y), d(x, gy), d(y, fx)\}=\alpha d(x, y)$$

so that $x=y$. Thus the first part of the theorem is proved. Let x be the common fixed point of f and g. Then

$$d(x, x_{2n})=d(fx, gx_{2n-1})\leq\alpha\max\{d(x, x_{2n-1}), d(x, x_{2n})\}$$

so that

$$d(x, x_{2n})\leq\alpha d(x, x_{2n-1}).$$

Similarly,

$$d(x, x_{2n+1})\leq\alpha d(x, x_{2n}) \quad \text{since} \quad gx=x.$$

Thus

$$d(x, x_n)\leq\alpha d(x, x_{n-1}) \quad \text{for} \quad n=1, 2,\cdots.$$

Hence $\{x_n\}$ converges to x.

Remark. The first part of the conclusion in Theorem 14 is valid even if, in the place of (I), we suppose

$$d(fx, gy)\leq\alpha\max\{d(x, y), d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}.$$

But the second part of the conclusion need not be true even when the maximum is considered over the last four only (see Example 7).

The following Theorem is easy to prove.

Theorem 15. *Let* $\alpha: X\times X\to[0, 1)$ *and* $f: X\to X$, $g: X\to X$ *be such that* $d(fx, gy)\leq\alpha(x, y)d(fx, y)$ *for all* x, y *in* X. *Then* f *is a constant map and* g *has a unique fixed point which is necessarily the common fixed point of* f *and* g.

Remark. The above Theorem may be viewed as a generalisation of the example preceeding Corollary 3 of Wong [2].

We conclude this section with the following

Problem. What can we say about the existence of common fixed points of f and g when they satisfy

$$d(fx, gy)\leq\alpha\max\{d(x, gy), d(y, fx)\}$$

for all x, y in X and for some $\alpha\in(0, 1)$?

§3. Examples

1. Let $X=\{1, 2, 3, 4\}$, $d(1, 2)=d(1, 3)=1$, $d(1, 4)=\frac{3}{2}$, $d(2, 3)=d(2, 4)=d(3, 4)=2$, $f: X\to X$ be given by $f1=1, f2=4, f3=4$ and $f4=1$. (Clearly f has a fixed point)

   Then

$$d(fx, fy) \leq \frac{3}{4} \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$$

for all x, y in X.

But there can not exist non-negative constants a, b, c such that $a+2b+2c \leq 1$ and

$$d(fx, fy) \leq ad(x, y)+bd(x, fx)+bd(y, fy)+cd(x, fy)+cd(y, fx)$$

for all x, y in X.

For, if such a, b, c exist, then

$$d(1, 4)=d(f1, f2) \leq ad(1, 2)+bd(1, 1)+bd(2, 4)+cd(1, 4)+cd(2, 1)$$

so that

$$\frac{3}{2} \leq a1+b0+b2+c\frac{3}{2}+c1$$
$$=(a+2b+2c)+\frac{c}{2} \leq 1+\frac{c}{2} \leq 1+\frac{1}{4}.$$

This is a contradiction.

2. Let $X=\{1, 2, 3, 4, 5\}$, $d(1, 2)=d(1, 3)=d(3, 5)=\frac{13}{8}$, $d(1, 4)=\frac{3}{2}$, $d(1, 5)=d(2, 4)=\frac{7}{4}$, $d(2, 3)=d(4, 5)=1$, $d(2, 5)=\frac{15}{8}$ and $d(3, 4)=2$, $f: X \to X$ be given by $f1=1$, $f2=4$, $f3=4$, $f4=1$ and $f5=2$. (Clearly f has afixed point).

Then

$$d(fx, fy) \leq \frac{14}{15} \max \left\{d(x, y), \frac{1}{2}[d(x, fx)+d(y, fy)], \frac{1}{2}[d(x, fy)+d(y, fx)]\right\}$$

for all x, y in X.

But there do not exist non-negative real valued functions a, b, c on $(0, \infty)$ such that

(i) $a(t)+2b(t)+2c(t)<1$ for each t in $(0, \infty)$

and

(ii) $$d(fx, fy) \leq a(t)d(x, y)+b(t)[d(x, fx)+d(y, fy)] + c(t)[d(x, fy)+d(y, fx)]$$

whenever $x, y \in X$ and $t=d(x, y)>0$.

For, if such functions a, b, c exist, for $t_0=\frac{13}{8}$ we have

$$\frac{3}{2}=d(1,4)=d(f1,f2)\leq \frac{13}{8}a(t_0)+\frac{7}{4}b(t_0)+\frac{25}{8}c(t_0)$$

and

$$\frac{7}{4}=d(4,2)=d(f3,f5)\leq \frac{13}{8}a(t_0)+\frac{31}{8}b(t_0)+2c(t_0)$$

which, when added and simplified, yield

$$26\leq 26a(t_0)+45b(t_0)+41\ c(t_0)$$

so that

$$26<26[a(t_0)+2b(t_0)+2c(t_0)]<26\,.$$

This is a contradiction.

3. Let $X=[1,\infty)$ with the usual metric, $f: X\to X$ be given by $fx=2x$.
Define $\alpha: (0,\infty)\to(0,1)$ by $\alpha(t)=\dfrac{2t}{1+2t}$
Then, clearly, α is continuous, and

$$|fx-fy|\leq\alpha(t)\max\{|x-y|,\ |x-fx|,\ |y-fy|,\ |x-fy|,\ |y-fx|\}$$

whenever $x,\ y\in X$ and $t=|x-y|>0$, but f has no fixed point.

4. Let $X=[0,1]$ with the usual metric,

$$f\colon [0,1]\longrightarrow[0,1] \text{ be given by } f(x)=\begin{cases}\dfrac{x}{2} & \text{if } 0<x\leq 1\\ 1 & \text{if } x=0\end{cases},$$

and

$$\alpha\colon (0,\infty)\longrightarrow(0,1) \text{ be defined by } \alpha(t)=\begin{cases}1-\dfrac{t}{2} & \text{if } 0<t\leq 1\\ \dfrac{1}{2} & \text{if } 1<t<\infty\end{cases}$$

Then, clearly α is continuous monotonically decreasing, and $|fx-fy|\leq\alpha(t)\cdot\max\{|x-y|,\ |x-fx|,\ |y-fy|\}$ whenever $x,\ y\in X$ and $t=|x-y|>0$, but f has no fixed point.

5. Let $X=\{1,2,3,4\}$, $d(1,2)=d(3,4)=2$, $d(1,3)=d(1,4)=d(2,3)=d(2,4)=1$

Define $f: X\to X$ by $f1=f4=2$ and $f2=f3=1$

and $g: X\to X$ by $g1=g3=4$ and $g2=g4=3$.

Then

$$d(fx, gy) \leq \frac{1}{2} \max \{d(x, fx), d(y, gy), d(x, gy), d(y, fx)\} \quad \forall\, x, y \text{ in } X$$

but neither f nor g has a fixed point.

6. Let $X = \{1, 2, 3, 4\}$, $d(1, 2) = d(3, 4) = 2$; $d(1, 3) = d(2, 4) = 1$; $d(1, 4) = d(2, 3) = \frac{3}{2}$.

Define f: $X \to X$ by $f1 = f4 = 2$; $f2 = f3 = 1$,

g: $X \to X$ by $g1 = g3 = 4$; $g2 = g4 = 3$.

Then

$$d(fx, gy) \leq \frac{3}{4} \max \{d(x, fx), d(x, gy), d(y, fx)\} \quad \forall\, x, y \in X.$$

Also

$$d(fx, gy) \leq \frac{3}{4} \max \{d(x, fx), d(y, gy), d(x, gy)\} \quad \forall\, x, y \in X.$$

But neither f nor g has a fixed point.

7. Let $X = \{1, 2, 3, 4, 5\}$; $d(1, 2) = d(1, 4) = d(2, 5) = d(3, 5) = 1$;

$$d(1, 5) = d(2, 3) = \frac{3}{2}; \; d(4, 5) = \frac{7}{4}; \; d(3, 4) = \frac{15}{8}; \; d(1, 3) = d(2, 4) = 2.$$

Define

f: $X \to X$ by $f1 = 2$, $f2 = f4 = f5 = 5$, $f3 = 4$

and

g: $X \to X$ by $g1 = g2 = 3$, $g3 = g5 = 5$, $g4 = 1$.

Then f and g have a common fixed point,

$$d(fx, gy) \leq \frac{15}{16} \max \{(d(x, fx), d(y, gy), d(x, gy), d(y, fx)\}$$

for all x, y in X, but neither the sequence $\{x_n\}$ nor the sequence $\{y_n\}$ of iterates corresponding to the point $x_0 = 1 = y_0$ converges,

where $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$,

$y_{2n+1} = gy_{2n}$ and $y_{2n+2} = fy_{2n+1}$ for $n = 0, 1, 2, \cdots$.

Acknowledgement. The authors are grateful to Dr. D. R. K. Sangameswara Rao, for his valuable help throughout the preparation of this paper.

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