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FIXED POINT THEOREMS FOR MULTI-MAPPINGS

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ABSTRACT: In this paper, some fixed point theorems are proved for multi-mappings defined on uniform spaces. These employ suitable continuity conditions and require certain sets to be nonempty. Some well known results for point-valued mappings including Banach's contraction principle are contained as special cases of the results obtained here.

Given a multi-mapping and a point-valued mapping defined on a Hausdorff locally convex space, conditions are sought, that yield a fixed point for the sum of two mappings. Certain theorems of Krasnoselskii-type are obtained. These contain some known results for point-valued mappings.

1. Introduction

A multi-mapping T on a set X is a correspondence such that $T(x)$ is a subset of X for each $x \in X$; and a fixed point of T is a point x satisfying $x \in T(x)$. In this paper, we extend the principal fixed point result (Theorems 2.1 and 2.4) of Wong [10] for point valued mappings on uniform spaces to multi-mappings. These yield certain interesting corollaries which are employed to obtain fixed point theorems of Krasnoselskii-type (cf. [1], p. 26) for multi-mappings on locally convex spaces. These contain as special cases some recent results of Cain Jr., and Nashed [2] for point-valued mappings.

2. Fixed points in uniform spaces

Let $(X, \mathscr{U})$ be a uniform space. For the terminology of uniform spaces, we refer the reader to Kelley [5]. Let $\mathscr{B}$ denote the base for $\mathscr{U}$ consisting of all closed symmetric entourages. Let 2^X (resp. $\mathrm{Cpt}(X)$) denote the family of nonempty closed (resp. compact) subsets of X. Let $T: X \to 2^X$ be a multi-mapping. Given $U \in \mathscr{U}$, let

$$U_T = \{x \in X : x \in U[T(x)]\} \quad \text{and}$$
$$U'_T = \{x \in X : (x, y) \in U \quad \text{for all} \quad y \in T(x)\}.$$

The families $\{U_T \times U_T \cup \Delta : U \in \mathscr{U}\}$ and $\{U'_T \times U'_T \cup \Delta : U \in \mathscr{U}\}$ constitute bases for uniformities on X. (Here Δ denotes the diagonal of $X \times X$). We denote these

uniformities by $\mathscr{U}_T$, $\mathscr{U}'_T$ respectively.
Given $U \in \mathscr{U}$, let

$$\hat{U} = \{(A, B) \in 2^X \times 2^X : A \times B \subset U\} \cup \Delta$$

and

$$\tilde{U} = \{(A, B) \in 2^X \times 2^X : A \subset U[B] \quad \text{and} \quad B \subset U[A]\}.$$

The families

$$\{\hat{U} : U \in \mathscr{U}\}, \{\tilde{U} : U \in \mathscr{U}\}$$

constitute bases for uniformities on 2^X. We denote these uniformities by $\hat{\mathscr{U}}$, $\tilde{\mathscr{U}}$ respectively. Let $\mathscr{D} = \{d_i : i \in I\}$ be a family of uniformly continuous pseudometrics on X such that the family $\{B(i, \varepsilon) : i \in I, \varepsilon > 0\}$, where $B(i, \varepsilon) = \{(x, y) : d_i(x, y) < \varepsilon\}$ is a base for $\mathscr{U}$. Such a family $\mathscr{D}$ is called an augmented associated family for the uniformity $\mathscr{U}$. It is well known that for each uniformity on X, there exists an augmented associated family $\mathscr{D}$ (cf. Thron [9], p. 177). For each $i \in I$ and A, $B \in 2^X$, let

$$\delta_i(A, B) = \sup\{d_i(a, b) : a \in A, b \in B\} \quad \text{and}$$
$$D_i(A, B) = \max\{\sup_{a \in A} \inf_{b \in B} d_i(a, b), \sup_{b \in B} \inf_{a \in A} d_i(a, b)\}.$$

Let

$$\hat{B}(i, \varepsilon) = \{(A, B) \in 2^X \times 2^X : \delta_i(A, B) < \varepsilon\} \cup \Delta$$
$$\tilde{B}(i, \varepsilon) = \{(A, B) \in 2^X \times 2^X : D_i(A, B) < \varepsilon\}$$

and let

$$\hat{\mathscr{B}} = \{\hat{B}(i, \varepsilon) : i \in I, \varepsilon > 0\},$$
$$\tilde{\mathscr{B}} = \{\tilde{B}(i, \varepsilon) : i \in I, \varepsilon > 0\}.$$

It is easily verified that the families $\hat{\mathscr{B}}$, $\tilde{\mathscr{B}}$ constitute bases for uniformities on 2^X that are uniformly equivalent to the uniformities $\hat{\mathscr{U}}$, $\tilde{\mathscr{U}}$ respectively.

Theorem 2.1. *Let $(X, \mathscr{U})$ be a nonempty complete uniform space. Suppose that $T: (X, \mathscr{U}_T) \to (2^X, \hat{\mathscr{U}})$ is uniformly continuous and satisfies* (i) $x \neq y$ and $T(x)$, $T(y)$ *are not singleton sets imply* $T(x) \neq T(y)$; (ii) U_T *is a nonempty closed subset of X for each $U \in \mathscr{B}$.*

Then T has a fixed point. Furthermore, if X is Hausdorff, then the fixed point is unique.

Proof. We consider the family $\mathscr{F} = \{U_T : U \in \mathscr{B}\}$. By hypothesis, $\mathscr{F}$ is a family of nonempty closed sets with finite intersection property. We assert that

$\mathscr{F}$ contains small sets. Let $W \in \mathscr{U}$. Pick up $U \in \mathscr{B}$ such that $U \circ U \circ U \subset W$. Since $T: (X, \mathscr{U}_T) \to (2^X, \hat{\mathscr{U}})$ is uniformly continuous, there is a $V \in \mathscr{B}$ such that $(u, v) \in V_T \times V_T \cup \Delta$ implies $(T(u), T(v)) \in \hat{U}$. Let $H = U \cap V$. Then $H_T \in \mathscr{F}$. Let $x, y \in H_T$. Then $(x, z_x) \in H$, $(y, z_y) \in H$ for some $z_x \in T(x)$ and $z_y \in T(y)$. Since $(x, y) \in V_T \times V_T$, $(T(x), T(y)) \in \hat{U}$ and hence $(z_x, z_y) \in U$. Thus $(x, y) \in H \circ U \circ H \subset W$. Therefore, $H_T \times H_T \subset W$ and $\mathscr{F}$ contains small sets. Since $(X, \mathscr{U})$ is complete, $\cap \mathscr{F} \neq \emptyset$. Let $x \in \cap \mathscr{F}$. Then $x \in \cap \{U[T(x)]: U \in \mathscr{B}\} = \overline{T(x)} = T(x)$ and x is a fixed point of T. Now suppose that X is Hausdorff and let u, v be fixed points of T. Given $V \in \mathscr{B}$, there is a $U \in \mathscr{B}$ such that $(T(x), T(y)) \in \hat{V}$ whenever $(x, y) \in U_T \times U_T \cup \Delta$. Since $(u, v) \in U_T \times U_T$, one has $(u, v) \in V$ for each $V \in \mathscr{B}$. Therefore $(u, v) \in \cap \mathscr{B} = \Delta$. Hence $u = v$ and the proof is complete.

Theorem 2.1 generalizes a point valued result of Wong ([8], p. 97) to multi-mappings.

It is easily seen that $T: (X, \mathscr{U}) \to (2^X, \hat{\mathscr{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\varepsilon > 0$, there is a $j \in I$ and $\delta(\varepsilon) > 0$ such that $x, y \in X$ and

$$d_j(x, T(x)) + d_j(y, T(y)) < \delta(\varepsilon)$$

imply either $T(x) = T(y)$ or $\delta_i(T(x), T(y)) < \varepsilon$. (Here $d_j(x, T(x)) = \inf\{d_j(x, y): y \in T(x)\}$).

We denote the class of multi-mappings $T: X \to \mathrm{Cpt}(X)$ satisfying the above criterion by $\hat{\mathscr{D}}(X)$. An easy application of Theorem 2.1 yields:

Theorem 2.2. *Let X be a nonempty complete uniform space. Suppose $T \in \hat{\mathscr{D}}(X)$ and satisfies:*

(i) *$x \neq y$ and $T(x)$, $T(y)$ are not singleton sets imply $T(x) \neq T(y)$;*

(ii) *The set $\{x \in X: d_i(x, T(x)) \leq \varepsilon\}$ is nonempty and closed for each $i \in I$ and $\varepsilon > 0$.*

Then T has a fixed point which is unique if X is Hausdorff.

Corollary 2.3. *Let $T: X \to \mathrm{Cpt}(X)$ and suppose that* (i) *For each $i \in I$, there is a constant k_i, $0 \leq k_i < 1$, such that given $x, y \in X$ arbitrarily with $T(x) \neq T(y)$ implies*

$$\delta_i(T(x), T(y)) \leq k_i d_i(x, y);$$

(ii) *$x \neq y$ and $T(x)$, $T(y)$ are not singleton sets imply $T(x) \neq T(y)$.*

Then T has a fixed point which is unique if X is Hausdorff.

Proof. Let $x, y \in X$. Then

$$d_i(x, y) \leq d_i(x, T(x)) + \delta_i(T(x), T(y)) + d_i(y, T(y)).$$

Thus either $T(x)=T(y)$ or

$$\delta_i(T(x), T(y)) \leq \frac{k_i}{1-k_i} d_i(x, T(x)) + d_i(y, T(y)).$$

Hence $T \in \hat{\mathscr{D}}(X)$. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Pick up $x_2 \in T(x_1)$. Then eitgher $T(x_0)=T(x_1)$ or $d_i(x_1, x_2) \leq k_i d_i(x_0, x_1)$. In the former case, x_1 is a fixed point of T. Proceeding thus, one obtains a sequence $\{x_n\}$ satisfying: $x_n \in T(x_{n-1})$ and $d_i(x_n, x_{n-1}) \leq k_i^{n-1} d_i(x_0, x_1)$. Hence the set $\{x \in X: d_i(x, T(x)) \leq \varepsilon\}$ is nonempty for each $i \in I$ and $\varepsilon > 0$; it is also closed. This follows from the continuity of $x \to d_i(x, T(x))$ which can be easily established from the hypothesis.

Theorem 2.4. *Let* $(X, \mathscr{U})$ *be a nonempty complete uniform space. Suppose* $T: (X, \mathscr{U}'_T) \to (2^X, \tilde{\mathscr{U}})$ *is uniformly continuous and that* U'_T *is a nonempty closed subset of* X *for each* $U \in \mathscr{B}$. *Then* T *has a fixed point.*

The proof of Theorem 2.4 is similar to that of Theorem 2.1 and hence is omitted.

Remark. If X is Hausdorff then $\cap \{U'_T: U \in \mathscr{B}\}$ contains a unique point u such that $T(u)=u$. However, in general, the uniqueness of fixed points is not ensured.

It is easily seen that $T: (X, \mathscr{U}'_T) \to (2^X, \tilde{\mathscr{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\varepsilon > 0$, there is a $j \in I$ and $\delta(\varepsilon) > 0$ such that $x, y \in X$ and $\delta_j(x, T(x)) + \delta_j(y, T(y)) < \delta(\varepsilon)$ imply $D_i(T(x), T(y)) < \varepsilon$. (Here $\delta_j(x, T(x)) = \sup \{d_j(x, y): y \in T(x)\}$). We denote the class of multi-mappings $T: X \to 2^X$ satisfying the above criterion by $\tilde{\mathscr{D}}'(X)$. From Theorem 2.4, we easily obtain:

Theorem 2.5. *Let* X *be a nonempty complete uniform space. Suppose* $T \in \tilde{\mathscr{D}}'(X)$ *and satisfies*:

The set $\{x \in X: \delta_i(x, T(x)) \leq \varepsilon\}$ *is nonempty and closed for each* $i \in I$ *and* $\varepsilon > 0$. *Then* T *has a fixed point.*

Corollary 2.6. *Let* $T: X \to 2^X$ *and suppose that for each* $i \in I$, *there is a constant* k_i, $0 \leq k_i < 1$, *such that given* $x, y \in X$ *arbitrarily either* $T(x)=T(y)$ *or* $\delta_i(T(x), T(y)) \leq k_i d_i(x, y)$. *Then* T *has a fixed point.*

Proof. It is easily observed that under the given hypothesis $\delta_i(T(x), T(y)) \leq \delta_i(x, T(x)) + \delta_i(y, T(y))$ holds for $x, y \in X$ satisfying $T(x)=T(y)$ and $\delta_i(T(x), T(y)) \leq \frac{k_i}{1-k_i} \delta_i(x, T(x)) + \delta_i(y, T(y))$ holds for $x, y \in X$ satisfying $T(x) \neq T(y)$. Hence $T \in \tilde{\mathscr{D}}'(X)$. Let $x_0 \in X$ and $x_1 \in T(x_0)$. Pick up $x_2 \in T(x_1)$. Then either $T(x_0) = T(x_1)$ or $\delta_i(x_1, T(x_1)) \leq \delta_i(T(x_0), T(x_1)) \leq k_i d_i(x_0, x_1)$. We can inductively obtain $\{x_n\}$ such that $x_n \in T(x_{n-1})$ and $\delta_i(x_n, T(x_n)) \leq k_i^n d_i(x_0, x_1)$. Therefore the set

$\{x \in X: \delta_i(x, T(x)) \leq \varepsilon\}$ is nonempty for each $i \in I$ and $\varepsilon > 0$; it is also closed. This follows from the continuity of the function $x \to \delta_i(x, T(x))$ which results from the inequality

$$|\delta_i(x, T(x)) - \delta_i(y, T(y))| \leqq 2d_i(x, y)$$

which is easily seen to hold for all $x, y \in X$.

We recall that a mapping $T: X \to 2^X$ is said to be upper semi continuous (u.s.c.) if $T^{-1}(K) = \{x \in X: T(x) \cap K \neq \emptyset\}$ is a closed set for each closed subset K of X. It is easily verified that if X is a compact uniform space then T is u.s.c. if and only if for each net $x_\lambda \to x_0$ and a net $y_\lambda \in T(x_\lambda)$ such that $y_\lambda \to y_0$, one has $y_0 \in T(x_0)$. The following result is essentially known (cf. Ky Fan [3], p. 128). We recall it here in the present setting for the sake of completeness.

Theorem 2.7. *Let $(X, \mathscr{U})$ be a nonempty compact uniform space. Suppose that T satisfies:*

(1) *$T: X \to 2^X$ is u.s.c.;*

(2) *For each $U \in \mathscr{B}$, $U_T \neq \emptyset$.*

Then T has a fixed point.

Proof. We consider the family $\mathscr{F} = \{\overline{U}_T: U \in \mathscr{B}\}$ of nonempty closed sets. Evidently, it has finite intersection property. Since X is compact, one has $\cap \mathscr{F} \neq \emptyset$. Let $x \in \cap \mathscr{F}$. We partially order the family $\{U(x): U \in \mathscr{B}\}$ of neighbourhoods of x by the reversed set inclusion. Pick up $W \in \mathscr{B}$ such that $W(x)$ is in this family. Let $U \in \mathscr{B}$ be such that $U \circ U \subset W$. Since $x \in \overline{U}_T$, there is $y_W \in U_T$ and $z_W \in Ty_W$ such that $(x, y_W) \in U$, $(y_W, z_W) \in U$ and consequently $(x, z_W) \in W$. The nets y_W and z_W converge to x. Since $z_W \in Ty_W$ and T is u.s.c. we have $x \in T(x)$.

In case X is a compact uniform space, we note that if $T(X, \mathscr{U}) \to (2^X, \tilde{\mathscr{U}})$ is uniformly continuous, then it is u.s.c.. This follows from the remark preceding Theorem 2.7. Again it is easily observed that $T: (X, \mathscr{U}) \to (2^X, \tilde{\mathscr{U}})$ is uniformly continuous if and only if for each $i \in I$ and $\varepsilon > 0$, there is $j \in I$ and $\delta(\varepsilon) > 0$ such that $D_i(T(x), T(y)) < \varepsilon$ whenever $d_j(x, y) < \delta(\varepsilon)$. Denote the class of multi-mappings $T: X \to 2^X$ satisfying this condition by $\tilde{\mathscr{D}}(X)$.

Corollary 2.8. *Let X be a nonempty compact uniform space. Suppose $T \in \tilde{\mathscr{D}}(X)$ and satisfies:*
The set $\{x \in X: d_i(x, T(x)) < \varepsilon\}$ is nonempty for each $i \in I$ and $\varepsilon > 0$. Then T has a fixed point.

Corollary 2.9. *Let X be a nonempty compact uniform space. Suppose* *that for each $i \in I$, there is a constant k_i, $0 \leq k_i \leq 1$ such that $D_i(T(x), T(y)) \leq k_i d_i(x, y)$ for all $x, y \in X$. Then T has a fixed point.*

Proof. Evidently $T \in \tilde{\mathscr{D}}(X)$. By using a technique due to Nadler (cf. [8], p. 479) one obtains a sequence $\{x_n\}$ satisfying $x_n \in T(x_{n-1})$ and

$$d_i(x_n, x_{n+1}) \leq k_i^n d_i(x_0, x_1) + nk_i^n \qquad \text{for} \quad n=1, 2, 3, \cdots.$$

Thus the set $\{x \in X: d_i(x, T(x)) < \varepsilon\}$ is nonempty for each $i \in I$ and $\varepsilon > 0$ and this completes the proof.

3. Fixed Points in locally convex spaces

Let $(X, \mathscr{T})$ be a locally convex linear topological space. Let $\mathscr{P} = \{p_i: i \in I\}$ be a family of semi-norms on X such that the family $\mathscr{N} = \{V(i, r): i \in I, r > 0\}$, where $V(i, r) = \{x: p_i(x) < r\}$, is a neighbourhood base for $\mathscr{T}$. Such a family $\mathscr{P}$ is called an augmented associated family for the topology $\mathscr{T}$. It is well known (cf. Kothe [6], p. 203) that for each locally convex topology on a linear space X, there exists an augmented associated family $\mathscr{P}$. For each $i \in I$, let d_i denote the pseudometric on X corresponding to p_i and let δ_i, D_i be as in § 2.

The next two Theorems are Krasnosekskii-type (cf. [1] p. 26) theorems for multi-mappings.

Theorem 3.1. *Suppose X is Hausdorff and let K be a nonempty complete convex subset of X. Suppose $T: K \times K \to 2^K$ is a multi-mapping and $A: K \to X$ is a continuous mapping satisfying:*

(i) *for each $i \in I$, there is a constant k_i, $0 \leq k_i < 1$ such that given x, x', $y \in K$, $x \neq x'$ implies*

$$\delta_i(T(x, y), T(x', y)) \leq k_i p_i(x - x');$$

and

(ii) *$A(K)$ is contained in a compact set and for each $i \in I$, $x, y, y' \in K$ $y \neq y'$ implies*

$$\delta_i(T(x, y), T(x, y')) \leq p_i(Ay - Ay').$$

Then there is a point $\bar{x} \in K$ such that $\bar{x} \in T(\bar{x}, \bar{x})$.

Proof. For each fixed $y \in K$, we consider the mapping $\hat{T}$ given by $\hat{T}(x) = T(x, y)$. By Corollary 2.6 if (i) is satisfied, $\hat{T}$ has a fixed point. Let Fy be the fixed point set for each $y \in K$. Define $P: K \to K$ as $Py = z$ for some $z \in Fy$.

We show that $P: K \to K$ is continuous. Let $i \in I$, then we have for $z_1 \neq z_2$:

$$\begin{aligned} p_i(z_1-z_2) &\le \delta_i(T(z_1, u), T(z_2, v) \qquad \text{for} \quad z_1 \in Fu,\ z_2 \in Fu \\ &\le \delta_i(T(z_1, u), T(z_1, v)) + \delta_i(T(z_1, v), T(z_2, v)) \\ &\le p_i(Au-Av) + k_i p_i(z_1-z_2). \end{aligned}$$

Thus

$$p_i(z_1-z_2) \le \frac{1}{1-k_i} p_i(Au-Av). \tag{3.1}$$

This establishes that P is continuous. We assert that $P(K)$ is contained in a compact set. Let $\{Px_r\}$ be a net in $P(K)$. Since $A(K)$ is contained in a compact set, $\{Ax_r\}$ has a convergent subnet $\{Ax'_r\}$. Thus $\{Ax'_r\}$ is a Cauchy net, and by (3.1) so also is $\{Px'_r\}$. Hence $P(K)$ is contained in a compact set. By a Singbal's version of of Schauder-Tychonoff theorem (cf. Bonsall [1], p. 169), P has a fixed point $\bar{x}$ in K and

$$\bar{x} = P\bar{x} \in T(P\bar{x}, \bar{x}) = T(\bar{x}, \bar{x}).$$

Theorem 3.2. *Let X be Hausdorff and K be a nonempty complete convex subset of X. Let $T: K \to 2^X$ and $S: K \to X$ be mappings such that $T(x)+Sy \subset K$ for each pair, $x, y \in K$. Assume that T and S satisfy the following conditions:*

(i) *for each $i \in I$, there is a constant k_i, $0 \le k_i < 1$, such that given $x, y \in K$ arbitrarily $x \neq y$ implies $\delta_i(T(x), T(y)) \le k_i p_i(x-y)$;*

(ii) *for each $x, y, y' \in K$ either $Sy = Sy'$ or $\delta_i(T(x)+Sy, T(x)+Sy) \le p_i(Sy - Sy')$;*

(iii) *S is continuous and $S(K)$ is contained in a compact set.*

Then there is a point $\bar{x}$ in K such that $\bar{x} \in T(\bar{x}) + S\bar{x}$.

Proof. Define for $x, y \in K$, $T(x, y) = T(x) + Sy$. By using the same argument as in the previous Theorem, for each $y \in K$ take $Py \in Fy$, the fixed point set corresponding to y. We show that $P: K \to K$ is continuous. For $Su \neq Sv$ and $Pu \neq Pv$ one can easily establish

$$p_i(Pu-Pv) \le \frac{1}{1-k_i} p_i(Su-Sv), \qquad \text{for} \quad i \in I.$$

Suppose $Su = Sv$ but $pu \neq Pv$.
Then

$$\begin{aligned} p_i(pu-pv) &\le \delta_i(T(pu)+Su, T(pv)+Sv) \\ &\le \delta_i(T(pu), T(pv)) \\ &\le k_i p_i(pu-pv), \end{aligned}$$

a contradiction.

Hence $Su=Sv$ implies $Pu=Pv$. Therefore, $P: K \to K$ is continuous.

Now the proof of the Theorem can be completed in exactly the same manner as in the previous theorem.

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