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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS OF NONEXPANSIVE TYPE

By

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Introduction. Browder [2], Gohde [3] and Kirk [4] have independently proved that a nonexpansive self mapping on a weakly compact convex subset of a Banach space with normal structure has a fixed point. In Section 1 of this paper we define the concept of normal structure of a bounded convex subset of a locally convex linear topological space and also the notion of a multivalued mapping of nonexpansive type on such a space. We then prove a fixed point theorem for such mappings which include the above mentioned theorem of [2], [3] and [4] and also a theorem of [5] as special cases. In Section 2 we give another definition of nonexpansive multivalued mapping and prove a fixed point theorem for such a mapping on a closed bounded interval of the real line.

§1. Throughout this section (E, τ) will denote a locally convex linear Hausdorff topological space where the topology τ is generated by the family $[p_\alpha: \alpha \in I]$ of seminorms on E.

The concept of normal structure of a bounded convex set in a Banach space was first introduced by Brodskii and Milman [1]. We have introduced below the same concept for a bounded convex subset of E.

A point x of a bounded subset K of E is said to be a p_α-diametral point of K if $\delta(K, \alpha)=\sup\{p_\alpha(x-y): y \in K\}$, where $\delta(K, \alpha)$ is the p_α-diameter of K, i.e. $\delta(K, \alpha)=\sup\{p_\alpha(x-y): x, y \in K\}$. A point y which is not a p_α-diametral point of K is called a p_α-nondiametral point of K.

Definition 1. A bounded convex subset K of E is said to have normal structure if every convex subset B of K containing more than one point has at least one p_α-nondiametral point of B for each $\alpha \in I$ satisfying $\delta(B, \alpha)>0$.

Example 1.1. Let K be a convex subset of E such that K is p_α-compact for each $\alpha \in I$. Then K has normal structure.

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For, suppose K does not have normal structure. Then there are a convex subset B of K containing more than one point and an $\alpha \in I$ with $\delta(B, \alpha) > 0$ such that B does not contain any nondiametral point.

Let $x_1 \in B$. Then we can find x_2 such that $p_\alpha(x_1 - x_2) = \delta(B, \alpha)$. Since B is convex, $\frac{x_1 + x_2}{2} \in B$. We can find $x_3 \in B$ such that $p_\alpha\left(x_3 - \frac{x_1+x_2}{2}\right) = \delta(B, \alpha)$. Continuing this process we obtain a sequence $\{x_n\}$ of points in B such that $p_\alpha\left(x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n}\right) = \delta(B, \alpha)$. Since $\delta(B, \alpha) = p_\alpha\left(x_{n+1} - \frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leq \frac{1}{n} \sum_{k=1}^{n} p_\alpha(x_{n+1} - x_k) \leq \delta(B, \alpha)$, it follows that $p_\alpha(x_{n+1} - x_k) = \delta(B, \alpha)$ for $k = 1, 2, \ldots, n$. This implies that $\{x_n\}$ has no p_α-Cauchy subsequence contradicting the assumption that K is p_α-compact.

Example 1.2. Let K be a τ-compact convex subset of E. Then K has normal structure.

Since K is τ-compact, it is p_α-compact for each $\alpha \in I$. Hence K has normal structure as shown in Example 1.1.

For any bounded subset K of E and $x \in K$ let

$$\gamma_x(K, \alpha) = \sup\{p_\alpha(x - y) : y \in K\},$$
$$\gamma(K, \alpha) = \inf\{\gamma_x(K, \alpha) : x \in K\},$$

and

$$C(K, \alpha) = \{x \in K : \gamma_x(K, \alpha) = \gamma(K, \alpha)\}.$$

Lemma 1.1. *Let K be a nonempty weakly compact convex subset of E. Then $C(K, \alpha)$ is a nonempty closed convex subset of K for each $\alpha \in I$.*

Proof. For each positive integer n and $x \in K$, let $K_n(x, \alpha) = \left\{y \in K : p_\alpha(x-y) \leq \gamma(K, \alpha) + \frac{1}{n}\right\}$. Obviously $K_n(x, \alpha)$ is nonempty, convex and p_α-closed. Let $C_n(\alpha) = \bigcap_{x \in K} K_n(x, \alpha)$. Clearly $C_n(\alpha)$ is convex and p_α-closed and hence τ-closed. $C_n(\alpha)$ is also nonempty. Indeed, there is a $z \in K$ such that $\gamma_z(K, \alpha) \leq \gamma(K, \alpha) + \frac{1}{n}$ i.e. $p_\alpha(z - x) \leq \gamma(K, \alpha) + \frac{1}{n}$ for all $x \in K$. Hence $z \in C_n(\alpha)$. Now since $\{C_n(\alpha) : n = 1, 2, \ldots\}$ is a decreasing sequence of τ-closed (hence weakly closed because each $C_n(\alpha)$ is convex), convex subsets of the weakly compact set K, it follows that $\bigcap_n C_n(\alpha)$ is nonempty, τ-closed and convex. We complete the proof by noting that $C(K, \alpha) = \bigcap_n C_n(\alpha)$.

Lemma 1.2. *Let K be as in Lemma 1.1. In addition assume that K has normal structure. Then $\delta(C(K, \alpha), \alpha) < \delta(K, \alpha)$ whenever $\delta(K, \alpha) > 0$.*

Proof. Since K has normal structure, there is a point $x \in K$ such that $\gamma_x(K, \alpha) < \delta(K, \alpha)$. If $u, v \in C(K, \alpha)$ then $p_\alpha(u-v) \leq \gamma_u(K, \alpha) = \gamma(K, \alpha)$. Hence $\delta(C(K, \alpha), \alpha) \leq \gamma(K, \alpha) \leq \gamma_x(K, \alpha) < \delta(K, \alpha)$.

Definition 1.2. Let K be a subset of E. A multivalued (or a single valued) mapping f: $K \to 2^K$ (nonempty subsets of K) is said to be nonexpansive type on K if f satisfies either of the following conditions:

(a) for each $\alpha \in I$, there are nonnegative real numbers $a_1(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$ with $a_1(\alpha) + a_2(\alpha) + a_3(\alpha) \leq 1$ such that for all $x, y \in K$, $p_\alpha(u-v) \leq a_1(\alpha) p_\alpha(x-y) + a_2(\alpha) p_\alpha(x-v) + a_3(\alpha) p_\alpha(y-u)$ whenever $u \in f(x)$ and $v \in f(y)$;

(b) given $x \in K$ and $u \in f(x)$, for each $v \in f(y)$, $y \in K$ and each $\alpha \in I$, there exists $v'(\alpha) \in f(y)$ such that $p_\alpha(u-v) \leq p_\alpha(x - v'(\alpha))$;

(c) given $x \in K$ and a real number $\varepsilon > 0$, there exists for each $\alpha \in I$ a real number $\delta(\alpha) \geq \varepsilon$ such that $p_\alpha(u-v) \leq \varepsilon$ whenever $u \in f(x)$, $v \in f(y)$, $y \in K$ and $p_\alpha(x-y) \leq \delta(\alpha)$.

Theorem 1.1. *Let K be a nonempty weakly compact convex subset of E. Assume that K has normal structure. Then for each multivalued mapping f of nonexpansive type on K, there is a point $x \in K$ such that $f(x) = \{x\}$ where $\{x\}$ denotes the set consisting of the single point x.*

Proof. By using weak compactness of K and Zorn's lemma we can find a minimal nonempty τ-closed convex subset F of K such that $f(F) \subseteq F$ (Cf. [2]).

We assert that F is a set consisting of a single point, by showing that $\delta(F, \alpha) = 0$ for each $\alpha \in I$. If possible, let us suppose that $\delta(F, \alpha) > 0$ for some $\alpha \in I$. Since F is weakly compact, by Lemma 1.1, $C(F, \alpha)$ is a nonempty, τ-closed convex subset of F. We now prove that $f(C(F, \alpha)) \subseteq C(F, \alpha)$. To this end let $u \in f(C(F, \alpha))$ [here we note that $f(A) = \bigcup_{x \in A} f(x)$ for any subset A of K]. Then there is a point $x \in C(F, \alpha)$ such that $u \in f(x)$. Let $S = \{y \in F: p_\alpha(u-y) \leq \gamma(F, \alpha)\}$. Clearly S is nonempty, convex and p_α-closed and hence τ-closed. Also $f(S) \subseteq S$. Let $y \in f(S)$. Then there is a $z \in S \subseteq F$ such that $y \in f(z)$. If f satisfies (a), then

$$\begin{aligned} p_\alpha(u-y) &\leq a_1(\alpha) p_\alpha(x-z) + a_2(\alpha) p_\alpha(x-y) + a_3(\alpha) p_\alpha(z-u) \\ &\leq a_1(\alpha)\gamma(F, \alpha) + a_2(\alpha)\gamma(F, \alpha) + a_3(\alpha)\gamma(F, \alpha) \leq \gamma(F, \alpha) \end{aligned}$$

because $z \in S \subseteq F$, $x \in C(F, \alpha)$ and $u, y \in F$. Hence $y \in S$ in this case. If f satisfies (b), then there is a $v'(\alpha) \in f(z) \subseteq F$ such that $p_\alpha(u-y) \leq p_\alpha(x - v'(\alpha)) \leq \gamma(F, \alpha)$ since $x \in C(F, \alpha)$ and $v'(\alpha) \in F$. Thus $y \in S$ in this case, too. Finally if f satisfies (c), there is a $\delta(\alpha)\geq\gamma(F, \alpha)$ such that $p_\alpha(u-y)\leq\gamma(F, \alpha)$ whenever $u\in f(x)$, $y\in f(z)$ and $p_\alpha(x-z)\leq\delta(\alpha)$. Now since for all $z\in F$, $p_\alpha(x-z)\leq\gamma(F, \alpha)\leq\delta(\alpha)$, we have $p_\alpha(u-y)\leq\gamma(F, \alpha)$. Thus $y\in S$ in this case also. Hence, we conclude that $f(S)\subseteq S$. Thus $S=F$ by the minimality of F.

Now $\gamma_u(F, \alpha)=\sup\{p_\alpha(u-y): y\in F\}=\sup\{p_\alpha(u-y): y\in S\}\leq\gamma(F, \alpha)$. Noting that $u\in F$, we obtain that $\gamma_u(F, \alpha)=\gamma(F, \alpha)$, i.e. $u\in C(F, \alpha)$. Hence $f(C(F, \alpha))\subseteq C(F, \alpha)$ as was to be shown. But then, $F=C(F, \alpha)$ by the minimality of F. Since $\delta(F, \alpha)>0$, it is impossible in view of our Lemma 1.2. Thus we have shown that $\delta(F, \alpha)=0$ for all $\alpha\in I$. Since E is Hausdorff, F is a set consisting of a single point $\{x\}$, say. Whence $f(x)=\{x\}$.

Remark. (i) If E is a Banach space with $\|\cdot\|$ and f is single valued and satisfies (a) with $a_2(\|\cdot\|)=a_3(\|\cdot\|)=0$, then above theorem reduces to the result of Browder [2], Gohde [3] and Kirk [4].

(ii) If E is a Banach space with norm $\|\cdot\|$, f is singlevalued and $C(K, \|\cdot\|)$ is a single point, then the above theorem reduces to a theorem of Wong [5].

The following corollary shows how a multivalued mapping of nonexpansive type can arise and an application of Theorem 1.1 gives rise to a common fixed point theorem.

Corollary 1.1. *Let E and K be as in Theorem* 1. *Let* $\{f_\gamma: \gamma\in J\}$ *be a family of single valued mappings on* K *(i.e.* f_γ *is a mapping of* K *into itself for each* $\gamma\in J$*) satisfying either of the following conditions:*

(i) *for each* $\alpha\in I$, *there ɒre nonnegative numbers* $a_1(\alpha)$, $a_2(\alpha)$, $a_3(\alpha)$ *with* $a_1(\alpha)+a_2(\alpha)+a_3(\alpha)\leq 1$ *such that for all* $x, y\in K$ *and all* $\gamma, \delta\in J$,

$$p_\alpha(f_\gamma(x)-f_\delta(y))\leq a_1(\alpha)p_\alpha(x-y)+a_2(\alpha)p_\alpha(x-f_\delta(y))+a_3(\alpha)p_\alpha(y-f_\gamma(x))\,;$$

(ii) *given* $x\in K$, $\alpha\in I$, $\gamma\in J$, *there exists for each pair* $(y\in K, \delta\in J)$ *a* $\delta'\in J$ *such that*

$$p_\alpha(f_\gamma(x)-f_\delta(y))\leq p_\alpha(x-f_{\delta'}(y))\,.$$

Then the family $\{f_\gamma: \gamma\in J\}$ *has a common fixed point.*

Proof. We define the multivalued mapping $f: K\to 2^K$ by $f(x)=\{f_\gamma(x): \gamma\in J\}=\bigcup_{\gamma\in J} f_\gamma(x)$, $x\in K$. We can easily verify that if $\{f_\gamma: \gamma\in J\}$ satisfies (i), then f is nonexpansive type of (a) in Definition 1.2 and if $\{f_\gamma: \gamma\in J\}$ satisfies (ii), then f is nonexpansive type of (b) in Definition 1.2. Hence by Theorem 1.1 in either case there is a point $x\in K$ such that $f(x)=\{x\}$. This implies that x is a common fixed point of $\{f_\gamma: \gamma\in J\}$.

§2. In this section we give another definition of nonexpansive multivalued mapping which, in single valued case, coincides with the usual definition of nonexpansive mapping.

Definition 2.1. Let C be a subset of a metric space (X, ρ). A multivalued mapping $f: C \to 2^C$ (nonempty subsets of C) is said to be nonexpanisve if given x and $u \in f(x)$ there is a $v_y \in f(y)$ for each $y \in C$, such that $\rho(u, v_y) \leq \rho(x, y)$.

Remark. This definition can obviously be extended in locally convex spaces in terms of seminorms.

Example 2.1. Let $\{f_\alpha: \alpha \in I\}$ be a family of single valued nonexpansive self mappings on a subset C of a metric space (X, ρ) [i.e. for each $\alpha \in I, f_\alpha: C \to C$ and $\rho(f_\alpha(x), f_\alpha(y)) \leq \rho(x, y)$ for all $x, y \in C$]. Then the multivalued mapping $f: C \to 2^C$ defined by $f(x) = \{f_\alpha(x): \alpha \in I\} (= \bigcup_{\alpha \in I} f_\alpha(x))$, $x \in C$ is clearly nonexpansive in the sense of our Definition 2.1.

We do not as yet know if a fixed point theorem similar to our Theorem 1.1 or the theorem of [2], [3] and [4] can be proved in general for such a nonexpansive mapping on a weakly compact subset (with normal structure) of a Banach space. However, we prove the following fixed point theorem on the subsets of the real line **R**.

Theorem 2.1. *Let C be a closed, convex and bounded subset (i.e. a closed and bounded interval) of the real line. Let f be a nonexpansive (in the sense of Definition 2.1) multivalued mapping on C with closed and convex subsets of C as values (i.e. $f(x)$ is closed and convex for each $x \in C$.) Then there is a point $x_0 \in C$ such that $x_0 \in f(x_0)$.*

Proof. Since C is compact, by using Zorn's lemma we can find a minimal nonempty closed bounded convex set $K \subseteq C$ such that $f(K) \subseteq K$, where as before, $f(K) = \bigcup_{x \in K} f(x)$, see [2]. Noting that K is a closed bounded interval, say [a, b], let z be the midpoint (centre) and r the radius of K, i.e. $r = |z-a| = |z-b|$. Let $N = N(f(z), r) = \{y \in K: |y-x| \leq r \text{ for some } x \in f(z)\}$. Since $f(z)$ and K are convex, it follows that N is convex. N is also closed. Indeed, if $y_n \in N$, $n = 1, 2, \cdots$ and $y_n \to y$, we can find $x_n \in f(z)$, $n = 1, 2, \ldots$ such that $|y_n - x_n| \leq r$. Since $f(z)$ is compact, we can select a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in f(z)$. Now from the triangle inequality

$$|y-x| \leq |y-y_{n_i}| + |y_{n_i} - x_{n_i}| + |x_{n_i} - x|,$$

it follows that $|y-x| \leq r$. Thus $x \in N$ and N is, therefore, closed. We show that $f(N) \subseteq N$. If $y \in f(N)$, then there is a $\omega \in N \subseteq K$ such that $y \in f(\omega)$. Now since f is nonexpansive, there is a $u \in f(z)$ such that $|u-y| \leq |z-\omega| \leq r$, z being the centre of K and ω being in K. Hence $y \in N$. Thus by the minimality of K, we have $K=N=[\text{a}, \text{b}]$. Hence $z \in f(z)$. For if $z \notin f(z)$, then since $f(z)$ is a closed and convex subset of $K=N=[a, b]$, it follows that either (i) $f(z) \subset [a, z)$, or (ii) $f(z) \subset (z, b]$. Clearly in case (i) $b \notin N$ and in case (ii) $\text{a} \notin N$. Thus in either case $N \neq K$ which is a contradiction.

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