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A RANDOM RENEWAL THEOREM

By

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1. Introduction. Let $\{\xi_n\}$ be a sequence of i.i.d. random probability measures on the real line R with the expectation measure $F=E\xi_1$ and the second moment measure $G=E(\xi_1\times\xi_1)$. The random renewal measure ζ for $\{\xi_n\}$ is defined by

$$\zeta=\delta_0+\sum_{n=1}^{\infty}\xi_1*\cdots*\xi_n\ .$$

The purpose of this note is to show that ζ satisfies an analogue of Blackwell type renewal theorem under some natural assumptions on F and G.

In §2 it is shown that ζ is locally finite a.s. iff F generates a transient random walk. In §3 we prove the following random renewal theorem: if F generates a transient random walk and if G is not supported by the diagonal of R^2 then $\lim_{t\to\infty}\zeta(I+t)=c\cdot\lambda(I)$ in probability, where I is a bounded interval of length $\lambda(I)$ and c is a constant. When ξ_n's are non-random this result reduces to the well-known classical renewal theorem. Whether the above result holds in the sense of almost sure convergence or not is an open question.

2. Random renewal measures. Let M denote the set of all locally finite nonnegative measures ϕ on $(R,\mathscr{R})$, where $\mathscr{R}$ is the Borel σ-algebra of R. The smallest σ-algebra of subsets of M which respect to which every mapping $\phi\to\phi(A)$, $A\in\mathscr{R}$, is measurable is denoted by $\mathscr{M}$. The subset of M consisting of all probability measures is denoted by M_1. Obviously $M_1=\{\phi;\phi(R)=1\}\in\mathscr{M}$. An M-valued random element ζ defined on a probability space $(\Omega,\mathscr{F},P)$ is called a random measure (r.m.). A r.m. ξ satisfying $P\{\xi\in M_1\}=1$ is called a random probability measure (r.p.m.). The expectation measure $E\zeta$ of a r.m. ζ is a (not necessarily locally finite) measure defined by $(E\zeta)(A)=E[\zeta(A)]$, $A\in\mathscr{R}$. If ξ is a r.p.m. then $E\xi\in M_1$. In this case write F_ξ for $E\xi$. We have

$$E\int f(t)\zeta(dt)=\int f(t)(E\zeta)(dt)$$

for every Baire function $f\geq 0$, where an integral sign without the limits means integration over R. Thus if f is integrable with respect to $E\zeta$ then $\int f d\zeta$ is finite a.s. The second moment measure $E(\zeta\times\zeta)$ of a r.m. ζ is a (not necessarily locally finite) measure on $(R,\mathscr{R})\times(R,\mathscr{R})$ defined by

$$[E(\zeta\times\zeta)](A\times A)=E[\zeta(A)\cdot\zeta(B)]\,,\qquad A,B\in\mathscr{R}\,.$$

If ξ is a r.p.m. then $G_\xi\equiv E(\xi\times\xi)$ is a probability measure.

The convolution $\xi*\eta$ of two r.m.'s ξ and η is not a r.m. in general. However if ξ and η are independent with $E\xi\in M$, $E\eta\in M$ and $(E\xi)*(E\eta)\in M$ then $\xi*\eta$ is a r.m. satisfying $E(\xi*\eta)=(E\xi)*(E\eta)$. In fact

$$\begin{aligned}E[\xi*\eta(A)]&=E\iint\chi_A(s+t)\xi(ds)\eta(dt)\\&=\iint\chi_A(s+t)(E\xi)(ds)(E\eta)(dt)\\&=(E\xi)*(E\eta)(A)\,,\qquad A\in\mathscr{R}\,,\end{aligned}$$

where χ_A is the indicator of A. In particular if ξ and η are independent r.p.m.'s then $\xi*\eta$ is a r.p.m. and we have $F_{\xi*\eta}=F_\xi*F_\eta$, $G_{\xi*\eta}=G_\xi*G_\eta$. The second equality follows from

$$\begin{aligned}E[\xi*\eta(A)\cdot\xi*\eta(B)]&=E\left[\int\xi(A-s)\eta(ds)\cdot\int\xi(B-t)\eta(dt)\right]\\&=E\left[\int(\xi\times\xi)((A-s)\times(B-t))(\eta\times\eta)(dsdt)\right]\\&=E[(\xi\times\xi)*(\eta\times\eta)(A\times B)]\,,\qquad A,B\in\mathscr{R}\,.\end{aligned}$$

Let $\{\xi_n\}$ be a sequence of i.i.d.r.p.m.'s with common distribution P. Let $F=E\xi_1$, $G=E(\xi_1\times\xi_1)$ and $U=\sum_{n=0}^{\infty}F^{n*}$, where $F^{0*}=\delta_0$ is the Dirac measure located at 0. We call F transient (recurrent) if F generates a transient (recurrent) random walk. Define the random renewal measure ζ by

$$\zeta=\delta_0+\sum_{n=1}^{\infty}\xi_1*\cdots*\xi_n\,.$$

Then we have

$$E\zeta=\delta_0+\sum_{n=1}^{\infty}E(\xi_1*\cdots*\xi_n)=\delta_0+\sum_{n=1}^{\infty}F^{n*}=U\,.$$

The following theorem is an extension of a classical result on the recurrence of random walks due to Chung and Fuchs [3]. In particular if $\xi_n=\delta_{X_n}$, where $\{X_n\}$ is a sequence of i.i.d. random variables and δ_a is the Dirac measure at a, then ζ reduces to the ordinary renewal process: $\zeta=\sum_{n=0}^{\infty}\delta_{S_n}$, $S_n=\sum_{i=1}^{n}X_i$, $S_0=0$, and the theorem reduces to the well-known result [3].

Theorem 1. *If F is recurrent then $\xi(I)=\infty$ a.s. for every neighborhood I of* 0. *If F is transient then $\zeta(I)<\infty$ a.s. for every bounded interval I.*

Proof. The second half is immediate. In fact if F is transient then $E\zeta(I)=U(I)<\infty$ and therefore $\zeta(I)<\infty$ a.s. for bounded I. To show the first half let $\{X_n\}$ be a sequence of i.i.d. random variables uniformly distributed over $[0, 1]$ and independent of $\{\xi_n\}$. Define a sequence $\{Y_n\}$ of random variables by $Y_n=\sup\{y;\, \xi_n(-\infty, y]\leqq X_n\}$. Then we have easily

$$P\{Y_n\leqq y\}=E[\xi_n(-\infty, y]]=F(-\infty, y]\,.$$

Thus $\{Y_n\}$ is a sequence of i.i.d. random variables with common distribution F. Let $T_n=\sum_{k=0}^{n} Y_k$, $T_0=0$. It is easy to see that

$$P\{T_n\in A|\xi_1, \xi_2, \cdots\}=\xi_1*\cdots*\xi_n(A) \quad \text{a.s.}$$

and therefore

$$\zeta(A)=P\{\sum_{n=0}^{\infty}\delta_{T_n}(A)|\xi_1, \xi_2, \cdots\} \quad \text{a.s.}$$

for every $A\in\mathscr{R}$. If F is recurrent then $\sum_{n=0}^{\infty}\delta_{T_n}(I)=\infty$ a.s. and therefore $\zeta(I)=\infty$ a.s. for every neighborhood I of 0.

3. A random renewal theorem. Throughout the rest suppose F is transient. Write μ for $\int tF(dt)$ if this integral exists. Let $D=\{(x, x);\, x\in R\}$ be the diagonal of R^2 and let M_d denote the set of all degenerate probability measures on R. Then the second moment measure G of a r.p.m. ξ is supported by D iff $P\{\xi\in M_d\}=1$.

Theorem 2. *Suppose that F is transient non-arithmetic. If $G=G_{\xi_1}$ is not supported by D then*

(1) $$\lim_{t\to\infty}\zeta(I+t)=c\cdot\lambda(I) \quad \textit{in probability}$$

for every bounded interval I, where λ is the Lebesgue measure, $c=\mu^{-1}$ if $0<\mu<\infty$ and $c=0$ otherwise. When $c>0$ the above condition on G is necessary for (1).

Proof. The last statement is obvious since if $0<\mu<\infty$ and if G is supported by D then ξ_n is written as $\xi_n=\delta_{X_n}$ where $\{X_n\}$ is a sequence of i.i.d. random variables. In this case it is obvious that $\zeta=\sum_{n=0}^{\infty}\delta_{S_n}$, $S_n=\sum_{k=1}^{n}X_k$, does not satisfy (1).

Let I be a bounded interval. It is well-known [4, 5] that if F is transient non-arithmetic then

(2) $$\lim_{t\to\infty} U(I_t)=\lim_{t\to\infty} E\zeta(I_t)=c\cdot\lambda(I)\ ,$$

where $I_t=I+t$. When $c=0$ this implies (1). Suppose $c>0$. In order to prove (1) it suffices to show

(3) $$\lim_{t\to\infty} E[\zeta(I_t)^2]=c^2\cdot\lambda(I)^2\ ,$$

because (2) and (3) imply that

$$\lim_{t\to\infty} E[\zeta(I_t)-c\lambda(I)]^2=0\ .$$

At first we show the following relation:

(4) $$E[\zeta(J)^2]=2\int U(J_{-s})V(J\times ds)-V(J\times J)$$

for every bounded interval J, where $V=\sum_{n=1}^{\infty} G^{n*}$. In fact

$$\begin{aligned}
E[\zeta(J)^2]&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} E[\xi_1*\cdots*\xi_j(J)\cdot\xi_1*\cdots*\xi_k(J)]\\
&=2\sum_{0\le j\le k} E[E[\xi_1*\cdots*\xi_j(J)\cdot\xi_1*\cdots*\xi_k(J)|\xi_1,\cdots,\xi_j]]\\
&\quad-\sum_{j=0}^{\infty} E[\xi_1*\cdots*\xi_j(J)^2]\\
&=2\sum_{0\le j\le k} E[\xi_1*\cdots*\xi_j(J)\cdot\xi_1*\cdots*\xi_j*F^{(k-j)*}(J)]\\
&\quad-\sum_{j=0}^{\infty} E[\xi_1*\cdots*\xi_j(J)^2]\\
&=2\sum_{0\le j\le k}\int G^{j*}(J\times J_{-t})F^{(k-j)*}(dt)-\sum_{j=0}^{\infty} G^{j*}(J\times J)\\
&=2\int V(J\times J_{-t})U(dt)-V(J\times J)\ .
\end{aligned}$$

The assumption on G implies

(5) $$\lim_{t\to\infty} V(J_t\times J_t)=0$$

for bounded J (see [2]). On the other hand since G generates a transient random walk on R^2 we have by a theorem of Bickel and Yahav [1] that

(6) $$\lim_{t\to\infty} V\{(x,y);\ \max(x,y)\in I_t\}=c\lambda(I)\ .$$

Since $\lim_{t\to\infty} V(I_t\times R)=\lim_{t\to\infty} U(I_t)=c\lambda(I)$ and since V is invariant under the permutation of coordinates, it follows from (5) and (6) that

(7) $$\lim_{t\to\infty} V(I_t\times(-\infty,t])=c\lambda(I)/2\ .$$

Let $\{m_t\}_{t>0}$ be the family of measures defined by $m_t(A)=V(I_t\times(-A)_t)$, $A\in\mathscr{R}$. Then $\{m_t\}$ is uniformly bounded and by (5) $\lim_{t\to\infty} m_t(A)=0$ for compact A. Thus for every bounded measurable g vanishing at $\pm\infty$ we have

$$\lim_{t\to\infty}\int g(t-s)V(I_t\times ds)=\lim_{t\to\infty}\int g(s)m_t(ds)=0\,.$$

The function $h(t)=U(I_t)-c\lambda(I)1_+(t)$, where 1_+ is the indicator of $[0,\infty)$, is bounded and vanishes at infinity. Therefore it follows from (7) that

$$\begin{aligned}&\lim_{t\to\infty}\int U(I_{t-s})V(I_t\times ds)\\&=\lim_{t\to\infty}\int h(t-s)V(I_t\times ds)+\lim_{t\to\infty}c\lambda(I)V(I_t\times(-\infty,t])\\&=c^2\lambda(I)^2/2\,.\end{aligned}$$

In view of (4) and (5) this proves (3) and therefore the theorem.

References

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