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# SOME REMARKS ON CROSSED PRODUCTS OF VON NEUMANN ALGEBRAS BY KAC ALGEBRAS

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ABSTRACT. Some supplementary results to our previous papers are given. The outer conjugacy between covariant systems corresponds to the conjugacy between covariant systems of the crossed products. The integrability of an action $\alpha$ with $\mathscr{M}^{\alpha}$ properly infinite is characterized by $\{\mathscr{M}\otimes\mathscr{L}(H), \bar{\alpha}\}\cong\{\mathscr{M}\otimes\mathscr{L}(H), \tilde{\alpha}\}_e$ for some projection $e$ in the crossed product $\mathscr{W}^*(\mathscr{M}, \boldsymbol{K}, \alpha)$. Every action (or co-action of a locally compact group) is implemented by a unitary whenever the von Neumann algebra is properly infinite and standard. The definition of inner tensor products $\alpha_1 * \alpha_2$ of actions $\alpha_1$ and $\alpha_2$ and some related properties are discussed. If $\alpha$ is dual, then $(\mathscr{M}^{\alpha})'\cap\mathscr{M}\subset\mathscr{M}^{\alpha}$ is equivalent to $\alpha(\mathscr{M})'\cap\mathscr{W}^*(\alpha)\subset\mathscr{W}^*(\alpha)'$.

**Introduction.**

Many of results concerning crossed products of von Neumann algebras by continuous actions of a locally compact group obtained by Connes and Takesaki [3], Landstad [8, 9], Nakagami [10, 11], Nakagami and Takesaki [13], Strătilă, Voiclescue and Zsidó [16], and Takesaki [17, 18] are extended to those for crossed products of von Neumann algebras by Kac algebras by Enock and Schwartz [4, 6]. In the present paper we shall use their terminologies and give some supplementary results on crossed products to our previous papers.

In § 1 we shall sketch briefly the definitions of a Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$, the fundamental operator $W$ of $\boldsymbol{K}$, a right action $\alpha$ of $\boldsymbol{K}$ on a von Neumann algebra $\mathscr{M}$, and a crossed product $\mathscr{W}^*(\mathscr{M}, \boldsymbol{K}, \alpha)$ or $\mathscr{W}^*(\alpha)$ of $\mathscr{M}$ by $\boldsymbol{K}$ in order to recall notations and basic facts.

In § 2 it is shown that a pair of covariant systems are outer conjugate if and only if a pair of covariant systems of the crossed products together with the dual right actions are conjugate.

In § 3 it is shown that a right action $\alpha$ of a Kac algebra on a von Neumann algebra $\mathscr{M}$ with $\mathscr{M}^{\alpha}$ properly infinite is integrable if and only if $\{\bar{\mathscr{M}}, \bar{\alpha}\}$ is conjugate to a reduction of $\{\bar{\mathscr{M}}, \tilde{\alpha}\}$ to some projection $e$ in $\bar{\mathscr{M}}^{\tilde{\alpha}}$, where $\bar{\mathscr{M}}=\mathscr{M}\otimes\mathscr{L}(H)$, $H$ is the underlying Hilbert space of $M$, $\bar{\alpha}=(\iota\otimes\sigma)\circ(\alpha\otimes\iota)$ and $\tilde{\alpha}=Ad_{1\otimes\sigma(W^*)}\circ\bar{\alpha}$. This is a generalization of [3, Theorem 3.2.12], [11, Theorems 4.1 and 4.3] and [13, Theorems III.3.1 and III.3.2]

In § 4 any right action of a Kac algebra on a properly infinite standard von Neumann algebra is shown to be unitarily implementable. This yields a partial answer to a conjecture in [13, Chapter III], that is, every co-action of a locally compact group on a properly infinite standard von Neumann algebra is unitarily implementable.

In § 5 we shall disscuss inner tensor product $\alpha_1 * \alpha_2$ of actions $\alpha_1$ and $\alpha_2$. The action $\tilde{\alpha}$ of $\boldsymbol{K}$ on $\tilde{\mathcal{M}}$ conjugate to $\hat{\hat{\alpha}}$ on $\mathscr{W}^*(\mathscr{W}^*(\alpha), \boldsymbol{K}^{\wedge\prime}, \hat{\alpha})$ by Takesaki's duality is naturally interpreted as an inner tensor product of $\alpha$ and the right action $\hat{\Gamma}$ of the dual Kac algebra $\boldsymbol{K}^{\wedge}$ on $\mathscr{L}(H)$. This generalizes the fact that for each action $\alpha$ of a locally compact group $G$ on a von Neumann algebra, $\tilde{\alpha}_t$ is of the form $\alpha_t \otimes \lambda_t$, where $\lambda_t$ is the action of $G$ on $\mathscr{L}(L^2(G))$ implemented by the left regular representation of $G$ on $L^2(G)$. If $\delta$ is a co-action of $G$ on a von Neumann algebra, the $\tilde{\delta}$ is an inner tensor product of $\delta$ and the co-action $\delta_G^c$ of $G$ on $\mathscr{L}(L^2(G))$ defined by $\delta_G^c(x) = W_G(x \otimes 1) W_G^*$.

In § 6 the relative commutant property due to Paschke is generalized to the context of a right action of a Kac algebra and discussed the equivalent conditions as in [14] by a different method from him.

In § 7 a necessary condition for a right action of a Kac algebra to be dual will be given. The converse is an open question.

## § 1. Preliminary.

All the contents in this section are taken from the works due to Enock and Schwartz, [4, 5, 6, 15].

For a locally compact group $G$ equipped with a right invariant Haar measure $dt$ we denote by $\mathscr{R}(G)$ the von Neumann algebra generated by the right regular representation $t \mapsto \rho(t)$ of $G$. For the left regular representation $t \mapsto \lambda(t)$ of $G$, we set $\lambda(f) = \int f(t)\lambda(t)dt$ for $f \in L^1(G)$. The duality of $G$ is then contained in the following diagram:

$$L^1(G) \to \mathscr{R}'(G) (\cong L^\infty(\hat{G}) \text{ for abelian } G)$$
$$L^\infty(G) \leftarrow \mathscr{R}'(G)_* (\cong L^1(\hat{G}) \text{ for abelian } G).$$

This is generalized into Kac algebra context by Takesaki [17] and then by Enock and Schwartz as the following:

$$M_* \xrightarrow{\lambda} \hat{M}$$
$$M \xleftarrow{\hat{\lambda}} \hat{M}_* .$$

Let's recall this procedure more closely. A triplet $(M, \Gamma, \kappa)$ consisting of a standard von Neumann algebra $M$ on a Hilbert space $H$, an isomorphism $\Gamma$ (Every isomorphism in this paper is assumed to be normal and $*$-preserving) of $M$ into $M\bar{\otimes}M$ satisfying coproduct condition

$$(\Gamma\otimes\iota)\circ\Gamma=(\iota\otimes\Gamma)\circ\Gamma \,, \tag{1.1}$$

and an involutive antiisomorphism $\kappa$ of $M$ into itself with

$$\sigma\circ\Gamma\circ\kappa=(\kappa\otimes\kappa)\circ\Gamma \tag{1.2}$$

is called an *involutive Hopf-von Neumann algebra,* where $\sigma$ is a symmetric isomorphism: $x\otimes y\rightarrow y\otimes x$. We define a product $*$ and an involution $\circ$ in the predual $M_*$ of $M$ by

$$\langle x, \omega*\omega'\rangle=\langle\Gamma(x), \omega\otimes\omega'\rangle$$
$$\langle x, \omega^\circ\rangle=\langle\kappa(x)^*, \omega\rangle^-$$

for $x\in M$ and $\omega$, $\omega'\in M_*$. Then $M_*$ turns out to be an involutive Banach algebra.

A faithful, semi-finite, normal weight $\phi$ on an involutive Hopf-von Neumann algebra $(M, \Gamma, \kappa)$ is called a *Haar weight,* if

(i) $(\iota\otimes\phi)(\Gamma(x))=\phi(x)1$ for $x\in M_+$, and

(ii) $(\iota\otimes\phi)((1\otimes y^*)\Gamma(x))=\kappa(\iota\otimes\phi)(\Gamma(y^*)(1\otimes x))$ for $x, y\in \mathfrak{n}_\phi$,

where $\mathfrak{n}_\phi=\{x\in M: \phi(x^*x)<\infty\}$. Any two Haar weights on an involutive Hopf-von Neumann algebra are proportional.

**Definition 1.1.** An involutive Hopf-von Neumann algebra $(M, \Gamma, \kappa)$ with a Haar weight $\phi$ is called a *Kac algebra* and denoted by $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$.

Through the GNS construction $\{\pi_\phi, \mathfrak{H}_\phi, \eta_\phi\}$ of $M$ with respect to $\phi$, $M$ is identified with $\pi_\phi(M)$: $\{M, H\}\cong\{\pi_\phi(M), \mathfrak{H}_\phi\}$. The intersection $\mathfrak{a}_\phi=\mathfrak{n}_\phi\cap\mathfrak{n}_\phi^*$ is a left Hilbert algebra. Every Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$ has the fundamental operator $W=W(\boldsymbol{K})$, i.e. a unitary $W$ on $H\otimes H$ with

$$W\eta_{\phi\otimes\phi}(x\otimes y)=\eta_{\phi\otimes\phi}(\Gamma(y)(x\otimes 1)) \,, \qquad x, y\in\mathfrak{n}_\phi \,.$$

Then $W$ satisfies $\Gamma(x)=W(1\otimes x)W^*$ and associativity condition

$$Ad_{1\otimes W^*}(W^*\otimes 1)=(W^*\otimes 1)(\iota\otimes\sigma)(W^*\otimes 1) \tag{1.3}$$

[5, Proposition 3.1.7]. The bounded linear operator $\lambda(\omega_{\xi_2,\xi_1})$ on $H$ defined by

$$(\eta_1|\lambda(\omega_{\xi_2,\xi_1})\eta_2)=(W(\xi_1\otimes\eta_1)|\xi_2\otimes\eta_2)$$

gives a mapping of $M_*$ into $\mathscr{L}(H)$ with $\lambda(\omega*\omega')=\lambda(\omega)\lambda(\omega')$ and $\lambda(\omega^\circ)=\lambda(\omega)^*$. We denote the von Neumann algebra generated by $\lambda(M_*)$ by $\hat{M}$. Then $\lambda$ is an isomorphism of $M_*$ into $\hat{M}$ and $W \in M \bar{\otimes} \hat{M}$. Let $J$ and $\hat{J}$ be the modular unitary involutions of $M$ and $\hat{M}$, respectively. Then

$$(\hat{J} \otimes J) W (\hat{J} \otimes J) = W^* . \tag{1.4}$$

Let $\mathscr{I}_\phi$ be the left ideal of $M_*$ generated by $\omega_{\xi,\eta}$ for $\xi, \eta \in \mathfrak{a}'_\phi$ and $A_* = \mathscr{I}_\phi \cap \mathscr{I}^*_\phi$. Then $A_*$ is an involutive subalgebra of $M_*$. For each $\omega \in M_*$ we set

$$\|\omega\|_\phi = \sup \{|\langle x^*, \omega\rangle| : \|x\|_\phi \leq 1, x \in \mathfrak{n}_\phi\} \qquad (\|x\|_\phi \equiv \|\eta_\phi(x)\|)$$

and $\mathscr{L}_\phi = \{\omega \in M_* : \|\omega\|_\phi < \infty\}$. Let $a$ be a mapping of $\mathscr{L}_\phi$ into $H$ given by

$$a(\omega_{\xi,\eta}) = \xi \eta^\flat , \qquad \xi, \eta \in \mathfrak{a}'_\phi .$$

Then $\mathfrak{b} = a(A_*)$ is a left Hilbert algebra with respect to a product and an involution:

$$a(\omega) a(\omega') = a(\omega * \omega')$$
$$a(\omega)^{\hat{\sharp}} = a(\omega^\circ) .$$

Let $\hat{\pi}$ be the left representation of $\mathfrak{b}$ on $H$. Then

$$\hat{\pi}(a(\omega)) = \lambda(\omega)$$

by identifying $\{\hat{\pi}(\mathfrak{b}), \mathfrak{H}_{\hat{\phi}}\}$ with $\{\hat{M}, H\}$ and $\hat{\pi}(\mathfrak{b})'' = \hat{M}$, where $\hat{\phi}$ is the canonical weight on $\hat{\pi}(\mathfrak{b})''$ with respect to $\mathfrak{b}$. Then $\eta_{\hat{\phi}}(\lambda(\omega)) = a(\omega)$. The mapping $\hat{\Gamma}$ of $\hat{M}$ into $\hat{M} \bar{\otimes} \hat{M}$ defined by

$$\hat{\Gamma}(x) = \sigma(W^*)(1 \otimes x) \sigma(W)$$

is a coproduct on $\hat{M}$. The mapping $\hat{\kappa}$ defined by

$$\hat{\kappa}(x) = J x^* J$$

is an involutive antiisomorphism of $\hat{M}$ into itself. Then we have $\lambda(\omega \circ \kappa) = \hat{\kappa}(\lambda(\omega))$ and $\kappa(\lambda_*(\hat{\omega})) = \hat{\lambda}(\hat{\omega})$ for $\hat{M}_*$. The quadruple $\boldsymbol{K}^\wedge = (\hat{M}, \hat{\Gamma}, \hat{\kappa}, \hat{\phi})$ turns out to be a Kac algebra, called the *dual* of $\boldsymbol{K}$. The fundamental operator $W(\boldsymbol{K}^\wedge)$ is $\sigma(W^*)$ and $\boldsymbol{K} = \boldsymbol{K}^{\wedge\wedge}$.

The *commutant* of a Kac algebra $\boldsymbol{K} = (M, \Gamma, \kappa, \phi)$ is constructed from

$$\Gamma'(x) = Ad_{J \otimes J} \circ \Gamma \circ Ad_J(x) , \qquad x \in M' ,$$
$$\kappa'(x) = Ad_J \circ \kappa \circ Ad_J(x) , \qquad x \in M' ,$$
$$\phi'(x) = \phi(JxJ) , \qquad x \in M'_+ ,$$

and denoted by $\boldsymbol{K}' = (M', \Gamma', \kappa', \phi')$. The fundamental operator $W(\boldsymbol{K}')$ is $(J \otimes J) W (J \otimes J)$ and $\boldsymbol{K} = \boldsymbol{K}''$.

The *reflection* $\boldsymbol{K}^\sigma$ of $\boldsymbol{K}$ is defined by $\boldsymbol{K}^\sigma=(M, \sigma\circ\Gamma, \kappa, \phi\circ\kappa)$. The fundamental operator $W(\boldsymbol{K}^\sigma)$ is $(\hat{J}\otimes\hat{J})W(\hat{J}\otimes\hat{J})$ and $\boldsymbol{K}=\boldsymbol{K}^{\sigma\sigma}$. Moreover, we have $\boldsymbol{K}'^\sigma=\boldsymbol{K}^{\sigma\prime}$, $\boldsymbol{K}'^\wedge=\boldsymbol{K}^{\wedge\sigma}$, $\boldsymbol{K}^{\sigma\wedge}=\boldsymbol{K}^{\wedge\prime}$ and $\boldsymbol{K}'^{\sigma\wedge}=\boldsymbol{K}^{\wedge\prime\sigma}$.

**Lemma 1.2.** *If $W$ is the fundamental operator of a Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$, then*

(i) $(\hat{J}\otimes\hat{J})\sigma(W^*)(\hat{J}\otimes\hat{J})$ *is the fundamental operator of* $\boldsymbol{K}^{\wedge\prime}$.

(ii) $Ad_{1\otimes\sigma(W)}(\sigma(W)\otimes 1)=(\sigma(W)\otimes 1)(\iota\otimes\sigma)(\sigma(W)\otimes 1)$ .

**Proof.** (i) Since $W(\boldsymbol{K}^\wedge)=\sigma(W^*)$ and $\hat{J}$ is the modular unitary involution of $\hat{M}$, we have $W(\boldsymbol{K}^{\wedge\prime})=(\hat{J}\otimes\hat{J})\sigma(W^*)(\hat{J}\otimes\hat{J})$.

(ii) The fundamental operator $V=W(\boldsymbol{K}^{\wedge\prime})$ satisfies the associativity condition (1.3):

$$Ad_{1\otimes V^*}(V^*\otimes 1)=(V^*\otimes 1)(\iota\otimes\sigma)(V^*\otimes 1) . \tag{1.5}$$

Applying $Ad_{\hat{J}\otimes\hat{J}\otimes\hat{J}}$ to both sides, we have (ii). Q.E.D.

Here we consider examples of a Kac algebra.

**Example 1.3.** Let $\alpha_G$ be the action of $G$ on $L^\infty(G)$, i.e. the coproduct on $L^\infty(G)$ with $(\alpha_G f)(s, t)=f(st)$, $\kappa$ the involutive antiisomorphism of $L^\infty(G)$ into itself with $(\kappa f)(t)=f(t^{-1})$ and $\mu'_G$ the faithful semi-finite mormal weight on $L(G)$ with $\mu'_G(f)=\int f(t)\Delta(t)dt$. Then

$$\boldsymbol{K}=\{L^\infty(G), \alpha_G, \kappa, \mu'_G\}$$

is a Kac algebra whose fundamental operator $W(\boldsymbol{K})$ is $W'^*_G$, where $(W'_G\xi)(s, t)=\Delta(s)^{1/2}\xi(s, s^{-1}t)$ for $\xi\in L^2(G\times G)$. In this case

$$\boldsymbol{K}'=\boldsymbol{K}\,;$$
$$\boldsymbol{K}^\sigma=(L^\infty(G), \alpha'_G, \kappa, \mu_G)\,, \qquad W(\boldsymbol{K}^\sigma)=W_G\,;$$

where $\mu_G=\mu'_G\circ\kappa$, $(\alpha'_G f)(s, t)=f(ts)$ and $(W_G\xi)(s, t)=\xi(s, ts)$. Of course, $\mu_G(f)=\int f(t)dt$. Let $\delta'_G$ be the co-action of $G$ on $\mathscr{R}'(G)$, i.e. the coproduct on $\mathscr{R}'(G)$ with $\delta'_G(\lambda(t))=\lambda(t)\otimes\lambda(t)$, $\hat{\kappa}$ the involutive antiisomorphism of $\mathscr{R}'(G)$ into itself with $\hat{\kappa}(\lambda(t))=\lambda(t)^*$ and $\psi'_G$ the faithful semi-finite normal weight on $\mathscr{R}'(G)$ with $\psi'_G(\lambda(f)^*\lambda(f))=\|f\|_2^2$ for $f\in\mathscr{K}(G)$. Then

$$\boldsymbol{K}^\wedge=(\mathscr{R}'(G), \delta'_G, \hat{\kappa}, \psi'_G)\,, \qquad W(\boldsymbol{K}^\wedge)=V'_G\,;$$

where $(V'_G\xi)(s, t)=\Delta(t)^{1/2}\xi(t^{-1}s, t)$. Moreover,

$$\boldsymbol{K}^{\wedge\prime}=(\mathscr{R}(G), \delta_G, \hat{\kappa}, \psi_G)\,, \qquad W(\boldsymbol{K}^{\wedge\prime})=V^*_G\,;$$

where $\delta_G(\rho(t))=\rho(t)\otimes\rho(t)$, $\tilde{\kappa}(\rho(t))=\rho(t)^*$, $\psi_G(\rho(f)^*\rho(f))=\|f\|_2^2$ for $f\in\mathscr{K}(G)$ and $(V_G\xi)(s, t)=\xi(st, t)$.

**Example 1.4.** If $\boldsymbol{K}=(\mathscr{R}(G), \delta_G, \tilde{\kappa}, \psi_G)$ and $W(\boldsymbol{K})=V_G^*$, then we have

$$\begin{aligned}
&\boldsymbol{K}'=(\mathscr{R}'(G), \delta_G', \tilde{\kappa}, \psi_G'), \qquad W(\boldsymbol{K}')=V_G';\\
&\boldsymbol{K}^o=\boldsymbol{K};\\
&\boldsymbol{K}^\wedge=(L^\infty(G), \alpha_G', \kappa, \mu_G), \qquad W(\boldsymbol{K}^\wedge)=W_G;\\
&\boldsymbol{K}^{\wedge\prime}=\boldsymbol{K}^{o\wedge}=\boldsymbol{K}^\wedge;\\
&\boldsymbol{K}^{\wedge o}=\boldsymbol{K}'^\wedge=(L^\infty(G), \alpha_G, \kappa, \mu_G'), \qquad W(\boldsymbol{K}^{\wedge o})=W_G'^*,
\end{aligned}$$

Now we shall generalize the action and the crossed product into Kac algebra context as the following:

**Definition 1.5.** A *right action* $\alpha$ of a Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$ on a von Neumann algebra $\mathscr{M}$ is an isomorphism of $\mathscr{M}$ into $\mathscr{M}\bar{\otimes}M$ satisfying

$$(\alpha\otimes\iota)\circ\alpha=(\iota\otimes\Gamma)\circ\alpha. \tag{1.6}$$

**Definition 1.6.** Let $\alpha$ be a right action of $\boldsymbol{K}$ on $\mathscr{M}$. The *crossed product* $\mathscr{W}^*(\mathscr{M}, \boldsymbol{K}, \alpha)$ or $\mathscr{W}^*(\alpha)$ of $\mathscr{M}$ by $\boldsymbol{K}$ with respect to $\alpha$ is the von Neumann algebra generated by $\alpha(\mathscr{M})$ and $\boldsymbol{C}\otimes\hat{M}'$.

The *dual* action $\hat{\alpha}$ of $\alpha$ is a right action of $\boldsymbol{K}^{\wedge\prime}$ on the crossed product $\mathscr{W}^*(\alpha)$ defined by

$$\hat{\alpha}(x)=Ad_{1\otimes(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})}(x\otimes 1), \tag{1.7}$$

[4, II.6. Proposition]. The Takesaki's duality is given by

$$\{\mathscr{W}^*(\mathscr{W}^*(\alpha), \boldsymbol{K}^{\wedge\prime}, \hat{\alpha}), \hat{\hat{\alpha}}\}\cong\{\mathscr{M}\bar{\otimes}\mathscr{L}(H), \tilde{\alpha}\},$$

where $\hat{\hat{\alpha}}$ is the dual action of $\hat{\alpha}$ and $\tilde{\alpha}$ is the right action of $\boldsymbol{K}$ on $\mathscr{M}\bar{\otimes}\mathscr{L}(H)$ defined by

$$\tilde{\alpha}(x)=Ad_{1\otimes\sigma(W^*)}\circ(\iota\otimes\sigma)\circ(\alpha\otimes\iota)(x), \tag{1.8}$$

[4, III.8. Theoreme; 6, Theoreme IV.3].

Throughout this paper we denote the underlying Hilbert spaces of $M$ and $\mathscr{M}$ by $H$ and $\mathfrak{H}$, respectively.

## §2. Outer conjugacy of actions.

In this section we shall show that the study of outer conjugate classes of covariant systems is reduced to that of conjugate classes of the dual right actions acting on the crossed products.

Let $\alpha$ be a right action of a Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$ on a von Neumann algebra $\mathcal{M}$. A unitary $u\in\mathcal{M}\bar{\otimes}M$ is called an *$\alpha$-cocycle,* if

$$(u\otimes 1)(\alpha\otimes\iota)(u)=(\iota\otimes\Gamma)(u)\ . \tag{2.1}$$

Denote the trivial right action of $\boldsymbol{K}$ on $\mathcal{M}$ by $\iota_{\boldsymbol{K}}^{\mathcal{M}}$, i.e. $\iota_{\boldsymbol{K}}^{\mathcal{M}}(x)=x\otimes 1$ for $x\in\mathcal{M}$. If $\alpha=\iota_{\boldsymbol{K}}^{\mathcal{M}}$, then every $\alpha$-cocycle $u$ satisfies the associativity condition

$$(u\otimes 1)(\iota\otimes\sigma)(u\otimes 1)=(\iota\otimes\Gamma)(u)\ , \tag{2.2}$$

and vice versa.

A pair of right actions $\alpha_j$ of Kac algebras $\boldsymbol{K}_j=(M_j, \Gamma_j, \kappa_j, \phi_j)$ on von Neumann algebras $\mathcal{M}_j$ is said to be *outer conjugate,* if there exist isomorphisms $\pi$ of $\mathcal{M}_1$ onto $\mathcal{M}_2$ and $\Psi$ of $K_1$ onto $K_2$ such that

$$\alpha_2\circ\pi(x)=(\pi\otimes\Psi)\circ Ad u\circ\alpha_1(x)\ ,\qquad x\in\mathcal{M}_1$$

for some $\alpha_1$-cocycle $u\in\mathcal{M}_1\bar{\otimes}M_1$. This is written in the form

$$\{\mathcal{M}_1, \boldsymbol{K}_1, \alpha_1\}\sim\{\mathcal{M}_2, \boldsymbol{K}_2, \alpha_2\}\qquad\text{or}\qquad \alpha_2\sim\alpha_1(\pi, \Psi, u)\ .$$

In case of $u=1\otimes 1$, $\alpha_1$ and $\alpha_2$ are said to be *conjugate* and denoted by

$$\{\mathcal{M}_1, \boldsymbol{K}_1, \alpha_1\}\cong\{\mathcal{M}_2, \boldsymbol{K}_2, \alpha_2\}\ .$$

**Proposition 2.1.** *If $\alpha_j$ $(j=1, 2)$ are right actions of $\boldsymbol{K}_j$ on $\mathcal{M}_j$, then the following two conditions are equivalent*:

(i) $\{\mathcal{M}_1, K_1, \alpha_1\}\sim\{\mathcal{M}_2, \boldsymbol{K}_2, \alpha_2\}$

(ii) $\{\mathcal{W}^*(\alpha_1), \widehat{\boldsymbol{K}_1}', \hat{\alpha}_1\}\cong\{\mathcal{W}^*(\alpha_2), \widehat{\boldsymbol{K}_2}', \hat{\alpha}_2\}$

**Lemma 2.2.** *Let $V$ denote the fundamental operator $(\hat{J}\otimes\hat{J})\sigma(W^*)(\hat{J}\otimes\hat{J})$ of $\boldsymbol{K}^{\wedge\prime}$. If $\alpha$ is a right action of $\boldsymbol{K}$ on $\mathcal{M}$ and $\hat{\alpha}$ is the dual right of $\boldsymbol{K}^{\wedge\prime}$ on $\mathcal{W}^*(\alpha)$, then*

$$(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}\otimes\iota)(1\otimes V^*)=(1\otimes V^*\otimes 1)(1\otimes 1\otimes\sigma(V^*))\ . \tag{2.3}$$

**Proof.** Since the dual right action $\hat{\alpha}$ is implemented by an $\iota_{\boldsymbol{K}^{\wedge\prime}}^{\mathcal{L}(\mathfrak{H}\otimes H)}$-cocycle $1\otimes((J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J}))$ as (1.7), we have

$$(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}\otimes\iota)(1\otimes V^*)=(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes((J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J}))\otimes 1}\circ(\iota\otimes\iota\otimes\sigma)(1\otimes V^*\otimes 1)\ .$$

To prove (2.3) it suffices to show

$$(\iota\otimes\sigma)\circ Ad_{U^*\otimes 1}\circ(\iota\otimes\sigma)(V^*\otimes 1)=(V^*\otimes 1)(1\otimes\sigma(V^*))\ ,$$

where $U=(J\hat{J}\otimes J\hat{J})W(J\hat{J}\otimes J\hat{J})$. This will be done as the following. Since $W^*=(\hat{J}\otimes J)W(\hat{J}\otimes J)$ and

$$
\begin{aligned}
&(1\otimes U^*)(\sigma(V^*)\otimes 1)(1\otimes U)\\
&\quad=(1\otimes J\hat{J}\otimes J\hat{J})(1\otimes W^*)(1\otimes J\hat{J}\otimes J\hat{J})\\
&\qquad\cdot(\hat{J}\otimes\hat{J}\otimes\hat{J})(W\otimes 1)(\hat{J}\otimes\hat{J}\otimes\hat{J})(1\otimes J\hat{J}\otimes J\hat{J})(1\otimes W)(1\otimes J\hat{J}\otimes J\hat{J})\\
&\quad=(1\otimes J\hat{J}\otimes J\hat{J})(1\otimes W^*)(\hat{J}\otimes J\otimes J)(W\otimes 1)(\hat{J}\otimes J\otimes J)(1\otimes W)(1\otimes J\hat{J}\otimes J\hat{J})\\
&\quad=(1\otimes J\hat{J}\otimes J\hat{J})(1\otimes W^*)(W^*\otimes 1)(1\otimes W)(1\otimes J\hat{J}\otimes J\hat{J})\\
&\quad=(1\otimes J\hat{J}\otimes J\hat{J})(W^*\otimes 1)\{(\iota\otimes\sigma)(W^*\otimes 1)\}(1\otimes J\hat{J}\otimes J\hat{J})\,, \qquad \text{by (1.3)},\\
&\quad=(\hat{J}\otimes\hat{J}\otimes\hat{J})(\hat{J}\otimes J\otimes J)(W^*\otimes 1)\{(\iota\otimes\sigma)(W^*\otimes 1)\}(\hat{J}\otimes J\otimes J)(\hat{J}\otimes\hat{J}\otimes\hat{J})\\
&\quad=(\hat{J}\otimes\hat{J}\otimes\hat{J})(W\otimes 1)\{(\iota\otimes\sigma)(W\otimes 1)\}(\hat{J}\otimes\hat{J}\otimes\hat{J})\\
&\quad=(\sigma(V^*)\otimes 1)(\iota\otimes\sigma)(\sigma(V^*)\otimes 1)\,,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
(\iota\otimes\sigma)\circ Ad_{U^*\otimes 1}\circ(\iota\otimes\sigma)(V^*\otimes 1)&=\{(\iota\otimes\sigma)(U^*\otimes 1)\}(V^*\otimes 1)(\iota\otimes\sigma)(U\otimes 1)\\
&=(\sigma\otimes\iota)((1\otimes U^*)(\sigma(V^*)\otimes 1)(1\otimes U))\\
&=(\sigma\otimes\iota)((\sigma(V^*)\otimes 1)(\iota\otimes\sigma)(\sigma(V^*)\otimes 1))\\
&=(V^*\otimes 1)(1\otimes\sigma(V^*))\,. \qquad \text{Q.E.D.}
\end{aligned}
$$

Now we shall go into the proof of our proposition.

**Proof of Proposition 2.1.** (i)⟹(ii): By [4, II.8. Proposition].

(ii)⟹(i): Since $\{\mathcal{M}_j, \boldsymbol{K}_j, \alpha_j\}\cong\{\alpha_j(\mathcal{M}_j), \boldsymbol{K}_j, \iota\otimes\Gamma_j\}$, we have only to show that $\{\alpha_1(\mathcal{M}_1), \boldsymbol{K}_1, \iota\otimes\Gamma_1\}$ is outer conjugate to $\{\alpha_2(\mathcal{M}_2), \boldsymbol{K}_2, \iota\otimes\Gamma_2\}$. Let $\pi$ (resp. $\Psi$) be the isomorphism of $\mathscr{W}^*(\alpha_1)$ (resp. $\widehat{\boldsymbol{K}_1}'$) onto $\mathscr{W}^*(\alpha_2)$ (resp. $\widehat{\boldsymbol{K}_2}'$) such that $(\pi\otimes\Psi)\circ\hat{\alpha}_1=\hat{\alpha}_2\circ\pi$. Let $W_j$ be the fundamental operators of $\boldsymbol{K}_j$ and $V_j$ the fundamental operators $(\hat{J}\otimes\hat{J})\sigma(W_j^*)(\hat{J}\otimes\hat{J})$ of $\widehat{\boldsymbol{K}_j}'$. Then $V_j$ implements $\Gamma_j$. Since $W_j\in M_j\bar{\otimes}\hat{M}_j$, we have $V_j\in\hat{M}_j'\bar{\otimes}M_j$ and

$$1\otimes V_j\in\mathscr{W}^*(\alpha_j)\bar{\otimes}M_j \qquad (j=1, 2)\,.$$

Here we set

$$w_1=(\pi\otimes\Psi)(1\otimes V_1) \qquad \text{and} \qquad w_2=1\otimes V_2\,.$$

Then $w_1$ and $w_2$ belong to $\mathscr{W}^*(\alpha_2)\bar{\otimes}M_2$. Since

$$
\begin{aligned}
(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_2\otimes\iota)(w_1^*)&=(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_2\otimes\iota)\circ(\pi\otimes\Psi)(1\otimes V_1^*)\\
&=(\iota\otimes\iota\otimes\sigma)\circ(\pi\otimes\Psi\otimes\Psi)\circ(\hat{\alpha}_1\otimes\iota)(1\otimes V_1^*)\\
&=(\pi\otimes\Psi\otimes\Psi)\circ(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_1\otimes\iota)(1\otimes V_1^*)\\
&=(\pi\otimes\Psi\otimes\Psi)((1\otimes V_1^*\otimes 1)(1\otimes 1\otimes\sigma(V_1^*)))\,, \qquad \text{by Lemma 2.2},\\
&=(w_1^*\otimes 1)(1\otimes 1\otimes\sigma(V_2^*))
\end{aligned}
$$

and

$$(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_2\otimes\iota)(w_2^*)=(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_2\otimes\iota)(1\otimes V_2^*)$$
$$=(1\otimes V_2^*\otimes 1)(1\otimes 1\otimes\sigma(V_2^*))$$
$$=(w_2^*\otimes 1)(1\otimes 1\otimes\sigma(V_2^*))\ ,$$

it follows that

$$(\iota\otimes\iota\otimes\sigma)\circ(\hat{\alpha}_2\otimes\iota)(w_1^*w_2)=w_1^*w_2\otimes 1\ .$$

This implies $w_2^*w_1\in\alpha_2(\mathcal{M}_2)\bar{\otimes}M_2$ by [6, Théorème IV.2]. Since $\Gamma_2$ is implemented by a unitary $w_2^*$, we have

$$(w_1^*w_2\otimes 1)(\iota\otimes\Gamma_2\otimes\iota)(w_1^*w_2)=(w_1^*w_2\otimes 1)Ad_{w_2^*\otimes 1}\circ(\iota\otimes\iota\otimes\sigma)(w_1^*w_2\otimes 1)$$
$$=(w_1^*\otimes 1)\{(\iota\otimes\iota\otimes\sigma)(w_1^*\otimes 1)\}\{(\iota\otimes\iota\otimes\sigma)(w_2\otimes 1)\}(w_2\otimes 1)\ ,$$

Since $(V_j^*\otimes 1)(\iota\otimes\sigma)(V_j^*\otimes 1)=(\iota\otimes\Gamma_j)(V_j^*)$ by Lemma 1.2, we have

$$(w_j^*\otimes 1)(\iota\otimes\iota\otimes\sigma)(w_j^*\otimes 1)=(\iota\otimes\iota\otimes\Gamma_2)(w_j^*)\ ,$$

and hence

$$(w_1^*w_2\otimes 1)(\iota\otimes\Gamma_2\otimes\iota)(w_1^*w_2)=(\iota\otimes\iota\otimes\Gamma_2)(w_1^*w_2)\ .$$

This means that $w_1^*w_2$ is an $(\iota\otimes\Gamma_2)$-cocycle in $\alpha_2(\mathcal{M}_2)\bar{\otimes}M_2$ and so $\{\alpha_2(\mathcal{M}_2), \boldsymbol{K}_2, \iota\otimes\Gamma_2\}$ is outer conjugate to $\{\alpha_2(\mathcal{M}_2), \boldsymbol{K}_2, Ad_{w_1^*}\}$. Since $\{\alpha_1(\mathcal{M}_1), \boldsymbol{K}_1, \iota\otimes\Gamma_1\}$ is conjugate to $\{\alpha_2(\mathcal{M}_2), \boldsymbol{K}_2, Ad_{w_1^*}\}$, it follows that $\{\alpha_1(\mathcal{M}_1), \boldsymbol{K}_1, \iota\otimes\Gamma_1\}$ is outer conjugate to $\{\alpha_2(\mathcal{M}_2), \boldsymbol{K}_2, \iota\otimes\Gamma_2\}$. Q.E.D.

**Corollary 2.3.** (a) *Let $\alpha_j$ be actions of G on $\mathcal{M}_j$. It is necessary and sufficient for $\{\mathcal{M}_1\times_{\alpha_1}G, \hat{\alpha}_1\}\cong\{\mathcal{M}_2\times_{\alpha_2}G, \hat{\alpha}_2\}$ that $\{\mathcal{M}_1, \alpha_1\}$ is outer conjugate to $\{\mathcal{M}_2, \alpha_2\}$.*

(b) *Let $\delta_j$ be co-actions of G on $\mathcal{N}_j$. It is necessary and sufficient for $\{\mathcal{N}_1\times_{\delta_1}G, \hat{\delta}_1\}\cong\{\mathcal{N}_2\times_{\delta_2}G, \hat{\delta}_2\}$ that $\{\mathcal{N}_1, \delta_1\}$ is outer conjugate to $\{\mathcal{N}_2, \delta_2\}$.*

## § 3. Integrable actions.

For a right action $\alpha$ of a Kac algebra $\boldsymbol{K}=(M,\Gamma,\kappa,\phi)$ on a von Neumann algebra $\mathcal{M}$, we can define a faithful normal $\mathcal{M}^\alpha$-valued weight $\mathcal{E}_\alpha$ on $\mathcal{M}$ by $(\iota\otimes\phi)\circ\alpha$, i.e.

$$\langle\mathcal{E}_\alpha(x), \omega\rangle=(\omega\otimes\phi)(\alpha(x))\ , \qquad x\in\mathcal{M}_+\ , \qquad \omega\in\mathcal{M}_*^+\ .$$

When $\mathcal{E}_\alpha$ is semi-finite, i.e. $\{x\in\mathcal{M}: \mathcal{E}_\alpha(x^*x)\text{ exists}\}$ is $\sigma$-weakly dense in $\mathcal{M}$, the right action $\alpha$ is said to be *integrable*, [6, Definition II.3] The following theorem is a generalization of [3, 11, 13], whose proof will proceed similarly as [13].

**Theorem 3.1.** *Assume the separability of underlying Hilbert spaces $H$ of $M$ and $\mathfrak{H}$ of $\mathcal{M}$. If $\mathcal{M}^\alpha=\{x\in\mathcal{M}: \alpha(x)=x\otimes 1\}$ is properly infinite, then the following three conditions are equivalent:*

(i) *$\alpha$ is integrable.*

(ii) *For any non zero projection $f\in\mathscr{M}^{\alpha}\otimes C$ there exists a non zero $y\in\mathscr{M}\bar{\otimes}\mathscr{L}(H)$ such that $y=fyf$ and*

$$\bar{\alpha}(y)=(1\otimes\sigma(W))(y\otimes 1)\,. \qquad (\bar{\alpha}\equiv(\iota\otimes\sigma)\circ(\alpha\otimes\iota)) \tag{3.1}$$

(iii) *$\{\bar{\mathscr{M}},\tilde{\alpha}\}\cong\{\bar{\mathscr{M}},\tilde{\alpha}\}_e$ for some projection $e\in\bar{\mathscr{M}}^{\tilde{\alpha}}$,*

*where $\bar{\mathscr{M}}=\mathscr{M}\bar{\otimes}\mathscr{L}(H)$ and $\tilde{\alpha}$ is defined by* (1.8).

Before entering the proof it should be noted that if $\alpha$ is integrable, so is the restriction $\alpha^e$ of $\alpha$ to $\mathscr{M}_e$ for any non zero projection $e$ in $\mathscr{M}^{\alpha}$.

**Proof.** (i)$\Rightarrow$(ii): As our assertion is true within a conjugate class, we may assume that $\mathscr{M}$ is standard. Suppose that $\alpha$ is integrable. There exists a faithful semi-finite normal weight $\psi$ on $\mathscr{M}$ which is $\hat{\Delta}^{-1}$-relatively invariant:

$$(\psi\otimes\iota)(\alpha(x))=\psi(x)\hat{\Delta}^{-1}\,, \qquad x\in\mathscr{M}_+\,;$$

$$(\psi\otimes\omega_{\hat{\Delta}^{1/4}\xi})(\alpha(y^*)(x\otimes 1))$$

$$=(\psi\otimes\omega_{\hat{\Delta}^{-1/4}\xi\circ\kappa})((y^*\otimes 1)\alpha(x))\,, \qquad x, y\in\mathfrak{n}_{\psi}\,, \qquad \xi\in\mathscr{D}(\hat{\Delta}^{1/4})\cap\mathscr{D}(\hat{\Delta}^{-1/4})\,,$$

where $\hat{\Delta}$ is the modular operator of $\{\hat{M},\hat{\phi}\}$, [6, Théorème III.4]. The integrability assures the existence of a non zero element $z\in\mathfrak{n}_{\psi}$. Fix a non zero element $d\in\mathfrak{n}_{\phi}$. If $a_j\in\mathfrak{n}_{\phi}$ and $x_j\in\mathfrak{n}_{\psi}$ for $j=1, 2, \dots, n$, then

$$\|\sum\phi(d^*a_j)x_j\|_{\psi}\le\|d\|_{\phi}\|\sum x_j\otimes a_j\|_{\psi\otimes\phi}$$

by universal property of tensor product, so that

$$\begin{aligned}\|\sum\phi(d^*a_j)\eta_{\psi\otimes\phi}(\alpha(z)(x_j\otimes 1))\|^2_{\psi\otimes\phi}&=((\sum_j\phi(d^*a_j)x_j)\psi(\sum_k\phi(d^*a_k)x_k)^*\otimes\phi)(\alpha(z^*z))\\&=\langle\mathscr{E}_{\alpha}(z^*z), (\sum_j\phi(d^*a_j)x_j)\psi(\sum_k\phi(d^*a_k)x_k)^*\rangle\\&=\|\mathscr{E}_{\alpha}(z^*z)\|\|\sum_j\phi(d^*a_j)x_j\|^2_{\psi}\\&\le\|\mathscr{E}_{\alpha}(z^*z)\|\|d\|^2_{\phi}\|\sum x_j\otimes a_j\|^2_{\psi\otimes\phi}\end{aligned}$$

Therefore a bounded linear operator $y$ on $\mathfrak{H}\otimes H$ is well defined by

$$y\eta_{\psi\otimes\phi}(\sum x_j\otimes a_j)=\sum\phi(d^*a_j)\eta_{\psi\otimes\phi}(\alpha(z)(x_j\otimes 1))\,. \tag{3.2}$$

Since $\mathfrak{n}_{\psi}$ is dense in $\mathscr{M}$ and $z\neq 0$, we have $\alpha(z)(x\otimes 1)\neq 0$ for some $x\in\mathfrak{n}_{\psi}$. Therefore $y\neq 0$. If $y'\in\mathscr{M}'$, then

$$\alpha(z)(y'x\otimes 1)=(y'\otimes 1)\alpha(z)(x\otimes 1)$$

and hence $y$ commutes with $y'\otimes 1$. Thus $y\in\mathscr{M}\bar{\otimes}\mathscr{L}(H)$.

It remains to show that $y$ satisfies (3.1). Since $\alpha$ is integrable and $\mathscr{M}$ is standard, there is an $\iota_K^{\mathscr{L}(\mathfrak{H})}$-cocycle $u\in\mathscr{L}(\mathfrak{H})\bar{\otimes}M$ implementing $\alpha$:

$$\text{(3.3)}\qquad u(\eta_\psi(x)\otimes\hat{\Delta}^{-1/2}\eta_\phi(a))=\eta_{\psi\otimes\phi}(\alpha(x)(1\otimes a))\ ,\qquad x\in\mathfrak{n}_\psi\ ,\qquad a\in\mathfrak{n}_\phi\cap\mathfrak{n}_{\phi\circ\kappa}\ ,$$

[6, Corollaire III.19]. Then $\bar{\alpha}=(\iota\otimes\sigma)\circ(\alpha\otimes\iota)$ satisfies

$$\bar{\alpha}(y)=Ad_{(\iota\otimes\sigma)(u\otimes1)}(y\otimes1)\ ,\qquad y\in\mathscr{M}\bar{\otimes}\mathscr{L}(H)\ .$$

Here we set $\omega=\psi\otimes\phi\otimes\phi$. For any $x\in\mathfrak{n}_\psi$ and $a,b\in\mathfrak{n}_\phi$, we have

$$\begin{aligned}&\bar{\alpha}(y)\eta_\omega(\bar{\alpha}(x\otimes1)(1\otimes a\otimes b))\\&\quad=\{(\iota\otimes\sigma)(u\otimes1)\}(y\otimes1)(\eta_{\psi\otimes\phi}(x\otimes a)\otimes\hat{\Delta}^{-1/2}\eta_\phi(b))\ ,\qquad\text{by (3.3)},\\&\quad=\phi(d^*a)(\iota\otimes\sigma)(u\otimes1)(\eta_{\psi\otimes\phi}(\alpha(z)(x\otimes1))\otimes\hat{\Delta}^{-1/2}\eta_\phi(b))\ ,\qquad\text{by (3.2)},\\&\quad=\phi(d^*a)\eta_\omega(\bar{\alpha}(\alpha(z)(x\otimes1))(1\otimes1\otimes b))\ ,\qquad\text{by (3.3)},\\&\quad=\phi(d^*a)\eta_\omega(\{(\iota\otimes\sigma)\circ(\iota\otimes\Gamma)(\alpha(z))\}\bar{\alpha}(x\otimes1)(1\otimes1\otimes b))\ .\end{aligned}$$

The fundamental operator $W=W(\boldsymbol{K})$ of $\boldsymbol{K}$ was defined by $W\eta_{\phi\otimes\phi}(a\otimes b)=\eta_{\phi\otimes\phi}(\Gamma(b)(a\otimes1))$, and so $\sigma(W)\eta_{\phi\otimes\phi}(b\otimes a)=\eta_{\phi\otimes\phi}(\sigma\circ\Gamma(b)(1\otimes a))$. Therefore

$$\begin{aligned}&\phi(d^*a)\eta_\omega(\{(\iota\otimes\sigma)\circ(\iota\otimes\Gamma)(\alpha(z))\}\bar{\alpha}(x\otimes1)(1\otimes1\otimes b))\\&\quad=\phi(d^*a)(1\otimes\sigma(W))\eta_\omega((\alpha(z)\otimes1)\bar{\alpha}(x\otimes1)(1\otimes1\otimes b))\\&\quad=(1\otimes\sigma(W))(y\otimes1)\eta_\omega(\bar{\alpha}(x\otimes1)(1\otimes a\otimes b))\ ,\qquad\text{by (3.2)}.\end{aligned}$$

Consequently, we have $\bar{\alpha}(y)=(1\otimes\sigma(W))(y\otimes1)$.

(ii)$\Rightarrow$(iii): Put $\bar{\mathscr{M}}=\mathscr{M}\bar{\otimes}\mathscr{L}(H)$ and $\bar{\alpha}=(\iota\otimes\sigma)\circ(\alpha\otimes\iota)$. Let $\mathscr{I}$ be the set of all $y\in\bar{\mathscr{M}}$ with (3.1). Since $\tilde{\alpha}=Ad_{1\otimes\sigma(W^*)}\circ\bar{\alpha}$, we have a right action $\beta$ of $\boldsymbol{K}$ on $\bar{\mathscr{M}}\otimes F_2$ defined by

$$\begin{aligned}&\beta=Ad_v\circ(\iota\otimes\iota\otimes\sigma)\circ(\bar{\alpha}\otimes\iota)\\&v=(\iota\otimes\iota\otimes\sigma)(1\otimes1\otimes1\otimes e_{11}+1\otimes\sigma(W^*)\otimes e_{22})\ ,\end{aligned}$$

where $F_2$ is a $\mathrm{I}_2$-factor and $e_{ij}$ are the matrix units. Therefore $\sum x_{ij}\otimes e_{ij}\in(\bar{\mathscr{M}}\otimes F_2)^\beta$ if and only if

$$x_{11}\in\bar{\mathscr{M}}^{\bar{\alpha}}\ ,\qquad x_{12}^*\in\mathscr{I}\ ,\qquad x_{21}\in\mathscr{I}\ ,\qquad x_{22}\in\bar{\mathscr{M}}^{\tilde{\alpha}}$$

Condition (ii) implies that the central support of $1\otimes1\otimes e_{11}$ in $(\bar{\mathscr{M}}\otimes F_2)^\beta$ is majorized by the central support of $1\otimes1\otimes e_{22}$ in $(\bar{\mathscr{M}}\otimes F_2)^\beta$. Since $\mathscr{M}^\alpha$ is properly infinite, $1\otimes1\otimes e_{22}$ is also properly infinite in $(\bar{\mathscr{M}}\otimes F_2)^\beta$. Since $1\otimes1\otimes e_{22}$ is $\sigma$-finite in $(\bar{\mathscr{M}}\otimes F_2)^\beta$, it follows that

$$1\otimes1\otimes e_{11}\prec1\otimes1\otimes e_{22}$$

in $(\bar{\mathscr{M}}\otimes F_2)^\beta$. Thus there exists an isometry $u\in\mathscr{I}$, because $u\otimes e_{21}\in(\bar{\mathscr{M}}\otimes F_2)^\beta$ implies $u\in\mathscr{I}$. Put $e=uu^*$. Then $e\in\bar{\mathscr{M}}^{\tilde{\alpha}}$, which yields condition (iii).

(iii)$\Rightarrow$(i): Since $\tilde{\alpha}$ is integrable [6, Proposition II. 4], so is $\tilde{\alpha}^e$ for any projection

$e\in\bar{\mathscr{M}}^{\bar{\alpha}}$. Therefore $\bar{\alpha}$ is integrable by (iii). If $p$ is a minimal projection in $\mathscr{L}(H)$, then $1\otimes p\in\bar{\mathscr{M}}^{\bar{\alpha}}$ and $\bar{\alpha}^{1\otimes p}$ is integrable on $\bar{\mathscr{M}}_{1\otimes p}$. Since $\{\mathscr{M},\alpha\}\cong\{\bar{\mathscr{M}},\bar{\alpha}\}_{1\otimes p}$, $\alpha$ is integrable on $\mathscr{M}$. Q.E.D.

## § 4. Unitary implementability of action

The unitary implementability of a co-action of a locally compaet group on a von Neumann algebra is our conjecture is [13, Chapter III]. It is already known that any right action of a Kac algebra on a standard von Neumann algebra is unitarily implementable if it is integrable, [6, 13]. Our aim is to remove this assumption for properly infinite von Neumann algebras.

**Theorem 4.1.** *Assume that $\mathscr{M}$ is standard and spatially isomorphic to $\mathscr{M}\bar{\otimes}\mathscr{L}(H)$. A right action $\alpha$ of a Kac algebra $\boldsymbol{K}=(M,\Gamma,\kappa,\phi)$ on $\mathscr{M}$ is implemented by an $\iota_{\boldsymbol{K}}^{\mathscr{L}(\mathfrak{H})}$-cocycle $u\in\mathscr{L}(\mathfrak{H})\bar{\otimes}M$ so that*

$$\alpha(x)=u(x\otimes 1)u^*\ ,\qquad x\in\mathscr{M}\ .$$

The following lemma is an immediate consequence of the associativity condition (1.3) of $W$.

**Lemma 4.2.** *The unitary $1\otimes\sigma(W)$ is an $\tilde{\alpha}$-cocycle.*

**Proof.** Put $\bar{\alpha}=(\iota\otimes\sigma)\circ(\alpha\otimes\iota)$. Then $1\otimes\sigma(W^*)$ is an $\bar{\alpha}$-cocycle in $\mathscr{M}\bar{\otimes}\mathscr{L}(H)\bar{\otimes}M$ by [4, III. 6. Proposition]. Since $\tilde{\alpha}=Ad_{1\otimes\sigma(W^*)}\circ\bar{\alpha}$, it follows that

$$\begin{aligned}(1\otimes\sigma(W)\otimes 1)(\tilde{\alpha}\otimes\iota)(1\otimes\sigma(W))&=\{(\bar{\alpha}\otimes\iota)(1\otimes\sigma(W))\}(1\otimes\sigma(W)\otimes 1)\\&=(\iota\otimes\iota\otimes\Gamma)(1\otimes\sigma(W))\ .\end{aligned}$$

This means that $1\otimes\sigma(W)$ is an $\tilde{\alpha}$-cocycle in $\mathscr{M}\bar{\otimes}\mathscr{L}(H)\bar{\otimes}M$. Q.E.D.

**Proof of Theorem 4.1.** Since $\{\mathscr{W}^*(\mathscr{W}^*(\alpha),\boldsymbol{K}^{\wedge\prime},\hat{\alpha}),\boldsymbol{K},\hat{\hat{\alpha}}\}$ is conjugate to $\{\mathscr{M}\bar{\otimes}\mathscr{L}(H),\boldsymbol{K},\tilde{\alpha}\}$ by Takesaki's duality, $\tilde{\alpha}=Ad_{1\otimes\sigma(W^*)}\circ(\iota\otimes\sigma)\circ(\alpha\otimes\iota)$ is integrable [6, Proposition II. 4]. Since $\mathscr{M}$ in standard and spatially isomorphic to $\mathscr{M}\bar{\otimes}\mathscr{L}(H)$ by assumption, $\tilde{\alpha}$ is implemented by a unitary $v$ in $\mathscr{L}(\mathfrak{H}\otimes H)\bar{\otimes}M$ with the associativity condition

$$(4.1)\qquad (v\otimes 1)(\iota\otimes\iota\otimes\sigma)(v\otimes 1)=(\iota\otimes\iota\otimes\Gamma)(v)\ ,$$

by [6, Corollaire III. 19]. That is

$$Ad_{1\otimes\sigma(W^*)}\circ(\iota\otimes\sigma)\circ(\alpha\otimes\iota)(x)=Ad_v(x\otimes 1)\ ,\qquad x\in\mathscr{M}\bar{\otimes}\mathscr{L}(H)$$

and hence, with $y\in\mathscr{L}(H)$,

$$Ad_{(1\otimes W)(\iota\otimes\sigma)(v)}(1\otimes 1\otimes y)=1\otimes 1\otimes y\ .$$

This means that $(1\otimes W)(\iota\otimes\sigma)(v)$ belongs to $\mathscr{L}(\mathfrak{H}\otimes H)\otimes \boldsymbol{C}$. As $1\otimes\sigma(W)$ and $v$ are elements in $\mathscr{L}(\mathfrak{H}\otimes H)\bar{\otimes}M$, there is a unitary $u$ in $\mathscr{L}(\mathfrak{H})\bar{\otimes}M$ such that

$$(1\otimes W)(\iota\otimes\sigma)(v)=u\otimes 1 .$$

For any $x\in\mathscr{M}$, we have

$$\begin{aligned}\alpha(x)\otimes 1&=(\alpha\otimes\iota)(x\otimes 1)=Ad_{u\otimes 1}\circ(\iota\otimes\sigma)(x\otimes 1\otimes 1)\\&=Ad_{u\otimes 1}(x\otimes 1\otimes 1)=(u(x\otimes 1)u^*)\otimes 1\end{aligned}$$

and so $\alpha(x)=u(x\otimes 1)u^*$ for $x\in\mathscr{M}$.

Finally we shall show that $u$ satisfies the associativity condition. By direct computation we have

$$\begin{aligned}&(\sigma\otimes\iota\otimes\iota)\{(1\otimes u\otimes 1)(\iota\otimes\iota\otimes\sigma)(1\otimes u\otimes 1)\}\\&=\{(\iota\otimes\sigma\otimes\iota)(u\otimes 1\otimes 1)\}(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)(u\otimes 1\otimes 1)\\&=\{(\iota\otimes\sigma\otimes\iota)((1\otimes W\otimes 1)(\iota\otimes\sigma\otimes\iota)(v\otimes 1))\}\\&\quad\cdot(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)((1\otimes W\otimes 1)(\iota\otimes\sigma\otimes\iota)(v\otimes 1))\\&=(1\otimes\sigma(W)\otimes 1)(v\otimes 1)(\iota\otimes\iota\otimes\sigma)((\iota\otimes\sigma(W)\otimes 1)(v\otimes 1))\\&=\{(1\otimes\sigma(W)\otimes 1)(\tilde{\alpha}\otimes\iota)(\iota\otimes\sigma(W))\}(v\otimes 1)(\iota\otimes\iota\otimes\sigma)(v\otimes 1)\\&=\{(\iota\otimes\iota\otimes\Gamma)(1\otimes\sigma(W))\}(\iota\otimes\iota\otimes\Gamma)(v)\ , \qquad \text{by Lemma 4.2 and (4.1),}\\&=(\iota\otimes\iota\otimes\Gamma)((1\otimes\sigma(W))v)\\&=(\iota\otimes\iota\otimes\Gamma)\circ(\iota\otimes\sigma)(u\otimes 1)\\&=(\iota\otimes\iota\otimes\Gamma)\circ(\sigma\otimes\iota)(1\otimes u)\\&=(\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\Gamma)(1\otimes u)\end{aligned}$$

and hence $(u\otimes 1)(\iota\otimes\sigma)(u\otimes 1)=(\iota\otimes\Gamma)(u)$. Q.E.D.

**Corollary 4.3.** *Let $\delta$ be a co-action of a lacally compact group $G$ on a von Neumann algebra $\{\mathscr{N},\mathfrak{K}\}$. If $\mathscr{N}$ is properly infinite and standard, then $\delta$ is unitarily implementable, i.e. $\delta(x)=w^*(x\otimes 1)w$ for some unitary $w\in\mathscr{L}(\mathfrak{K})\bar{\otimes}\mathscr{R}(G)$ with $(w\otimes 1)(\iota\otimes\sigma)(w\otimes 1)=(\iota\otimes\delta_G)(w)$.*

## § 5. Inner tensor product of actions.

Given two actions $\alpha^j (j=1,2)$ of locally compact groups $G_j$ on $\mathscr{M}_j$, we can construct an action $\alpha^1\times\alpha^2$:

$$(s,t)\in G_1\times G_2\mapsto\alpha^1_s\otimes\alpha^2_t$$

of a locally compact group $G_1\times G_2$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$, that is, an isomorphism $(\iota\otimes\sigma\otimes\iota)\circ(\alpha^1\otimes\alpha^2)$ of $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ into $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2\bar{\otimes}\mathscr{R}(G_1\times G_2)$ satisfying (1.6). When $G_1=G_2(=G)$, we can construct another action $\alpha^1 * \alpha^2$:

$$t\in G\mapsto\alpha^1_t\otimes\alpha^2_t$$

of $G$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$. We shall generalize these tensor products to the Kac algebra context.

Let $\boldsymbol{K}_j=(M_j, \Gamma_j, \kappa_j, \phi_j)$ be Kac algebras. For any right actions $\alpha_1$ of $\boldsymbol{K}_1$ on $\mathscr{M}_1$ and $\alpha_2$ of $\boldsymbol{K}_2$ on $\mathscr{M}_2$, the isomorphism

$$\alpha\times\alpha_2=(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2) \tag{5.1}$$

of $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ into $(\mathscr{M}_1\bar{\otimes}\mathscr{M}_2)\bar{\otimes}(M_1\bar{\otimes}M_2)$ turns out to be a right action of the Kac algebra $\boldsymbol{K}_1\otimes\boldsymbol{K}_2$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$, [15, Chapitre VI], which will be called the *outer tensor product* of $\alpha_1$ and $\alpha_2$. When $\boldsymbol{K}_1=\boldsymbol{K}_2(=\boldsymbol{K})$, we can construct an inner tensor product of $\alpha_1$ and $\alpha_2$, i.e. a right action of $\boldsymbol{K}$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ as follows: For any $\iota_{\boldsymbol{K}}^{\mathscr{L}(\mathfrak{H}_2)}$-cocycle $u$ implementing $\alpha_2$ we set

$$\alpha_1 *_u \alpha_2=Ad_{1\otimes u}\circ(\iota\otimes\sigma)\circ(\alpha_1\otimes\iota)\ . \tag{5.2}$$

It is easy to see that $\alpha_1 *_u\alpha_2$ is a right action of $\boldsymbol{K}$ on $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$. Indeed,

$$\begin{aligned}
&((\alpha_1 *_u\alpha_2)\otimes\iota)\circ(\alpha_1 *_u\alpha_2)\\
&\quad=Ad_{1\otimes u\otimes 1}\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\iota\otimes\iota)\circ Ad_{1\otimes u}\circ(\iota\otimes\sigma)\circ(\alpha_1\otimes\iota)\\
&\quad=Ad_{1\otimes u\otimes 1}\circ(\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes 1\otimes u}\circ(\iota\otimes\iota\otimes\sigma)\circ(\alpha_1\otimes\iota\otimes\iota)\circ(\alpha_1\otimes\iota)\\
&\quad=Ad_{(\iota\otimes\iota\otimes\Gamma)(1\otimes u)}\circ(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\Gamma\otimes\iota)\circ(\alpha_1\otimes\iota)\ , \qquad \text{by (2.2)},\\
&\quad=Ad_{(\iota\otimes\iota\otimes\Gamma)(1\otimes u)}\circ(\iota\otimes\iota\otimes\Gamma)\circ(\iota\otimes\sigma)\circ(\alpha_1\otimes\iota)\\
&\quad=(\iota\otimes\iota\otimes\Gamma)\circ Ad_{1\otimes u}\circ(\iota\otimes\sigma)\circ(\alpha_1\otimes\iota)\\
&\quad=(\iota\otimes\iota\otimes\Gamma)\circ(\alpha_1 *_u\alpha_2)\ ,
\end{aligned}$$

where the fourth equality follows from

$$(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\Gamma\otimes\iota)=(\iota\otimes\iota\otimes\Gamma)\circ(\iota\otimes\sigma)\ . \tag{5.3}$$

Therefore $\alpha_1 *_u\alpha_2$ is one of the candidates of inner tensor product. However, it is desirable to define it independently from a given $\iota_{\boldsymbol{K}}^{\mathscr{L}(\mathfrak{H}_2)}$-cocycle $u$ implementing $\alpha_2$. Notice that $\{\mathscr{M}_2, \boldsymbol{K}, \alpha_2\}$ is conjugate to $\{\alpha_2(\mathscr{M}_2), \boldsymbol{K}, \iota\otimes\Gamma\}$ and $\iota\otimes\Gamma$ is implemented by a unitary $1\otimes V^*$, where $V=W(\boldsymbol{K}^{\frown\prime})$. As $1\otimes V^*\in\mathscr{L}(\mathfrak{H}_2\otimes H)\bar{\otimes}M$ satisfies the associativity condition by Lemma 1.2, the isomorphism $\alpha_1 *_{(1\otimes V^*)}(\iota\otimes\Gamma)$ is a right action of $\boldsymbol{K}$ on $\mathscr{M}_1\bar{\otimes}\alpha_2(\mathscr{M}_2)$ by (5.2). Therefore

$$(\alpha_1 *_{(1\otimes V^*)}(\iota\otimes\Gamma))(\mathscr{M}_1\bar{\otimes}\alpha_2(\mathscr{M}_2))\subset\mathscr{M}_1\bar{\otimes}\alpha_2(\mathscr{M}_2)\bar{\otimes}M\ .$$

This inclusion assures the following definition.

**Definition 5.1.** For any right actions $\alpha_1$ and $\alpha_2$ of $\boldsymbol{K}$ on $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively, we denote

$$\alpha_1 *\alpha_2=(\iota\otimes\alpha_2^{-1}\otimes\iota)\circ(\alpha_1 *_{(1\otimes V^*)}(\iota\otimes\Gamma))\circ(\iota\otimes\alpha_2)\ , \tag{5.4}$$

which is called the *inner tensor product* of $\alpha_1$ and $\alpha_2$.

It is immediate from the definition that

$$(\iota\otimes\alpha_2\otimes\iota)\circ(\alpha_1*\alpha_2)=Ad_{1\otimes 1\otimes V^*}\circ(\iota\otimes\iota\otimes\sigma)\circ(\alpha_1\times\alpha_2)\ , \tag{5.5}$$

which connects inner and outer tensor products.

**Proposition 5.2.** *If $\alpha_1$ and $\alpha_2$ are right actions of $\boldsymbol{K}$ on $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then $\alpha_1*\alpha_2$ is a right action of $\boldsymbol{K}$ on $\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$. If $x\in\mathcal{M}_2$, then* $(\alpha_1*\alpha_2)(1\otimes x)=1\otimes\alpha_2(x)$.

**Proof.** We set isomorphism $\beta_1$, $\beta$ and $\gamma$ by

$$\beta_1=Ad_{1\otimes 1\otimes V^*\otimes 1}\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ Ad_{1\otimes 1\otimes 1\otimes V^*}$$
$$\beta=\beta_1\circ(\alpha_1\otimes\iota\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2)$$

and

$$\gamma=(\iota\otimes\iota\otimes\iota\otimes\Gamma)\circ Ad_{1\otimes 1\otimes V^*}\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2)\ .$$

Then we have

$$((\alpha_1*\alpha_2)\otimes\iota)\circ(\alpha_1*\alpha_2)=(\iota\otimes\alpha_2^{-1}\otimes\iota\otimes\iota)\circ\beta$$

and

$$(\iota\otimes\iota\otimes\Gamma)\circ(\alpha_1*\alpha_2)=(\iota\otimes\alpha_2^{-1}\otimes\iota\otimes\iota)\circ\gamma\ .$$

It suffices to show $\beta=\gamma$. Since $(\sigma\otimes\iota)\circ(\iota\otimes\sigma)(V^*\otimes 1)=1\otimes V^*$, we have

$$\beta_1=Ad_{1\otimes 1\otimes V^*\otimes 1}\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes 1\otimes 1\otimes V^*}\circ(\iota\otimes\sigma\otimes\iota\otimes\iota)$$
$$=Ad_{1\otimes 1\otimes V^*\otimes 1}\circ Ad_{1\otimes 1\otimes(\iota\otimes\sigma)(V^*\otimes 1)}\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\sigma\otimes\iota\otimes\iota)\ .$$

Since

$$\beta=\beta_1\circ(\iota\otimes\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ\{((\alpha_1\otimes\iota)\circ\alpha_1)\otimes\alpha_2\}$$
$$=\beta_1\circ(\iota\otimes\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\Gamma\otimes\iota\otimes\iota)\circ(\alpha_1\otimes\alpha_2)$$

and since

$$(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\Gamma\otimes\iota\otimes\iota)$$
$$=(\iota\otimes\iota\otimes\iota\otimes\Gamma)\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)$$

by (5.3), it follows that

$$\beta=Ad_{1\otimes 1\otimes(V^*\otimes 1)(\iota\otimes\sigma)(V^*\otimes 1)}\circ(\iota\otimes\iota\otimes\iota\otimes\Gamma)\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2)$$
$$=(\iota\otimes\iota\otimes\iota\otimes\Gamma)\circ Ad_{1\otimes 1\otimes V^*}\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2)$$
$$=\gamma\ ,$$

where the second equality follows from Lemma 4.2. Q.E.D.

**Remark.** If $M$ is abelian, then $\alpha_1 * \alpha_2 = \alpha_1 *_u \alpha_2$. Indeed, if $M$ is abelian, then $\alpha_1(x)\otimes y\otimes 1$ commutes with $(\iota\otimes\sigma\otimes\iota)(1\otimes 1\otimes\sigma(V))$ for all $x\in\mathcal{M}_1$ and $y\in\mathcal{M}_2$, and hence

$$\alpha_1(x)\otimes\alpha_2(y) = Ad_{1\otimes 1\otimes u}\circ(\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes 1\otimes\sigma(V)}\circ(\iota\otimes\sigma\otimes\iota)(\alpha_1(x)\otimes y\otimes 1)\ .$$

Therefore

$$\begin{aligned} &Ad_{1\otimes 1\otimes V^*}\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)\circ(\alpha_1\otimes\alpha_2)(x\otimes y) \\ &= Ad_{1\otimes 1\otimes V^*}\circ Ad_{(1\otimes u\otimes 1)(1\otimes 1\otimes V)}\circ(\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\sigma\otimes\iota)(\alpha_1(x)\otimes y\otimes 1) \\ &= Ad_{1\otimes u\otimes 1}\circ(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes u\otimes 1}\circ(\iota\otimes\sigma\otimes\iota)(\alpha_1(x)\otimes y\otimes 1) \\ &= (\iota\otimes\alpha_2\otimes\iota)\circ Ad_{1\otimes u}\circ(\iota\otimes\sigma)(\alpha_1(x)\otimes y)\ . \end{aligned}$$

**Corollary 5.3.** *If $\alpha_j (j=1, 2, 3)$ are right actions of $\boldsymbol{K}$ on $\mathcal{M}_j$, then*

$$(\alpha_1 *_u \alpha_2) *_v \alpha_3 = \alpha_1 *_{(u*v)} (\alpha_2 *_v \alpha_3)\ ,$$

*where $u*v=(1\otimes v)(\iota\otimes\sigma)(u\otimes 1)$.*

**Example 5.4.** If $\alpha$ is an action of $G$ on $\mathcal{M}$, then $\{(\mathcal{M}\times_\alpha G)\times_{\hat\alpha}\hat G, \tilde\alpha\}$ is conjugate to $\{\mathcal{M}\bar\otimes\mathcal{L}(L^2(G)), \tilde\alpha\}$ by Takesaki's duality, [9, 10, 16]. The action $\tilde\alpha$ is of the form $\alpha*\lambda$, i.e. $\tilde\alpha_t=\alpha_t\otimes\lambda_t$, where $\lambda$ is the action of $G$ on $\mathcal{L}(L^2(G))$ defined by $\lambda_t(x)=\lambda(t)x\lambda(t)^*$.

**Example 5.5.** If $\delta$ is a co-action of $G$ on $\mathcal{N}$, then $\{(\mathcal{N}\times_\delta G)\times_{\hat\delta} G, \tilde\delta\}$ is conjugate to $\{\mathcal{N}\bar\otimes\mathcal{L}(L^2(G)), \tilde\delta\}$. The co-action $\tilde\delta$ is of the form $\delta *_{W_G}\delta_G^c$, where $\delta_G^c$ is the co-action of $G$ on $\mathcal{L}(L^2(G))$ defined by $\delta_G^c(x)=Ad_{W_G}(x\otimes 1)$.

**Example 5.6.** If $\alpha$ is a right action of a Kac algebra $\boldsymbol{K}=(M, \Gamma, \kappa, \phi)$ on $\mathcal{M}$, then $\tilde\alpha=\alpha *_{\sigma(W^*)}\hat\Gamma$, where $\boldsymbol{K}\hat{}=(\hat M, \hat\Gamma\,\hat\kappa, \hat\phi)$. Indeed, the fundamental operator of $\boldsymbol{K}$ is $\sigma(W^*)$.

It should be noted that $\alpha_1*\alpha_2$ is not symmetric, i.e. $\{\mathcal{M}_1\bar\otimes\mathcal{M}_2, \boldsymbol{K}, \alpha_1*\alpha_2\}$ is not necessarily conjugate to $\{\mathcal{M}_2\bar\otimes\mathcal{M}_1, \boldsymbol{K}, \alpha_2*\alpha_1\}$. However, if we set

$$\pi_j=(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes 1\otimes V}\circ(\iota\otimes\alpha_j\otimes\iota)\ ,$$

then, by Definition 5.1,

$$\alpha_2*\alpha_1=\pi_2^{-1}\circ\gamma\circ\pi_1\circ(\alpha_1*\alpha_2)\ ,$$

where $\gamma=(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ(\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\sigma\otimes\iota)$.

**Proposition 5.7.** *Let $\alpha_j (j=1,2)$ be right actions of $\boldsymbol{K}$ on $\mathcal{M}_j$. If $\alpha_2$ is integrable, then the inner tensor product $\alpha_1*\alpha_2$ is also integrable.*

**Proof.** Let $\mathcal{E}_{\alpha_2}$ and $\mathcal{E}_{\alpha_1*\alpha_2}$ be the faithful normal operator valued weight on

$\mathscr{M}_2$ and $\mathscr{M}_1\bar{\otimes}\mathscr{M}_2$ defined by $(\iota\otimes\phi)\circ\alpha_2$ and $(\iota\otimes\iota\otimes\phi)\circ(\alpha_1*\alpha_2)$, respectively. Let $\mathfrak{q}_{\alpha_2}=\{x\in\mathscr{M}_2\colon \mathscr{E}_{\alpha_2}(x^*x)$ exists$\}$ and $\mathfrak{q}_{\alpha_1*\alpha_2}=\{y\in\mathscr{M}_1\bar{\otimes}\mathscr{M}_2;\ \mathscr{E}_{\alpha_1*\alpha_2}(y^*y)$ exists$\}$. If $x\in\mathfrak{q}_{\alpha_2}$, then

$$\begin{aligned}\mathscr{E}_{\alpha_1*\alpha_2}(1\otimes x^*x)&=(\iota\otimes\iota\otimes\phi)\circ(\alpha_1*\alpha_2)(1\otimes x^*x)\\&=(\iota\otimes\iota\otimes\phi)\circ(\iota\otimes\alpha_2)(1\otimes x^*x)\ ,\qquad\text{by Proposition 5.2,}\\&=(\iota\otimes\mathscr{E}_{\alpha_2})(1\otimes x^*x)\ ,\end{aligned}$$

and so $C\otimes\mathfrak{q}_{\alpha_2}\subset\mathfrak{q}_{\alpha_1*\alpha_2}$. Since $\alpha_2$ is integrable, so is $\alpha_1*\alpha_2$. Q.E.D.

**Proposition 5.8.** *If $\alpha_j(j=1,2)$ are right actions of $\boldsymbol{K}$ on $\mathscr{M}_j$* and $\mathscr{N}_j=\mathscr{W}^*(\alpha_j)$, *then*

$$\{\mathscr{W}^*(\mathscr{M}_1\bar{\otimes}\mathscr{M}_2,\alpha_1*\alpha_2),(\alpha_1*\alpha_2)\hat{}\}\cong\{\mathscr{N},(\iota_{\mathscr{N}_1}\otimes\bar{\sigma})\circ(\hat{\alpha}_1\otimes\iota_{\mathscr{N}_2})\}\ ,$$

*where $\bar{\sigma}$ is the symmetric isomorphism of $\hat{M}'\bar{\otimes}\mathscr{N}_2$ onto $\mathscr{N}_2\bar{\otimes}\hat{M}'$ and $\mathscr{N}$ is the von Neumann subalgebra of $\mathscr{N}_1\bar{\otimes}\mathscr{N}_2$ generated by $\alpha_1(\mathscr{M}_1)\bar{\otimes}\alpha_2(\mathscr{M}_2)$ and $(\iota\otimes\sigma\otimes\iota)(C\otimes C\otimes Ad_{\sigma(v)}(\hat{M}'\otimes C))$*

**Proof.** We set

$$\pi=(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes1\otimes V}\circ(\iota\otimes\alpha_2\otimes\iota)\ .$$

Then we have

$$\pi\circ(\alpha_1*\alpha_2)(x_1\otimes x_2)=\alpha_1(x_1)\otimes\alpha_2(x_2)\ ,\qquad x_j\in\mathscr{M}_j$$

by Definition 5.1 and

$$\text{(5.5)}\qquad\begin{aligned}\pi(1\otimes1\otimes y)&=(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes1\otimes V}(1\otimes1\otimes1\otimes y)\\&=(\iota\otimes\sigma\otimes\iota)(1\otimes1\otimes Ad_{\sigma(V)}(y\otimes1))\ ,\qquad y\in\hat{M}'\ .\end{aligned}$$

The conjugacy is immediate from the following:

$$\begin{aligned}&(\pi\otimes\iota)\circ(\alpha_1*\alpha_2)\hat{}(\alpha_1*\alpha_2(x_1\otimes x_2))=(\pi\otimes\iota)(\alpha_1*\alpha_2(x_1\otimes x_2)\otimes1)\\&\quad=\alpha_1(x_1)\otimes\alpha_2(x_2)\otimes1=(\iota_{\mathscr{N}_1}\otimes\bar{\sigma})\circ(\hat{\alpha}_1\otimes\iota_{\mathscr{N}_2})(\alpha_1(x)\otimes\alpha_2(x_2))\end{aligned}$$

and

$$\begin{aligned}&(\pi\otimes\iota)\circ(\alpha_1*\alpha_2)\hat{}(1\otimes1\otimes y)=(\pi\otimes\iota)(1\otimes1\otimes\hat{\Gamma}'(y))\\&\quad=(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes1\otimes V\otimes1}(1\otimes1\otimes1\otimes\hat{\Gamma}'(y))\ ,\qquad\text{by (5.5)}\\&\quad=(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes1\otimes V\otimes1}\circ Ad_{1\otimes1\otimes1\otimes(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})}(1\otimes1\otimes1\otimes y\otimes1)\ .\end{aligned}$$

Since $W\in M\bar{\otimes}\hat{M}$, it follows that

$$V\in\hat{M}'\bar{\otimes}M\qquad\text{and}\qquad(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})\in M'\bar{\otimes}\hat{M}'\ .$$

Therefore, the right hand side equals to

$$
\begin{aligned}
&(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ Ad_{1\otimes 1\otimes 1\otimes(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})}\circ Ad_{1\otimes 1\otimes V\otimes 1}(1\otimes 1\otimes 1\otimes y\otimes 1)\\
&\quad=(\iota\otimes\sigma\otimes\iota\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\hat{\alpha}_1)\circ Ad_{1\otimes 1\otimes V}(1\otimes 1\otimes 1\otimes y)\\
&\quad=(\iota\otimes\iota\otimes\iota\otimes\sigma)\circ(\iota\otimes\iota\otimes\sigma\otimes\iota)\circ(\hat{\alpha}_1\otimes\iota\otimes\iota)\circ(\iota\otimes\sigma\otimes\iota)\circ(\iota\otimes\iota\otimes\sigma)\circ Ad_{1\otimes 1\otimes V}(1\otimes 1\otimes 1\otimes y)\\
&\quad=(\iota_{\mathcal{N}_1}\otimes\bar{\sigma})\circ(\hat{\alpha}_1\otimes\iota_{\mathcal{N}_2})\circ\pi(1\otimes 1\otimes y)\,. \qquad \text{Q.E.D.}
\end{aligned}
$$

**Corollary 5.9.** *If $\delta_j(j=1,2)$ are co-actions of $G$ on $\mathcal{N}_j$ and $\mathcal{M}_j=\mathcal{N}_j\times_{\delta_j}G$, then*

$$\{(\mathcal{N}_1\bar{\otimes}\mathcal{N}_2)\times_{\delta_1*\delta_2}G,\ (\delta_1*\delta_2)\hat{}_t\}\cong\{\mathcal{M},\ (\delta_1)\hat{}_t\otimes\iota\}\,,\qquad t\in G\,,$$

*where $\mathcal{M}$ is the von Neumann subalgebra of $\mathcal{M}_1\bar{\otimes}\mathcal{M}_2$ generated by $\delta_1(\mathcal{N}_1)\bar{\otimes}\delta_2(\mathcal{N}_2)$ and $(\iota\otimes\sigma\otimes\iota)(\boldsymbol{C}\otimes\boldsymbol{C}\otimes\alpha_G(L^\infty(G)))$.*

## § 6. Relative commutant property.

We shall generalize Paschke's results on relative commutant property [14] to the context on crossed products by Kac algebras. The proof will owe mainly to Takesaki's duality.

**Theorem 6.1.** *Let $\alpha$ be a dual right action of $\boldsymbol{K}^{\wedge\prime}$ on $\mathcal{M}$.*

(i) $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\alpha(\mathcal{M})}$ *if and only if* $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}}$.

(ii) $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\mathcal{W}^*(\alpha)}$ *if and only if* $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}^\alpha}$.

**Proof.** As our theorem is true up to conjugacy, we may assume that $\mathcal{M}$ is standard. If $\alpha$ is dual, there exists a right action $\beta$ of $\boldsymbol{K}$ on a von Neumann algebra $\{\mathcal{N},\mathfrak{K}\}$ such that

$$\{\mathcal{M},\boldsymbol{K}^{\wedge\prime},\alpha\}\cong\{\mathcal{W}^*(\mathcal{N},\boldsymbol{K},\beta),\boldsymbol{K}^{\wedge\prime},\hat{\beta}\}\,.$$

(If $\{\mathcal{N},\boldsymbol{K},\beta\}$ is conjugate to $\{\mathcal{N}_1,\boldsymbol{K},\beta_1\}$, then $\{\mathcal{W}^*(\beta),\boldsymbol{K}^{\wedge\prime},\hat{\beta}\}$ is conjugate to $\{\mathcal{W}^*(\beta_1),\boldsymbol{K}^{\wedge\prime},\hat{\beta}_1\}$.) We may assume that $\beta$ is implemented by a unitary $u\in\mathcal{L}(\mathfrak{K})\bar{\otimes}M$ with the associativity condition (2.2). If $\psi$ is a faithful, semi-finite, normal weight on $\mathcal{N}$ and $\tilde{\psi}$ is the dual weight $\psi\circ\beta^{-1}\circ\mathcal{E}_{\hat{\beta}}$ on $\mathcal{W}^*(\beta)$, then the modular unitary involution is given by

$$J_{\tilde{\psi}}=u(J_\psi\otimes\hat{J})(=\tilde{J})$$

and $\tilde{J}\mathcal{W}^*(\beta)\tilde{J}=\mathcal{W}^*(\beta)'$, [6, Corollaire IV. 8]. Here we recall Takesaki's duality [6, Théorème IV. 3] and the fixed point property of dual action [6, Théorème IV. 2]. Then

$$\{W^*(\alpha),\boldsymbol{K},\hat{\alpha}\}\cong\{\mathcal{N}\bar{\otimes}\mathcal{L}(H),\boldsymbol{K},\tilde{\beta}\} \tag{6.1}$$

and

$$(\mathcal{N}\bar{\otimes}\mathcal{L}(H))^{\tilde{\beta}}=\mathcal{W}(\beta)\,. \tag{6.2}$$

Now we shall go into the proof of our statements (i) and (ii).

(i) According to (6.1) and (6.2) the condition $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\alpha(\mathcal{M})}$ is equivalent to

$$\mathcal{W}^*(\beta)'\cap(\mathcal{N}\bar{\otimes}\mathcal{L}(H))=\mathcal{C}_{\mathcal{W}^*(\beta)}\,.$$

Applying $Ad\bar{J}$ to both sides, we find that (6.3) is equivalent to

$$\mathcal{W}(\beta)\cap w^*(\mathcal{N}'\bar{\otimes}\mathcal{L}(H))w=\mathcal{C}_{\mathcal{W}^*(\beta)}\,,$$

which is equivalent to $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}}$ by (7.1)

(ii) By the same reason as (i), $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\alpha(\mathcal{M})}$ is equivalent to

$$\mathcal{W}^*(\beta)'\cap(\mathcal{N}\bar{\otimes}\mathcal{L}(H))=\mathcal{C}_{\mathcal{N}\bar{\otimes}\mathcal{L}(H)}\,.$$

Applying $Ad\bar{J}$, we have

$$\mathcal{W}^*(\beta)\cap w^*(\mathcal{N}'\bar{\otimes}\mathcal{L}(H))w=w^*(\mathcal{C}_{\mathcal{N}}\otimes\boldsymbol{C})w\,,$$

which is equivalent to $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}^\alpha}$. Q.E.D.

In what follows, $\alpha$ is said to satisfy the *property* $(R)$ or the *property* $(\hat{R})$, if it satisfies

$$\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\alpha(\mathcal{M})}\,,$$

or

$$\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\mathcal{W}^*(\alpha)}\,,$$

respectively.

**Corollary 6.2.** *Let $\bar{\mathcal{M}}=\mathcal{M}\bar{\otimes}\mathcal{L}(H)$. If the property $(R)$ (or $(\hat{R})$) holds for $\{\mathcal{M},\boldsymbol{K},\alpha\}$, so does for $\{\bar{\mathcal{M}},\boldsymbol{K},\hat{\alpha}\}$.*

**Proof.** If $(R)$ is true for $\{\mathcal{M},\boldsymbol{K},\alpha\}$, then $(\hat{R})$ is true for $\{\mathcal{W}^*(\alpha),\hat{\boldsymbol{K}}',\hat{\alpha}\}$ by Theorem 6.1. ii. Therefore $(R)$ is true for $\{\bar{\mathcal{M}},\boldsymbol{K},\tilde{\alpha}\}$ by Theorem 6.1.i. The similar argument does hold for $(\hat{R})$. Q.E.D.

**Theorem 6.3.** *Assume that the underlying Hilbert spaces are separable. Let $\alpha$ be an integrable right action of $\boldsymbol{K}$ on $\mathcal{M}$.*

(i) $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\alpha(\mathcal{M})}$ *implies* $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}}$.

(ii) $\alpha(\mathcal{M})'\cap\mathcal{W}^*(\alpha)=\mathcal{C}_{\mathcal{W}^*(\alpha)}$ *implies* $(\mathcal{M}^\alpha)'\cap\mathcal{M}=\mathcal{C}_{\mathcal{M}^\alpha}$.

**Proof.** (i) If the property $(R)$ is true for $\{\mathcal{M},\boldsymbol{K},\alpha\}$, then it is true for $\{\bar{\mathcal{M}},\boldsymbol{K},\tilde{\alpha}\}$. Since $\tilde{\alpha}$ is dual, $(R)$ for $\{\bar{\mathcal{M}},\tilde{\alpha}\}$ implies

$$(\bar{\mathcal{M}}^{\tilde{\alpha}})'\cap\bar{\mathcal{M}}=\mathcal{C}_{\bar{\mathcal{M}}}\tag{6.3}$$

by Theorem 6.1. As $\alpha$ is integrable, $\{\bar{\mathscr{M}}, \bar{\alpha}\} \cong \{\bar{\mathscr{M}}, \tilde{\alpha}\}_e$ for some projection $e$ in $\bar{\mathscr{M}}^{\tilde{\alpha}}$ by Theorem 3.1, condition (6.3) implies

$$(\bar{\mathscr{M}}^{\tilde{\alpha}})'_e \cap (\bar{\mathscr{M}})_e = (\mathscr{C}_{\bar{\mathscr{M}}})_e \qquad \text{and} \qquad (\bar{\mathscr{M}}^{\tilde{\alpha}})_e = (\bar{\mathscr{M}}_e)^{\tilde{\alpha}^e}$$

and hence, by above conjugacy,

$$(\bar{\mathscr{M}}^{\bar{\alpha}})' \cap \bar{\mathscr{M}} = \mathscr{C}_{\bar{\mathscr{M}}}$$

Consequently, we have $(\mathscr{M}^{\alpha})' \cap \mathscr{M} = \mathscr{C}_{\mathscr{M}}$.

(ii) By the same reason as above, the property $(\hat{R})$ for $\{\mathscr{M}, \alpha\}$ implies

$$(\bar{\mathscr{M}}^{\tilde{\alpha}})' \cap \bar{\mathscr{M}} = \mathscr{C}_{\bar{\mathscr{M}}^{\tilde{\alpha}}}$$

by Theorem 6.1. Therefore the integrability of $\alpha$ implies

$$(\bar{\mathscr{M}}^{\bar{\alpha}})' \cap \bar{\mathscr{M}} = \mathscr{C}_{\bar{\mathscr{M}}^{\bar{\alpha}}} ,$$

which means

$$(\mathscr{M}^{\alpha})' \cap \mathscr{M} = \mathscr{C}_{\mathscr{M}^{\alpha}} . \qquad \text{Q.E.D.}$$

## §7. On a dual action.

In this section we shall given a necessary condition for a right action of a Kac algebra to be dual.

**Proposition 7.1.** *If $\beta$ is a right action of a Kac algebra $\boldsymbol{K}^{\frown\prime}$ on a von Neumann algebra $\mathscr{N}$ such that*

$$\{\mathscr{N}, \boldsymbol{K}^{\frown\prime}, \beta\} \cong \{\mathscr{W}^*(\mathscr{M}, \boldsymbol{K}, \alpha), \boldsymbol{K}^{\frown\prime}, \hat{\alpha}\} \tag{7.1}$$

*for some covariant system $\{\mathscr{M}, \boldsymbol{K}, \alpha\}$, then there is an $\iota_{\boldsymbol{K}}^{\mathscr{L}(\mathfrak{K})}$-cocycle $u \in \mathscr{N} \bar{\otimes} M$ such that*

$$\bar{\beta}(u) = (u \otimes 1)(1 \otimes \sigma(V^*)) , \qquad (\bar{\beta} = (\iota \otimes \sigma) \circ (\beta \otimes \iota)) , \tag{7.2}$$

*where*

$$V = (\hat{J} \otimes \hat{J}) \sigma(W^*)(\hat{J} \otimes \hat{J}) .$$

**Proof.** As $V$ is the fundamental operator of $\boldsymbol{K}^{\frown\prime}$, it satisfies

$$(V^* \otimes 1)(\iota \otimes \sigma)(V^* \otimes 1) = Ad_{1 \otimes V^*}(V^* \otimes 1) = (\iota \otimes \Gamma)(V^*) , \tag{7.3}$$

the unitary $U = 1_{\mathfrak{H}} \otimes V^*$ is an $\iota_{\boldsymbol{K}}^{\mathscr{L}(\mathfrak{H} \otimes H)}$-cocycle: $(U \otimes 1)(\iota \otimes \iota \otimes \sigma)(U \otimes 1) = (\iota \otimes \iota \otimes \Gamma)(U)$. Since $V \in \hat{M}' \bar{\otimes} M$, we have $U \in \mathscr{W}^*(\alpha) \bar{\otimes} M$. In order to show

$$(\iota \otimes \iota \otimes \sigma) \circ (\hat{\alpha} \otimes \iota)(U) = (U \otimes 1)(1 \otimes 1 \otimes \sigma(V^*)) , \tag{7.4}$$

it suffices to check

(7.5) $$(\iota\otimes\sigma)\circ Ad_{(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})\otimes 1}\circ(\iota\otimes\sigma)(V^*\otimes 1)=(V^*\otimes 1)(1\otimes\sigma(V^*))$$

The associativity condition (7.3) implies

$$(V\otimes 1)(1\otimes V)(V^*\otimes 1)=(1\otimes V)(\iota\otimes\sigma)(V\otimes 1)\ .$$

Put $\tilde{J}=J\otimes J\otimes\hat{J}$. Since $(J\otimes\hat{J})=V(J\otimes\hat{J})V^*$, if we apply $Ad_{\tilde{J}}$ to both sides, we have

$$\tilde{J}(V\otimes 1)\tilde{J}(1\otimes V^*)\tilde{J}(V^*\otimes 1)\tilde{J}=(1\otimes V^*)(\iota\otimes\sigma)(V^*\otimes 1)\ .$$

Applying $(\iota\otimes\sigma)\circ(\sigma\otimes\iota)$ to both sides, we have

$$Ad_{(\iota\otimes\sigma)(\tilde{J}(\sigma(V)\otimes 1)\tilde{J})}(V^*\otimes 1)=(V^*\otimes 1)(1\otimes\sigma(V^*))$$

Since $(J\otimes J)\sigma(V)(J\otimes J)=(J\hat{J}\otimes J\hat{J})W^*(J\hat{J}\otimes J\hat{J})$, this yields (7.5) and hence (7.4).

Our assumption (7.1) gives us an isomorphism $\pi$ of $\mathscr{W}^*(\alpha)$ onto $\mathscr{N}$ such that $\beta\circ\pi=(\pi\otimes\iota)\circ\hat{\alpha}$. Setting $u=(\pi\otimes\iota)(U)$, we find that $u$ is an $\iota_K^{\mathscr{L}(\mathfrak{K})}$-cocycle in $\mathscr{N}\bar{\otimes}M$ satisfying (7.2) by equation (7.4). Q.E.D.

When $\beta$ is an action or a co-action of a locally compact group, the converse of the above proposition does hold as shown in [8, 10, 13, 16].

**Problem 7.2.** Does the converse of Proposition 7.1 hold?

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