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DIRICHLET PROBLEM FOR MINIMAL SURFACE EQUATIONS IN A RIEMANNIAN MANIFOLD I.

By

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0. Introduction. Let $\tilde{M}$ be a Riemannian manifold of dimension $n(n\geqq 3)$ and M a hypersurface of $\tilde{M}$ with smooth boundary ∂M such that $\bar{M}=M\cup\partial M$ is compact. Let X be a vector field in $\tilde{M}$ which is transversal to $\bar{M}$. Let ε be a positive number such that the map Φ: $\bar{M}\times(-\varepsilon, \varepsilon)\rightarrow\tilde{M}$ defined by $\Phi(m, t)=\varphi_t(m)$, $(m, t)\in\bar{M}\times(-\varepsilon, \varepsilon)$, is imbedding where $\{\varphi_t\}$ denotes the local 1-parameter subgroup of local transformations generated by X. We consider the following problem:

Let f be a given real-valued smooth function on ∂M such that $|f|<\varepsilon$. Then find a minimal hypersurface of $\tilde{M}$ whose boundary is $\partial S_f:=\Phi(\{(m, f(m))\in\partial M\times(-\varepsilon, \varepsilon); m\in\partial M\})$.

We put $F=\{u\in C^2(\bar{M});\ |u|<\varepsilon$ in M, $u=f$ on $\partial M\}$ and for a $u\in F$ we let $S(u)=\Phi(\{(m, u(m)); m\in M\})$, $\Sigma_f=\{S(u); u\in F\}$. In Σ_f we want to find a minimal hypersurface of $\tilde{M}$ with given boundary ∂S_f. Let $S(u)$, $u\in F$, be a minimal hypersurface of $\tilde{M}$. Then we see that u is a solution of a quasilinear elliptic partial differential equation of second order in M. Therefore our problem stated above can be reduced to the Dirichlet problem for a quasilinear elliptic partial differential equation of second order in M.

In this paper we study our problem for the case where $\tilde{M}$ is a simply connected space form and $\bar{M}$ is contained in a hyperplane of $\tilde{M}$ and moreover X is a Killing vector field normal to $\bar{M}$. In Section 1, we give a minimal surface equation (see (1.8)) in a Riemannian manifold M with boundary ∂M. In order to find a solution of the Dirichlet problem for the minimal surface equations, we apply the method used in Serrin's paper [9]. In Section 2, we prove the maximum principle for solutions of the minimal surface equation (1.8). Supposing that the mean curvature (with respect to the inward direction) of boundary ∂M is non-negative, in Section 3, we give gradient estimates on ∂M for solutions of the Dirichlet problem for equation (1.8). In Section 4, we study the minimal surface equation (4.2) in a Riemannian manifold represented by an isothermal coordinate system. Applying the results due to Serrin [9], we obtain global gradient estimates for solutions of the Dirichlet problem for equation (4.2). In virtue of the estimates for solutions of the minimal surface equations obtained in Sections 2, 3 and 4, we can solve the Dirichlet problem for the minimal surface equation (4.2). Our problem stated above will be solved in Section 5 (see Theorem 5.1).

Throughout this paper we assume that Riemannian manifolds and apparatus on them are of class C^∞ and that manifolds are connected, unless otherwise stated.

1. Let $\bar{M}$ be a compact $n(\geqq 2)$ dimensional Riemannian manifold with boundary ∂M and interior M. We denote by $C^k(\bar{M})$ the set of real-valued functions of class C^k on $\bar{M}$ where k is a non-negative integer. Let ρ be a given positive valued function in $C^3(\bar{M})$. We now consider a Riemannian manifold $\tilde{M}:=M\times(-\varepsilon, \varepsilon)$, $0<\varepsilon\leqq\infty$, whose line element is expressed by

$$ds^2=\sum_{i,j=1}^{n} g_{ij}dx_i dx_j+\rho dt^2 \tag{1.1}$$

where g_{ij} $(1\leqq i, j\leqq n)$ is the Riemannian metric of $\bar{M}$. We shall denote by $\langle\ ,\ \rangle$ the Riemannian metric tensor of $\tilde{M}$ defined above.

Let $(U, (x_1, \cdots, x_n))$* be a local coordinate system about a point of $\bar{M}$ and let $g_{ij}=\langle\partial/\partial x_i, \partial/\partial x_j\rangle$, $1\leqq i, j\leqq n$. For a $u\in C^2(\bar{M})$ we put

$$u_i=\partial u/\partial x_i\ , \qquad u_{ij}=\partial^2 u/\partial x_i\partial x_j\ , \qquad Du=(u_1, \cdots, u_n)$$

and for a vector $p=(p_1, \cdots, p_n)\in R^n$ we put

$$p^i=\sum_{j=1}^{n} g^{ij}p_j\ , \qquad \|p\|=(\sum_{i,j=1}^{n} g^{ij}p_ip_j)^{1/2}$$

where g^{ij} is the (i, j)-component of the inverse matrix of (g_{ij}).

Now for a $u\in C^2(M)$ such that $|u|<\varepsilon$ in M let S be a non-parametric hypersurface in $\tilde{M}$ defined by

$$S=\{(m, u(m))\in\tilde{M};\ m\in M\}\ . \tag{1.2}$$

Let $(U, (x_1, \cdots, x_n))$ be a local coordinate system of M. We put $X_i=\partial/\partial x_i+u_i\partial/\partial t$, $1\leqq i\leqq n$. Then $X_1, \cdots, X_n$ are linearly independent tangent vector fields on S. We set

$$\tilde{g}_{ij}=g_{ij}+\rho u_iu_j\ , \quad 1\leqq i,\ j\leqq n\ . \tag{1.3}$$

Let η be a unit normal vector field to S given by

$$\eta=\frac{1}{\sqrt{G}}\{\sum_{i=1}^{n}(-\rho u^i)\partial/\partial x_i+\partial/\partial t\} \tag{1.4}$$

* Throughout this paper, we always assume that all local coordinate neighborhoods are homeomorphic to an open unit ball in the Euclidean space.

where

(1.5) $$G=\rho(1+\rho\|Du\|^2)\,.$$

Then the mean curvature $\mathscr{H}$ of S with respect to η is defined by

(1.6) $$\mathscr{H}=\frac{1}{n}\sum_{i,j=1}^{n}\tilde{g}^{ij}\langle \nabla_{X_i}X_j,\ \eta\rangle$$

where ∇ stands for the Riemannian connection of $\tilde{M}$ and $\tilde{g}^{ij}$ is the (i, j)-component of the inverse matrix of $(\tilde{g}_{ij})$ which is given by

(1.7) $$\tilde{g}^{ij}=g^{ij}-(\rho u^i u^j)/(1+\rho\|Du\|^2)\,.$$

From (1.6) we see that $\mathscr{H}$ is a continuous function on S. S is said to be minimal if $\mathscr{H}$ is identically zero. Suppose now S is minimal, then by (1.6) we see that u is a solution of the following quasilinear elliptic partial differential equation of second order:

(1.8) $$\mathscr{L}(u)=\sum_{i,j=1}^{n}A_{ij}(x,\ Du)u_{ij}-B(x,\ Du)\equiv 0\,.$$

where

(1.9) $$\begin{aligned}&A_{ij}(x,\ p)=(1+\rho\|p\|^2)g^{ij}-\rho p^i p^j\\ &B(x,\ p)=\sum_{i,j,k=1}^{n}\{(1+\rho\|p\|^2)g^{ij}-\rho p^i p^j\}\Gamma^k_{ij}p_k-(\sum_{k=1}^{n}\rho_k p^k)/2\rho\end{aligned}$$

where Γ^k_{ij} denotes the Christoffel's symbol with respect to g_{ij}. The operator $\mathscr{L}$ defined by (1.8) is independent of the choice of local coordinate systems. From now on we will call equation (1.8) the minimal surface equation in M.

Though we get equation (1.8) under the condition $|u|<\varepsilon$ in M, but the operator $\mathscr{L}$ defined by (1.8) can be defined for all $u\in C^2(M)$.

In the following sections we will use the notations defined in this section without any statement.

2. Let $\bar{M}$ be a compact n-dimensional Riemannian manifold with boundary ∂M and interior M, $n\geqq 2$.

Lemma 2.1. *Let u, v be functions in $C^2(M)\cap C^0(\bar{M})$ satisfying $\mathscr{L}(u)=0$ and $\mathscr{L}(v)\leqq 0$ ($\mathscr{L}(v)\geqq 0$) in M. Suppose that $u\leqq v$ ($u\geqq v$) on ∂M. Then $u\leqq v$ ($u\geqq v$) in M.*

Proof. Let $w=u-v$ and $k=\sup_M w$. Supposing $\mathscr{L}(v)\leqq 0$ in M and $u\leqq v$ on ∂M, we shall show $u\leqq v$ in M. Suppose for contradiction that k is positive. Then there exists a point of M at which w takes its maximum value k. Put $M'=\{m\in M;\ w(m)=k\}$. Let m be a point of M' and let $(U, (x_1, \cdots, x_n))$ be a local coordinate system about m such that $U\subset M$. Since $\mathscr{L}(u)=0$ and $\mathscr{L}(v)\leqq 0$ on U, subtracting the latter from the former and applying the mean value theorem, we get

$$\sum_{i,j=1}^{n} a_{ij}(x)w_{ij}+\sum_{i=1}^{n} a_i(x)w_i\geqq 0$$

where $a_{ij}(x)=A_{ij}(x, Du(x))$ and a_i is a continuous function on U, $1\leqq i\leqq n$. Then by E. Hopf's maximum principle we see $w\equiv k$ in U, so $U\subset M'$. Hence we have proved that M' is open in M. Since M' is closed in M and M is connected, $w\equiv k$ in M. But the continuity of w implies $w=k>0$ on ∂M, which is contradiction. Thus we get $u\leqq v$ in M. Note $\mathscr{L}(u)=\mathscr{L}(-u)=0$ and $\mathscr{L}(-v)\leqq 0$ if $\mathscr{L}(v)\geqq 0$. Therefor the second inequality follows from the first.

The following is an immediate consequence of the above lemma.

Proposition 2.1. *Let u be a function in $C^2(M)\cap C^0(\bar{M})$ satisfying $|u|\leqq m$ on ∂M. Suppose that u is a solution of the minimal surface equation* (1.8) *in M. Then $|u|\leqq m$ in M.*

Furthermore, using a similar argument as in the proof of Lemma 2.1, we have

Proposition 2.2. *Let u and v be functions in $C^2(M)$ satisfying the condition: $\mathscr{L}(u)\geqq 0$, $\mathscr{L}(v)\leqq 0$ in M. Suppose that $u\leqq v$ in M and at a point m of M $u(m)=v(m)$. Then $u=v$ in M.*

3. Let $\bar{M}$ be a compact n-dimensional Riemannian manifold with boundary ∂M and interior M. For a positive r we put $\perp_r(\partial M)=\{(m, \eta);\ \eta$ is an inward normal vector to ∂M at $m\in\partial M$ and $0\leqq\|\eta\|<r\}$. Since ∂M is compact, we can take a positive r_0 such that $\exp|_{\perp_{2r_0}(\partial M)}: \perp_{2r_0}(\partial M)\to\bar{M}$ is imbedding where exp denotes the exponential map. In the following we let $N_r=\exp(\perp_r(\partial M))$, $0<r\leqq r_0$, and in particular we put $N=N_{r_0}$ when $r=r_0$. For each point m of ∂M, we can take a local coordinate system $(V_m\times[0, r_0), (x_1, \cdots, x_{n-1}, r))$ about m in $\bar{M}$ which has the following properties:

(1) $(V_m, (x_1, \cdots, x_{n-1}))$ is a local coordinate system about m in ∂M and V_m is homeomorphic to an unit open ball in R^{n-1} and $x_1, \cdots, x_{n-1}, r$ are defined on an open neighborhood of the closure of $V_m\times[0, r_0)$.

(3.1) (2) $g_{ij}(m)=\delta_{ij}$, $g_{in}=0$, $g_{nn}=1$ where we put $g_{ij}=\langle\partial/\partial x_i, \partial/\partial x_j\rangle$, $\partial/\partial x_n=\partial/\partial r$.

(3) For any contravariant unit vector $\sigma=(\sigma^1, \cdots, \sigma^n)\in R^n(\|\sigma\|=1)$

$$\sum_{i=1}^{n}\{g^{ii}-(\sigma^i)^2\}\geqq 1/2 \qquad \text{on} \quad V_m\times[0, r_0)\,.$$

Here we note $\sum_{i=1}^{n}\{g^{ii}-(\sigma^i)^2\}=n-1$ on $\{m\}\times[0, r_0]\times\{\sigma\in R^n;\ \|\sigma\|=1\}$. Since ∂M is compact, ∂M is covered by finitely many local coordinate neighborhoods having the properties stated above. We let $\partial M \subset \bigcup_{\alpha=1}^{k} \tilde{V}_\alpha$ where each $\tilde{V}_\alpha := V_\alpha \times [0, r_0)$ has the properties (1), (2) and (3) in (3.1). In what follows we fix these $\tilde{V}_\alpha$, $1 \leqq \alpha \leqq k$.

Let f be a given function in $C^3(\bar{M})$. We put

$$c_0 = \sup_{\bar{M}} |f| , \quad c_1 = \sup_{\bar{M}} \|Df\| , \quad c_2 = \max \{\sup_{\tilde{V}_\alpha} \|D^2 f\|, 1 \leqq \alpha \leqq k\} \tag{3.2}$$

where $\|D^2 f\|$ denotes the norm of hessian of f $(\|D^2 f\| = (\sum_{i,j=1}^{n} (f_{ij})^2)^{1/2})$.

We now take a local coordinate system $(\tilde{V}_\alpha, (x_1 \cdots, x_{n-1}, r))$. We denote a point of $\tilde{V}_\alpha$ by $x = (x_1, \cdots, x_{n-1}, r)$. We put $g_{ij} = \langle \partial/\partial x_i, \partial/\partial x_j \rangle$ $(1 \leqq i, j \leqq n)$, $\partial/\partial x_n = \partial/\partial r$. On $V_\alpha \times (0, r_0)$ we rewrite equation (1.8) as follows:

$$\sum_{i,j=1}^{n} \mathscr{A}_{ij}(x, Du) u_{ij} = \mathscr{B}(x, Du) \tag{3.3}$$

where

$$\begin{aligned} &\mathscr{A}_{ij}(x, p) = A_{ij}(x, p)/\mathrm{trace}\,(A_{ij}(x, p)) , \quad 1 \leqq i, j \leqq n , \\ &\mathscr{B}(x, p) = B(x, p)/\mathrm{trace}\,(A_{ij}(x, p)) , \quad p = (p_1, \cdots, p_n) \in R^n . \end{aligned} \tag{3.4}$$

We note that $\mathscr{A}_{ij}$ and $\mathscr{B}$ are defined on an open neighborhood of the closure of $\tilde{V}_\alpha$ and that the equation (3.3) is not invariant for the choice of local coordinate systems. It is easy to see that there exist positive constants λ_α, Λ_α satisfying

$$\lambda_\alpha \leqq \sum_{i,j=1}^{n} \mathscr{A}_{ij}(x, p) p_i p_j \leqq \Lambda_\alpha \tag{3.5}$$

for all $x \in \tilde{V}_\alpha$ and all $p = (p_1, \cdots, p_n) \in R^n$ such that $\|p\| \geqq 1$. By (3.5) the equation (3.3) is of genre two in the sense of Bernstein. Since the eigenvalues of $(\mathscr{A}_{ij}(x, p))$ all lie in the open interval (0, 1), we have

$$\sum_{i,j=1}^{n} \mathscr{A}_{ij}(x, p) f_{ij} \leqq c_2 \quad \text{on} \quad \tilde{V}_\alpha \times R^n . \tag{3.6}$$

For non-zero vector p of R^n we put

$$\mathscr{C}(x, p) = \mathscr{B}(x, p)/\|p\| \tag{3.7}$$

and we put

$$\mathscr{A}_{0ij}(x, \sigma) = (g^{ij} - \sigma^i \sigma^j)/\{\sum_{k=1}^{n} (g^{kk} - (\sigma^k)^2)\} , \quad 1 \leqq i, j \leqq n , \tag{3.8}$$

$$\mathscr{C}_0(x, \sigma) = \{\sum_{i,j,k=1}^{n} (g^{ij} - \sigma^i \sigma^j) \Gamma_{ij}^k \sigma_k\}/\{\sum_{k=1}^{n} (g^{kk} - (\sigma^k)^2)\} \tag{3.9}$$

where $\sigma = p/\|p\|$, $\sigma^i = \sum_{j=1}^{n} g^{ij} \sigma_j$. Then by a direct calculation we can show

(3.10) $$|\mathscr{A}_{ij}(x, p)-\mathscr{A}_{0ij}(x, \sigma)|\leqq C_\alpha\|p\|^{-2}, \quad 1\leqq i, j\leqq n,$$

and

(3.11) $$|\mathscr{C}(x, p)-\mathscr{C}_0(x, \sigma)|\leqq C_\alpha\|p\|^{-2}$$

for all $x\in\tilde{V}_\alpha$ and all $p\in R^n$ such that $\|p\|\geqq 2c_1$ where C_α depends only on the bounds of ρ, $|g_{ij}|$ and $|\Gamma_{ij}^k|$ on $\tilde{V}_\alpha$. We put

$$\mathscr{F}(x, p)=\sum_{i,j=1}^n \mathscr{A}_{ij}(x, p)(p_i-p_{0i})(p_j-p_{0j}), \quad (x, p)\in\tilde{V}_\alpha\times R^n$$

where p_0 is a fixed vector of R^n such that $\|p_0\|\leqq c_1$. Then by (3.5), (3.10) and (3.11) there exists a positive constant μ_α satisfying

(3.12) $$\mathscr{F}(x, p)\geqq\mu_\alpha$$

for all $x\in\tilde{V}_\alpha$ and all $p\in R^n$ such that $\|p\|\geqq 2c_1$ (see p. 437 in Serrin [9]).

For each $x\in\tilde{V}_\alpha$ we denote by $\bar{x}$ the unique nearest point of ∂M from x to ∂M. Clearly, $\bar{x}$ is contained in V_α. We let $\nu=(0, \cdots, 0, 1)\in R^n$. Using (3.11) we have

(3.13) $$|\mathscr{C}(x, p)-\mathscr{C}_0(\bar{x}, \nu)|\leqq\tilde{C}_\alpha(\|p\|^{-1}+r)$$

for all $x=(x_1, \cdots, x_{n-1}, r)\in\tilde{V}_\alpha$ and all $p\in R^n$ such that $\|p\|\geqq 2c_1$ where $\tilde{C}_\alpha$ depends only on c_1, C_α the bounds of ρ, $|g_{ij}|$, $|\Gamma_{ij}^k|$ and the C^1 norm of $\mathscr{C}_0$.

Let m_1 be a given positive constant. We put

(3.14) $$c=\max\{2(c_2+\tilde{C}_\alpha(2m_1+1))\mu_\alpha^{-1};\ 1\leqq\alpha\leqq k\}$$

and

(3.15) $$s_0=\max\{3c_1, m_1/r_0\}.$$

We now define positive constants s_1, $r_1 (r_1\leqq r_0)$ by

(3.16) $$cm_1=\int_{s_0}^{s_1}\frac{s-c_1}{s^2}ds, \qquad cr_1=\int_{s_0}^{s_1}\frac{s-c_1}{s^3}ds.$$

We set

(3.17) $$c\bar{h}=\int_{s}^{s_1}\frac{s-c_1}{s^2}ds, \qquad cr=\int_{s}^{s_1}\frac{s-c_1}{s^3}ds, \quad s_0\leqq s\leqq s_1.$$

Then $h(r):=\bar{h}(s(r))$ is a function in $C^0([0, r_1])\cap C^2((0, r_1))$ satisfying

(3.18) $$h(0)=0, \quad h(r_1)=m_1, \qquad h'(r)=s\geqq s_0, \quad h''(r)<0.$$

Let v be a function in $C^0(\overline{N_{r_1}})$ defined by

(3.19) $$v=f+h\circ r$$

where r denotes the distance from each point of N to ∂M. For a point $\bar{x}$ of ∂M

we denote by $H(\bar{x})$ the mean curvature (with respect to the inward direction) of ∂M at $\bar{x}$. Then by (3.9) we have

$$\mathscr{C}_0(\bar{x}, \nu)=\{\sum_{i,j=1}^{n-1} g^{ij}(\bar{x})\Gamma_{ij}^n(\bar{x})\}/(\sum_{k=1}^{n-1} g^{kk}(\bar{x}))=(n-1)H(\bar{x})/(\sum_{k=1}^{n-1} g^{kk}(\bar{x})) \tag{3.20}$$

where $\bar{x} \in V_\alpha$ and $\nu=(0, \cdots, 0, 1) \in R^n$.

Under the circumstances stated above we shall prove the following.

Lemma 3.1. *Suppose $H \geqq 0$ on ∂M. Then v defined by* (3.19) *satisfies $\mathscr{L}(v) \leqq 0$ in N_{r_1}.*

Proof. We take a local coordinate system $(\tilde{V}_\alpha, (x_1, \cdots, x_{n-1}, r))$ and we put

$$\tilde{\mathscr{L}}(v)=\mathscr{L}(v)/\mathrm{trace}\,(A_{ij}(x, Dv)) .$$

Putting $p=Dv$, $p_0=Df$, by (3.2), (3.15) and (3.18) we have

$$2c_1 \leqq h'-c_1 \leqq \|p\| \leqq 2h' \tag{3.21}$$

where we note $p=p_0+h'\nu$, $\nu=(0, \cdots, 0, 1)$. From (3.13) we get

$$\mathscr{B}(x, p)=\|p\|\mathscr{C}(x, p) \geqq \|p\|\{\mathscr{C}_0(\bar{x}, \nu)-\tilde{C}_\alpha(\|p\|^{-1}+r)\} \geqq -2h'(r)\tilde{C}_\alpha(\|p\|^{-1}+r)$$

where we used (3.21) and $\mathscr{C}_0(\bar{x}, \nu) \geqq 0$ (see (3.20)). Since $h''(r)<0$ $(0<r<d_1)$, from (3.18) and (3.21)

$$r \leqq 2\|p\|^{-1}h'r \leqq 2m_1\|p\|^{-1} .$$

Thus we get

$$\mathscr{B}(x, p) \geqq -2\tilde{C}_\alpha(2m_1+1)h'(r)\|p\|^{-1} .$$

This estimate and (3.6) imply

$$\tilde{\mathscr{L}}(v) \leqq \mathscr{A}_{nn}(x, p)h''+c_2+2\tilde{C}_\alpha(2m_1+1)h'(r)\|p\|^{-1} .$$

Since $\mathscr{A}_{nn}(x, p)=\mathscr{F}(x, p)/(h'(r))^2$ and $\mathscr{F}(x, p) \geqq \mu_\alpha$ (see (3.12)), we have

$$\tilde{\mathscr{L}}(v) \leqq h'\mathscr{F}(x, p)[h''/(h')^3+2\{c_2+\tilde{C}_\alpha(2m_1+1)\}\mu_\alpha^{-1}\|p\|^{-1}] .$$

Finally, by (3.14) and (3.21), we have

$$\tilde{\mathscr{L}}(v) \leqq h'\mathscr{F}(x, p)[h''/(h')^3+c/(h'-c_1)] .$$

Then by the definition of h we see that the right-hand side of the last inequality vanishes. Thus we have proved $\tilde{\mathscr{L}}(v) \leqq 0$, so $\mathscr{L}(v) \leqq 0$ on $\tilde{V}_\alpha \cap N_{r_1}$ $(1 \leqq \alpha \leqq k)$. Hence we complete the proof.

Theorem 3.1. *Let $\bar{M}$ be a compact n-dimensional Riemannian manifold with* *boundary ∂M and interior M. Assume that the mean curvature (with respect to the inward direction) of ∂M is non-negative everywhere. Let f be a given function in $C^3(\bar{M})$. Suppose that $u \in C^2(\bar{M})$ is a solution of equation* (1.8) *such that $u=f$ on ∂M. Then we have $\|Du\| \leqq L$ on ∂M where L depends only on the bounds of ρ, $\|D\rho\|$, c_0, c_1, c_2, C^1-norms of $\mathscr{C}_0$ on V_α $(1 \leqq \alpha \leqq k)$ and the quantity determined by the Riemannian metric of $\bar{M}$.*

Proof. By Proposition 2.1, $|u| \leqq \sup_{\partial M} |f| = c_0$ in M. We now put $m_1 = 2c_0$ in (3.13). For this m_1 v defined by (3.19) satisfies

$$\mathscr{L}(v) \leqq 0 \quad \text{in } N_{r_1} \text{ (by Lemma 3.1)}$$

and

$$u=f=v \quad \text{when } r=0\,, \qquad u \leqq c_0 \leqq f+2c_0=v \quad \text{when } r=r_1\,.$$

Thus by Lemma 2.1 we have $u \leqq v$ in N_{r_1}, which implies $\partial u/\partial r \leqq \partial v/\partial r$ on ∂M. By a similar method we get $\partial u/\partial r \geqq -\partial v/\partial r$ on ∂M: Hence we obtain $\|Du\| \leqq \|Dv\|$ on ∂M. Putting $L := \sup_{\partial M} \|Dv\|$, we complete the proof.

4. Let Ω be a bounded domain in R^n with boundary $\partial\Omega$ of class C^3. Let a and ρ be given positive valued functions in $C^3(\bar{\Omega})$. For a ε, $0<\varepsilon \leqq \infty$, we let $\tilde{M} = \bar{\Omega} \times (-\varepsilon, \varepsilon)$. We now give a Riemannian metric on $\tilde{M}$ such that

$$ds^2 = a^2(dx_1^2 + \cdots + dx_n^2) + \rho dt^2 \tag{4.1}$$

where $x_1, \cdots, x_n$ are canonical coordinate functions on R^n. Then by (1.8) the minimal surface equation on Ω is expressed as follows:

$$\sum_{i,j=1}^{n} A_{ij}(x, Du)u_{ij} = B(x, Du) \tag{4.2}$$

where

$$\begin{aligned} A_{ij}(x, p) &= (a^2 + \rho|p|^2)\delta_{ij} - \rho p_i p_j\,, \quad 1 \leqq i,\ j \leqq n\,, \\ B(x, p) &= \frac{1}{a}\{a^2(2-n) + \rho(1-n)|p|^2\}(Da \cdot p) - a^2(D\rho \cdot p)/2\rho\,, \quad (x, p) \in \bar{\Omega} \times R^n\,, \end{aligned} \tag{4.3}$$

where $|p| = (\sum_{i=1}^{n} (p_i)^2)^{1/2}$ and the dot stands for the standard inner product in R^n. We rewrite equation (4.2) as follows:

$$\sum_{i,j=1}^{n} \mathscr{A}_{ij}(x, Du)u_{ij} = \mathscr{B}(x, Du) \tag{4.4}$$

where

$$\begin{aligned} \mathscr{A}_{ij}(x, p) &= A_{ij}(x, p)/\{a^2 n + (n-1)\rho|p|^2\}\,, \quad 1 \leqq i,\ j \leqq n\,, \\ \mathscr{B}(x, p) &= B(x, p)/\{a^2 n + (n-1)\rho|p|^2\}\,. \end{aligned} \tag{4.5}$$

We put

(4.6) $\mathscr{C}(x, p)=\mathscr{B}(x, p)/|p|$, $p\neq 0$, $\mathscr{C}_0(x, \sigma)=-(Da\cdot\sigma)/a$, $\sigma=p/|p|$.

Then we have

(4.7) $$|\mathscr{R}(x, p)|:=|\mathscr{C}(x, p)-\mathscr{C}_0(x, \sigma)|\leqq C_1|p|^{-2}$$

for all $x\in\bar{\Omega}$ and all $p\in R^n$ such that $|p|\geqq 1$ where C_1 depends only on the bounds of a, ρ, $|Da|$ and $|D\rho|$ on $\bar{\Omega}$.

We put

$$\mathscr{E}(x, p)=\sum_{i,j=1}^{n}\mathscr{A}_{ij}(x, p)p_i p_j\ ,\quad (x, p)\in\bar{\Omega}\times R^n\ .$$

Then $\mathscr{E}(x, p)=a^2|p|^2/\{a^2 n+(n-1)\rho|p|^2\}$.

For a function $\Phi(x, p)$ of class C^1 on $\bar{\Omega}\times R^n$ we put

$$\Phi_x=(\partial\Phi/\partial x_1, \cdots, \partial\Phi/\partial x_n)\ ,\qquad \Phi_p=(\partial\Phi/\partial p_1, \cdots, \partial\Phi/\partial p_n)\ .$$

It is easy to show the following.

Lemma 4.1.

(1) *There exist positive constants* μ $(0<\mu<1)$ *and* k_1 *such that*

$$p\cdot\mathscr{E}_p\leqq(1-\mu)\mathscr{E}\ ,\qquad \mathscr{E}\geqq\mu$$

for all $x\in\bar{\Omega}$ *and all* $p\in R^n$, $|p|\geqq k_1$ *where* μ *depends only on* k_1 *and the bounds of* a *and* ρ *on* $\bar{\Omega}$.

(2) *There exists a positive constant* C_2 *such that*

$$|\mathscr{E}_x|+|\mathscr{E}_p|\leqq C_2\mathscr{E}\ ,$$
$$|\mathscr{R}|+|\mathscr{R}_x|+|p||\mathscr{R}_p|\leqq C_2\mathscr{E}/|p|$$

for all $x\in\bar{\Omega}$ *and all* $p\in R^n$, $|p|\geqq 1$, *where* C_2 *depends only on the bounds of* a, ρ, $|Da|$ *and* $|D\rho|$ *on* $\bar{\Omega}$.

By the above lemma we can get global gradient estimates of solutions of equation (4.2). By virtue of the theorem ([9], p. 449) due to Serrin we have the following.

Proposition 4.1. *Let* $u\in C^3(\Omega)\cap C^1(\bar{\Omega})$ *be a solution of equation* (4.2) *satisfying* $|u|\leqq m$ *in* Ω *and* $|Du|\leqq L$ *on* $\partial\Omega$. *Then* $|Du|\leqq K$ *in* Ω *where* K *depends only on* μ, k_1, m, L, *the bounds of* a, ρ, $|Da|$, $|D\rho|$, *and the* C^1 *norm of* $\mathscr{C}_0$.

Now we consider the Dirichlet problem for equation (4.2). Since each function in $C^3(\partial\Omega)$ has a C^3 extention into $\bar{\Omega}$ and any function in $C^3(\bar{\Omega})$ gives C^3 boundary data by restriction, we may assume without loss of genelarity that all boundary data are in $C^3(\bar{\Omega})$. Regarding $\partial\Omega$ as a hypersurface in the Riemannian manifold $\bar{\Omega}$ with Riemannian metric $ds^2=a^2(dx_1^2+\cdots+dx_n^2)$, we let H the mean curvature of $\partial\Omega$ with respect to the inward normal direction. From Proposition 2.1, Theorem 3.1 and Proposition 4.1 we have the following.

Proposition 4.2. *Let f be a given function in $C^3(\bar{\Omega})$ and let τ be an arbitrary real number in* [0, 1]. *Suppose $H\geqq 0$ on $\partial\Omega$. Then there exists a positive constant K, independent of τ, such that the conditions*

(1) $u\in C^2(\bar{\Omega})$

(2) $u=\tau f$ *on* $\partial\Omega$

(3) *u is a solution of equation* (4.2) *in* Ω

imply $\sup\{|u|+|Du|\}\leqq K$.

(Using linear theory for elliptic partial differential equations, we see that the condition (3) guarantees $u\in C^3(\Omega)$. Therefore Proposition 4.1 is applicable.)

Making use of Schauder's theorem, Hölder estimates for gradients of solutions of quasilinear elliptic equations due to Ladyzhenskaya and Uralteseva [7] and the Leray-Schauder fixed point theorem, we see that Proposition 4.2 implies the following.

Theorem 4.1. *Let Ω be a bounded domain in R^n with boundary $\partial\Omega$ of class C^3. Let a and ρ be given positive valued functions in $C^3(\bar{\Omega})$. Regarding $\partial\Omega$ as a hypersurface of Riemannian manifold $\bar{\Omega}$ whose line element is $ds^2=a^2(dx_1^2+\cdots+dx_n^2)$, assume that the mean curvature* (*with respect to the inward direction*) *of $\partial\Omega$ is non-negative everywhere. Then for a given function f in $C^3(\partial\Omega)$ there exists a solution u of equation* (4.2) *in Ω such that $u=f$ on $\partial\Omega$. The solution is unique if exists.*

The uniqueness follows from Lemma 2.1.

5. In this section we give an application of Theorem 4.1. Let c be a constant. Throughout this section let $M(c)$ be a complete, simply connected $(n+1)$-dimensional space form with curvature c and let Ω be a bounded domain with boundary $\partial\Omega$ of class C^3 in a closed totally geodesic hypersurface M' of $M(c)$. In the case c is positive, we assume that $\bar{\Omega}=\Omega\cup\partial\Omega$ is contained in an open hemisphere in the n-dimensional Euclidean sphere $S^n(1/c)$ of radius $1/c$. Moreover, let X be a Killing vector field defined on an open set in $M(c)$ including $\bar{\Omega}$ such that X is normal to it. We denote by $\{\varphi_t\}$ the local 1-parameter subgroup of local isometries generated by X. Since $\bar{\Omega}$ is compact, we can take a ε $(0<\varepsilon\leqq\infty)$ such that the map Φ: $\Omega_1\times(-\varepsilon, \varepsilon)\rightarrow M(c)$ defined by $\Phi(x, t)=\varphi_t(x)$, $(x, t)\in\Omega_1\times(-\varepsilon, \varepsilon)$, is C^3-imbedding where Ω_1 is an open neighborhood of $\bar{\Omega}$ in M'. In what follows, we put

$$\mathscr{N}=\Phi(\Omega\times(-\varepsilon, \varepsilon)) \tag{5.1}$$

and for a $u\in C^2(\Omega)$ such that $|u|<\varepsilon$ in Ω we put

$$S(u)=\Phi(\{(x, u(x))\in\Omega\times(-\varepsilon, \varepsilon);\ x\in\Omega\}) \tag{5.2}$$

and also for a $f\in C^0(\partial\Omega)$ such that $|f|<\varepsilon$ on $\partial\Omega$ we put

$$\partial S_f=\Phi(\{(x, f(x))\in\partial\Omega\times(-\varepsilon, \varepsilon);\ x\in\partial\Omega\})\ . \tag{5.3}$$

Let u be a function in $C^2(\Omega)$ such that $|u|<\varepsilon$ in Ω. Suppose that $S(u)$ is a minimal hypersurface in $M(c)$. Then we see that u is a solution of the following quasilinear elliptic partial differential equation of second order:

$$\begin{aligned}&\sum_{i,j=1}^{n}\{(a^2+\rho|Du|^2)\delta_{ij}-\rho u_i u_j\}u_{ij}\\ &\quad=\frac{1}{a}\{a^2(2-n)+\rho(1-n)|Du|^2\}(Da\cdot Du)-a^2(D\rho\cdot Du)/2\rho\end{aligned} \tag{5.4}$$

where $a(x)=4/(4+c|x|^2)$, $\rho(x)=\|X\|^2(x)$, $x\in\bar{\Omega}$, and the dot denotes the inner product of R^n (where $\|\ \|$ denotes the norm defined by the Riemannian metric of $M(c)$).

By Theorem 4.1 and Proposition 2.1, we have

Theorem 5.1. *Let f be a given function iu $C^3(\partial\Omega)$ such that $|f|<\varepsilon$ on $\partial\Omega$. Assume that the mean curvature (with respect to the inward direction) of $\partial\Omega$ is non-negative everywhere. Then there exists a minimal hypersurface S in $M(c)$ such that the boundary of S is ∂S_f and $S=S(u)$ where u is a unique solution of equation* (5.4) *in Ω such that $u=f$ on $\partial\Omega$ and $|u|<\varepsilon$ in Ω.*

In what follows we assume that $\bar{\Omega}$ is homeomorphic to a closed unit ball $\bar{D}=\{x\in R^n;\ |x|\leqq 1\}$ in R^n. We denote by D (resp. ∂D) the interior (resp. the boundary) of $\bar{D}$. For a given function f in $C^0(\partial\Omega)$ such that $|f|<\varepsilon$ on $\partial\Omega$ let Ψ: $\bar{D}\to M(c)$ be a continuous map having the following properties:

(5.5)
(1) $\Psi_{|D}$: $D\to M(c)$ is a minimal immersion of class C^2.
(2) $\Psi_{|\partial D}$: $\partial D\to\partial S_f$ is a homeomorphism.

We now induce a Riemannian metric on $\tilde{M}:=\bar{\Omega}\times R$ such that

$$ds^2=a^2(dx_1^2+\cdots+dx_n^2)+\rho dt^2$$

where R denotes the real line and $a(x)=4/(4+c|x|^2)$, $\rho(x)=\|X\|^2(x)$, $x\in\bar{\Omega}$. Let π_1 (resp. π_2) be the natural projection (we ignore the metric) from $\tilde{M}$ to $\bar{\Omega}$ (resp. R).

For a given function f in $C^0(\partial\Omega)$ such that $|f|<\varepsilon$ on $\partial\Omega$, let Ψ: $\bar{D}\to M(c)$ be a continuous map having the properties in (5.5). Suppose that $\Psi(D)\subset\mathcal{N}$. Then we define a continuous map Ψ_1: $\bar{D}\to\tilde{M}$ by $\Psi_1=\Phi^{-1}\circ\Psi$.

Under the circumstances above, we shall prove the following.

Lemma 5.1. *Let u be a unique solution of equation* (5.4) *such that $u=f$ on $\partial\Omega$. Then $\Psi_1(D)=\{(x, u(x));\ x\in\Omega\}$ and Ψ_1: $D\to\tilde{M}$ is imbedding.*

Proof. Put $S_1=\{(x, u(x));\ x\in\Omega\}$. Let h be a continuous function on $\bar{D}$ defined by

$$h(y)=|u(\pi_1\circ\Psi_1(y))-\pi_2\circ\Psi_1(y)|\ ,\quad y\in\bar{D}\ .$$

Suppose $\Psi_1(D)\not\subset S_1$. Then there exists a point y_0 of D such that $h(y_0)=m:=\sup_D h>0$ (note that by the condition (2) in (5.5) $h\equiv 0$ on ∂D). Now assume $\pi_2\circ\Psi_1(y_0)=u(x_0)+m$ where $x_0=\pi_1\circ\Psi_1(y_0)$. We let $S_2=\{(x, u(x)+m);\ x\in\Omega\}$. Since S_1 is minimal in $\tilde{M}$, S_2 is also minimal in $\tilde{M}$ whose boundary is $\{(x, f(x)+m);\ x\in\partial\Omega\}$. We note $\Psi_1(D)\subset\{(x, t);\ t\leqq u(x)+m\}$. Since $\partial/\partial t$ is transversal to S_2 and $\Psi_1(D)$ is tangent to S_2 at $\Psi_1(y_0)$, by the theorem of implicit function, $\Psi_1(D)$ is locally expressed by a graph of a function v of class C^2 defined on an open ball U centred at x_0. Then v satisfies the condition: $v(x_0)=u(x_0)+m$, $v\leqq u+m$ in U. Moreover, since $\Psi_1(D)$ is minimal in $\tilde{M}$, v is a solution of equation (5.4). Of course, $u+m$ is also a solution of equation (5.4). Applying Proposition 2.2, we have $v=u+m$ in U. Thus there exists an open neighborhood V of y_0 such that $\Psi_1(V)\subset S_2$. This fact implies that $D':=\{y\in D;\ \Psi_1(y)\in S_2\}$ is open in D. It is clear that D' is closed in D. Thus $\Psi_1(D)\subset S_2$, so $\Psi_1(\bar{D})\subset\bar{S}_2$. But this is a contradiction because by the condition (2) in (5.5) $\Psi_1(\partial D)=\{(x, f(x));\ x\in\partial\Omega\}$. Hence we have proved $\pi_2\circ\Psi_1(y_0)\neq u(x_0)+m$. Therefore it must be $\pi_2\circ\Psi_1(y_0)=u(x_0)-m$. In this case we consider a minimal hypersurface $\{(x, u(x)-m);\ x\in\Omega\}$ in $\tilde{M}$. We note $\Psi_1(D)\subset\{(x, t);\ t\geqq u(x)-m\}$. Using the same argument as above, we also get a contradiction. Thus we have proved $\Psi_1(D)=S_1$. Since by the condition (1) in (5.5) Ψ_1 is locally homeomorphic, we see that Ψ_1 is imbedding.

Theorem 5.1 and Lemma 5.1 imply the following.

Theorem 5.2. *In Theorem* 5.1, *assume that $\bar{\Omega}$ is homeomorphic to a closed unit ball $\bar{D}=\{x\in R^n;\ |x|\leqq 1\}$ in R^n. For f let Ψ: $\bar{D}\to M(c)$ be a continuous map having the properties in* (5.5). *Suppose $\Psi(D)\subset\mathcal{N}$. Then $\Psi(D)=S(u)$ and Ψ: $D\to M(c)$ is imbedding where u is a unique solution of equation* (5.4) *in Ω such that $u=f$ on $\partial\Omega$.*

Remark 1. In Theorem 5.1, in the case $c=0$, $\bar{\Omega}\subset R^n$ and $X=\partial/\partial x_{n+1}$ the theorem was proved by Jenkins and Serrin [6].

Remark 2. In Theorem 5.2, suppose that the mean curvature (with respect to the inward direction) of $\Psi(\partial\Omega\times(-\varepsilon, \varepsilon))$ is non-negative everywhere. Then under the assumption $\Psi(D)\subset\Phi(\bar{\Omega}\times(-\varepsilon, \varepsilon))$ the theorem also holds. (See Theorem 4.1 and Lemma 5.1 in [5].)

In this paper we have studied the Dirichlet problem for minimal surface equations in simply connected space forms. We want to investigate this problem in more general Riemannian manifolds. It will be carried out in the next papers.

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