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CONNECTING ELECTRICAL CIRCUITS: TRANSVERSALITY AND WELL-POSEDNESS

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1. Introduction.

An electrical circuit $\mathscr{C}$ consists of three kinds of elements, inductors, capcitors and resistors mutually connected. A state of an electrical circuit is specified by a current vector $\boldsymbol{i}=(i_1, i_2, \cdots, i_b) \in R^b$ and a voltages vector $\boldsymbol{v}=(v_1, v_2, \cdots, v_b) \in R^b$. Let G be the (oriented) graph determined naturally by the circuit $\mathscr{C}$. We can regard $\boldsymbol{i}$ and $\boldsymbol{v}$ as a real 1-chain and 1-cochain of G, i.e., $\boldsymbol{i} \in C_1(G)$ and $\boldsymbol{v} \in C^1(G)$.* The Kirchhoff current (voltage) law restricts $\boldsymbol{i}(\boldsymbol{v})$ to belong to a linear subspace $K_c(\mathscr{C})=\operatorname{Ker} \partial \subset C_1(G)$ $(K_v(\mathscr{C})=\operatorname{Im} \partial^* \subset C^1(G))$, where $\partial: C_1(G) \to C_0(G)$ $(\partial^*: C^0(G) \to C^1(G))$ is boundary (coboundary) operator. Another restriction of possible states is the restraint of resistive characteristics. We admit couplings between same kind of elements. The resistive characteristics are represented by a ρ-dimensional smooth submanifold $\Lambda_R \subset C_1(G_R) \times C^1(G_R)$, where ρ is the number of resistive elements and $C_1(G_R)((C^1(G_R))$ is ρ-dimensional eucledian space consisting of resistive currents (voltages).

Thus the currents and voltages $(\boldsymbol{i}, \boldsymbol{v})=(\boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{i}_R, \boldsymbol{v}_L, \boldsymbol{v}_C, \boldsymbol{v}_R)$ must belong to the configuration space;

$$\Sigma=K \cap \Lambda\,, \qquad K=K_c(G) \times K_v(G)\,, \qquad \Lambda=\pi_R^{-1}(\Lambda_R)\,,$$

where π_R: $C_1(G) \times C^1(G) \to C_1(G_R) \times C^1(G_R)$ is the natural projection, i.e., $\pi_R(\boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{i}_R, \boldsymbol{v}_L, \boldsymbol{v}_C, \boldsymbol{v}_R)=(\boldsymbol{i}_R, \boldsymbol{v}_R)$.

For dynamics to be defined on whole of Σ, two things are needed. One is the transversality of K and Λ which assures Σ to be smooth submanifold. Since K is b-dimensional, $\dim \Sigma=2b-(b+\rho)=b-\rho$ which is equal to the number of inductors and capacitors. Another is the regularity of the map

$$\pi_{LC}\colon \Sigma \to C_1(G_L) \times C^1(G_C)\,,$$

which is the restriction of the natural projection to Σ, i.e.,

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* It is no suffering, however, to regard simply $C_1(G)$ and $C^1(G)$ as b-dimensional eucledian space in the sequel.

$$\pi_{LC}(\boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{i}_R, \boldsymbol{v}_L, \boldsymbol{v}_C, \boldsymbol{v}_R)=(\boldsymbol{i}_L, \boldsymbol{v}_C) .$$

If $D\pi_{LC}(x)$ is non-singular at any point $x \in \Sigma$, we can determine the smooth vector field X on Σ which describes the dynamics of the circuit ([5], [3], [2]). Then any solution curve is locally a solution of certain differential equation. Therefore so called 'jumping phenomena' does not occur. We call such a circuit *well-posed.*

Consider a circuit $\mathscr{C}$ whose graph $G=G(\mathscr{C})$ has a proper tree. Recall a *proper tree* means a tree which contains all the capacitors and no inductor. Since in a cicuit without proper tree the map π_{LC} is singular at any point of Σ, we assume the existence of proper tree ([3]). To assure Σ to be a submanifold, we need the transversality of the characterictic submanifold Λ and the Kirchhoff space K. Now, we define a somewhat stronger condition than the transversality condition. We say Λ and K are *everywhere transverse* if for every $x \in \Lambda$, $T_x(\Lambda)$ is transverse to K in $C_1(G)\times C^1(G)$. We call such a circuit itself everywhere transverse circuit. In the same way, we will define a notion of *strongly well-posedness* in section 3, these notions play a central role in this paper.

Now, we consider a connection of two circuits by two different kinds of (uncoupled) elements. For given circuits $\mathscr{C}_1$ and $\mathscr{C}_2$, connecting a node p of $\mathscr{C}_1$ with a node r of $\mathscr{C}_2$ by an uncoupled element e_1 and connecting a node q of $\mathscr{C}_1$ with a node s of $\mathscr{C}_2$ by an uncoupled element e_2, we obtain a new circuit $\mathscr{C}$.

We will discuss in § 2 the transversality problem, in § 3 the well-posedness problem of the new circuit $\mathscr{C}$ obtained by connecting two circuits as stated above and in § 4 some examples of strongly well-posed circuits. For the reason of connecting by different kinds of elements, see § 4.

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2. Transversality.

Let $\mathscr{C}$ be the circuit obtained by connecting two circuits $\mathscr{C}_i$, $i=1, 2$, by two uncoupled elements e_1 and e_2 as stated before.

Theorem 1. *Suppose the circuit* $\mathscr{C}_i$ *is everywhere transverse,* $i=1, 2$. *Then the new circuit* $\mathscr{C}$ *is also everywhere transverse.*

We say a node p is *L-connected* with q in $\mathscr{C}_1$ if there exists a path from p to q in $\mathscr{C}_1$ which is consisting of only inductor branches.

Theorem 2. *Suppose the circuit* $\mathscr{C}_i$ *is transverse,* $i=1, 2$, p *is L-connected with* q *in* $\mathscr{C}_1$ *and* r *is L-connected with* s *in* $\mathscr{C}_2$. *Then the new circuit* $\mathscr{C}$ *is also transverse.*

Proof of Theorem 1. First, we deal with the following case.

Case 1. The element e_1 is a capacitor and e_2 is an inductor.

Let $\mathscr{T}_i$ be a tree for $\mathscr{C}_i$, $i=1,2$. Put $\mathscr{T}=\mathscr{T}_1\cup\{e_1\}\cup\mathscr{T}_2$ and clearly $\mathscr{T}$ is a tree for $G(\mathscr{C})$. Let $C_1(\mathscr{C}_j)$ and $C^1(\mathscr{C}_j)$ denote the current and voltage spaces of $\mathscr{C}_j$, respectively, $j=1,2$. Since we do not add a new resistive element, the characteristic submanifold of the new circuit $\mathscr{C}$ is the following;

$$(1)\qquad \Lambda(\mathscr{C})=\Lambda(\mathscr{C}_1)\times\Lambda(\mathscr{C}_2)\times C_1(e_1)\times C^1(e_1)\times C_1(e_2)\times C^1(e_2)\subset C_1(G(\mathscr{C}))\times C^1(G(\mathscr{C}))\,.$$

The Kirchhoff space $K(\mathscr{C})$ of $\mathscr{C}$ is represented as follows. Let B_j denote the fundamental loop matrix for $\mathscr{C}_j$ corresponding to the tree $\mathscr{T}_j$, $j=1,2$. Then the fundamental loop matrix B for corresponding to $\mathscr{T}$ has the following form;

$$(2)\qquad B=[I\colon A]=\begin{array}{cc} \begin{array}{cccccc} \mathscr{L}_1 & \mathscr{L}_2 & e_2 & \mathscr{T}_1 & \mathscr{T}_2 & e_1 \end{array} & \\ \left[\begin{array}{cccccc} I & & & A_1 & & \\ & I & & & A_2 & \\ & & 1 & A_{21} & A_{22} & -1 \end{array}\right] & \begin{array}{c} \mathscr{L}_1 \\ \mathscr{L}_2 \\ e_2 \end{array} \end{array},$$

where $B_j=[I\colon A_j]$ and $A_{21}(A_{12})$ has non-zero element (± 1) at the tree branches in $\mathscr{C}_1(\mathscr{C}_2)$ contained in the path connecting $p(r)$ with $q(s)$ in $\mathscr{T}_1(\mathscr{T}_2)$. The current Kirchhoff space $K_c(\mathscr{C})$ of $\mathscr{C}$ is the image of the map

$$B^t\colon C_1(\mathscr{L})\to C_1(G(\mathscr{C}))\,,$$

where $C_1(\mathscr{L})$ denotes the current space of the link branches in $\mathscr{C}$. And the voltage Kirchhoff space $K_v(\mathscr{C})$ of $\mathscr{C}$ is the image of the map

$$Q^t\colon C^1(\mathscr{T})\to C^1(G(\mathscr{C}))\,,$$

where Q is the fundamental cutset matrix for $\mathscr{C}$ and has the form ([4], [1]);

$$(3)\qquad Q=[-A^t\colon I]\,,$$

and $C^1(\mathscr{T})$ denotes the voltage space of the tree branches in $\mathscr{C}$. Thus the Kirchhoff space $K(\mathscr{C})$ of $\mathscr{C}$ is represented as follows;

$$(4)\qquad K(\mathscr{C})=\operatorname{Im}B^t\times\operatorname{Im}Q^t\subset C_1(G(\mathscr{C}))\times C^1(G(\mathscr{C}))\,.$$

Now, we prove the everywhere transversality of $\Lambda(\mathscr{C})$ and $K(\mathscr{C})$. Take a point $(\boldsymbol{i},\boldsymbol{v})\in\Lambda(\mathscr{C})$. We must show that $T_{(i,v)}\Lambda(\mathscr{C})$ is transverse to $K(\mathscr{C})$. To show this, for each vector $\boldsymbol{w}\in T_{(i,v)}C_1(G(\mathscr{C}))\times C^1(G(\mathscr{C}))$ we will find two vectors $\boldsymbol{w}_\Lambda\in T_{(i,v)}\Lambda(\mathscr{C})$ and $\boldsymbol{w}_K\in K(\mathscr{C})$ such that $\boldsymbol{w}=\boldsymbol{w}_\Lambda+\boldsymbol{w}_K$.

Corresponding to the decomposition;

$$\begin{aligned} C_1(G(\mathscr{C}))\times C^1(G(\mathscr{C}))&=C_1(G(\mathscr{C}_1))\times C^1(G(\mathscr{C}_1))\times C_1(G(\mathscr{C}_2))\times C^1(G(\mathscr{C}_2))\times C_1(e_1)\\ &\quad\times C^1(e_1)\times C_1(e_2)\times C^1(e_2)\ ,\\ &=C_1(\mathscr{T}_1)\times C_1(\mathscr{T}_2)\times C_1(\mathscr{L}_1)\times C_1(\mathscr{L}_2)\times C_1(e_1)\times C_1(e_2)\\ &\quad\times C^1(\mathscr{T}_1)\times C^1(\mathscr{T}_2)\times C^1(\mathscr{L}_1)\times C^1(\mathscr{L}_2)\times C^1(e_1)\times C^1(e_2)\ , \end{aligned}$$

we write

$$\begin{aligned} (\boldsymbol{i},\boldsymbol{v})&=(\boldsymbol{i}_1,\boldsymbol{v}_1,\boldsymbol{i}_2,\boldsymbol{v}_2,i_{e_1},v_{e_1},i_{e_2},v_{e_2})\\ &=(\boldsymbol{i}_{\mathscr{T}_1},\boldsymbol{i}_{\mathscr{T}_2},\boldsymbol{i}_{\mathscr{L}_1},\boldsymbol{i}_{\mathscr{L}_2},i_{e_1},i_{e_2},\boldsymbol{v}_{\mathscr{T}_1},\boldsymbol{v}_{\mathscr{T}_2},\boldsymbol{v}_{\mathscr{L}_1},\boldsymbol{v}_{\mathscr{L}_2},v_{e_1},v_{e_2}) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{w}&=(\boldsymbol{w}_1,\boldsymbol{w}_2,w_{e_1},w_{e_2})\\ &=(\boldsymbol{w}(\boldsymbol{i}_{\mathscr{T}_1}),\boldsymbol{w}(\boldsymbol{i}_{\mathscr{T}_2}),\boldsymbol{w}(\boldsymbol{i}_{\mathscr{L}_1}),\boldsymbol{w}(\boldsymbol{i}_{\mathscr{L}_2}),w(i_{e_1}),w(i_{e_2}),\\ &\quad\ \boldsymbol{w}(\boldsymbol{v}_{\mathscr{T}_1}),\boldsymbol{w}(\boldsymbol{v}_{\mathscr{T}_2}),\boldsymbol{w}(\boldsymbol{v}_{\mathscr{L}_1}),\boldsymbol{w}(\boldsymbol{v}_{\mathscr{L}_2}),w(v_{e_1}),w(v_{e_2}))\ . \end{aligned}$$

Then by (1), $(\boldsymbol{i}_j,\boldsymbol{v}_j)\in\Lambda(\mathscr{C})$, $j=1,2$. Since $\Lambda(\mathscr{C}_j)$ is everywhere transverse to $K(\mathscr{C}_j)$, there exist vectors $\boldsymbol{w}_{\Lambda_j}\in T_{(\boldsymbol{i}_j,\boldsymbol{v}_j)}\Lambda(\mathscr{C})$ and $\boldsymbol{w}_{K_j}\in K(\mathscr{C}_j)=T_{(\boldsymbol{i}_j,\boldsymbol{v}_j)}(K(\mathscr{C}_j)+(\boldsymbol{i}_j,\boldsymbol{v}_j))$ such that $\boldsymbol{w}_j=\boldsymbol{w}_{K_j}+\boldsymbol{w}_{\Lambda_j}$, $j=1,2$. Put $\boldsymbol{w}_\Lambda=(\boldsymbol{w}_{\Lambda_1},\boldsymbol{w}_{\Lambda_2},w_\Lambda(i_{e_1}),w_\Lambda(i_{e_2}),w_\Lambda(v_{e_1}),w_\Lambda(v_{e_2}))$ then $\boldsymbol{w}_\Lambda\in T_{(\boldsymbol{i},\boldsymbol{v})}\Lambda(\mathscr{C})$ for any $w(i_{e_j})$ and $w(v_{e_j})$, $j=1,2$, by (1). Put $\boldsymbol{w}_K=(\boldsymbol{w}_{K_1},\boldsymbol{w}_{K_2},w_K(i_{e_1}),w_K(i_{e_2}),w_K(v_{e_1}),w_K(v_{e_2}))$ and decompose,

$$\boldsymbol{w}_{K_j}=(\boldsymbol{w}_{K_j}(\boldsymbol{i}),\boldsymbol{w}_{K_j}(\boldsymbol{v}))\ ,\quad j=1,2\ ,$$

and

$$\begin{aligned} \boldsymbol{w}_{K_j}(\boldsymbol{i})&=(\boldsymbol{w}_K(\boldsymbol{i}_{\mathscr{T}_j}),\boldsymbol{w}_K(\boldsymbol{i}_{\mathscr{L}_j}))\ ,\quad j=1,2\ ,\\ \boldsymbol{w}_{K_j}(\boldsymbol{v})&=(\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_j}),\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{L}_j}))\ ,\quad j=1,2\ . \end{aligned}$$

Then by (2), (3) and (4), $\boldsymbol{w}_K$ belongs to $K(\mathscr{C})$ if and only if

$$\begin{aligned} \boldsymbol{w}_K(\boldsymbol{i}_{\mathscr{T}_j})&=A_j^t\boldsymbol{w}_K(\boldsymbol{i}_{\mathscr{L}_j})+A_{2j}^t w_K(i_{e_2})\ ,\quad j=1,2\ ,\\ w_K(i_{e_1})&=-w_K(i_{e_2})\ , \end{aligned}$$

(5) and

$$\begin{aligned} \boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{L}_j})&=-A_j\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_j})\ ,\quad j=1,2\ ,\\ w_K(v_{e_2})&=-A_{21}\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_1})-A_{22}\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_2})+w_K(v_{e_1})\ . \end{aligned}$$

Thus, if we put $w_K(i_{e_2})=w_K(i_{e_1})=0$, $w_K(v_{e_1})=0$, $w_K(v_{e_2})=-A_{21}\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_1})-A_{22}\boldsymbol{w}_K(\boldsymbol{v}_{\mathscr{T}_2})$, $w_\Lambda(i_{e_1})=w(i_{e_1})$, $w_\Lambda(i_{e_2})=w(i_{e_2})$, $w_\Lambda(v_{e_1})=w(v_{e_1})$, and $w_\Lambda(v_{e_2})=w(v_{e_2})-w_K(v_{e_2})$, then $\boldsymbol{w}=\boldsymbol{w}_\Lambda+\boldsymbol{w}_K$ and $\boldsymbol{w}_\Lambda\in T_{(\boldsymbol{i},\boldsymbol{v})}\Lambda(\mathscr{C})$, $\boldsymbol{w}_K\in K(\mathscr{C})$. This proves the everywhere transversality of $\Lambda(\mathscr{C})$ and $K(\mathscr{C})$.

Case 2. The element e_1 is a linear resistor and the element e_2 is an inductor or a capacitor.

Put $\mathscr{T}=\mathscr{T}_1\cup\{e_1\}\cup\mathscr{T}_2$ and $\mathscr{T}$ is a tree. Let $\Lambda(e_1)$ denote the characteristics of e_1. Since e_1 is a linear resistor, we have

$$\Lambda(e_1)=\{(i_{e_1}, v_{e_1}):\ v_{e_1}=i_{e_1}\cdot r_{e_1}\}\subset C_1(e_1)\times C^1(e_1)\ ,$$

The characteristic submanifold $\Lambda(\mathscr{C})$ of the new circuit has the following form;

$$\Lambda(\mathscr{C})=\Lambda(\mathscr{C}_1)\times\Lambda(\mathscr{C}_2)\times\Lambda(e_1)\ .$$

The Kirchhoff space $K(\mathscr{C})$ is quite the same as in case 1. For a point $(\boldsymbol{i},\boldsymbol{v})\in\Lambda(\mathscr{C})$, we will show the transversality of $T_{(i,v)}\Lambda(\mathscr{C})$ and $K(\mathscr{C})$. As in case 1, for each $\boldsymbol{w}\in T_{(i,v)}(C_1(G(\mathscr{C}))\times C^1(G(\mathscr{C}))$ we must find $\boldsymbol{w}_\Lambda\in T_{(i,v)}\Lambda(\mathscr{C})$ and $\boldsymbol{w}_K\in K(\mathscr{C})$ such that $\boldsymbol{w}=\boldsymbol{w}_K+\boldsymbol{w}_\Lambda$. The only difference from case 1 is that the vector $(w_\Lambda(i_{e_1}), w_\Lambda(v_{e_1}))$ should belong to $T_{(i_{e_1},v_{e_1})}\Lambda(e_1)=\Lambda(e_1)$, i.e., $w_\Lambda(i_{e_1})\cdot r_{e_1}=w_\Lambda(v_{e_1})$. Besides (5) put

$$w_\Lambda(i_{e_1})=w(i_{e_1})\ ,\qquad w_\Lambda(v_{e_1})=w_\Lambda(i_{e_1})\cdot r_{e_1}\ ,\qquad w_K(v_{e_1})=w(v_{e_1})-w_\Lambda(v_{e_1})\ ,$$

and

$$w_\Lambda(v_{e_2})=w(v_{e_2})-w_K(v_{e_2})\ .$$

Then $\boldsymbol{w}_K$ and $\boldsymbol{w}_\Lambda$ have the required properties.

Proof of Theorem 2. The proof of Theorem 2 proceeds in the same way as that of Theorem 1 except one point that if $(\boldsymbol{i}_j, \boldsymbol{v}_j)$ does not belong to $K(\mathscr{C}_j)$ then $T_{(i_j,v_j)}\Lambda(\mathscr{C}_j)$ may not transverse to $K(\mathscr{C}_j)$. This point is rescued by the assumption of L-connectedness as follows.

Since $(\boldsymbol{i},\boldsymbol{v})$ belongs to $K(\mathscr{C})$, $(\boldsymbol{i},\boldsymbol{v})$ satisfies (5) or

$$\begin{aligned}
&\boldsymbol{i}_{\mathscr{T}_j}=A_j^t\boldsymbol{i}_{\mathscr{L}_j}+A_{2j}^t i_{e_2}\ ,\quad j=1,2\ ,\\
&i_{e_1}=-i_{e_2}\ ,\\
&\boldsymbol{v}_{\mathscr{L}_j}=-A_j\boldsymbol{v}_{\mathscr{T}_j}\ ,\quad j=1,2\ ,\\
&v_{e_2}=-A_{21}\boldsymbol{v}_{\mathscr{T}_1}-A_{22}\boldsymbol{v}_{\mathscr{T}_2}+v_{e_1}\ .
\end{aligned}$$

The assumption that p and q are connected by the path consisting of only inductors means the following. There exists a link currents vector $\tilde{\boldsymbol{i}}_{\mathscr{L}_j}\in C_1(G(\mathscr{L}_j))$ such that $\boldsymbol{i}_{\mathscr{T}_j}=A_j^t\tilde{\boldsymbol{i}}_{\mathscr{L}_j}$, $j=1,2$, and the difference between $\boldsymbol{i}_{\mathscr{L}_1}(\boldsymbol{i}_{\mathscr{L}_2})$ and $\tilde{\boldsymbol{i}}_{\mathscr{L}_1}(\tilde{\boldsymbol{i}}_{\mathscr{L}_2})$ occurs only in the link inductor branches connecting $p(r)$ with $q(s)$ in $\mathscr{C}_1(\mathscr{C}_2)$. If we put $\boldsymbol{i}_{\mathscr{L}_j}=(\boldsymbol{i}_{\mathscr{L}(L)_j}, \boldsymbol{i}_{\mathscr{L}(R)_j})$ then $\tilde{\boldsymbol{i}}_{\mathscr{L}_j}=(\tilde{\boldsymbol{i}}_{\mathscr{L}(L)_j}, \boldsymbol{i}_{\mathscr{L}(R)_j})$. Put $(\boldsymbol{i}'_j, \boldsymbol{v}_j)=(\boldsymbol{i}_{\mathscr{T}_j}, \tilde{\boldsymbol{i}}_{\mathscr{L}_j}, \boldsymbol{v}_{\mathscr{L}_j}, \boldsymbol{v}_{\mathscr{T}_j})$ and $(\boldsymbol{i}'_j, \boldsymbol{v}_j)\in K(\mathscr{C}_j)$ and $(\boldsymbol{i}'_j, \boldsymbol{v}_j)\in\Lambda(C_j)$. Since there is no coupling between different kinds of elements, $\Lambda(\mathscr{C}_j)$ has the form; $\Lambda(\mathscr{C}_j)=\pi_{R_j}^{-1}(\Lambda_{R_j})$, $j=1,2$, where π_{R_j}: $C_1(G(\mathscr{C}_j))\times C^1(G(\mathscr{C}_j))\rightarrow C_1(G_R(\mathscr{C}_j))\times C^1(G_R(\mathscr{C}_j))$ is the natural projection to the resistive current and voltage space, $j=1,2$. Therefore $T_{(i'_j,v_j)}\Lambda(\mathscr{C}_j)=T_{(i_j,v_j)}\Lambda(\mathscr{C}_j)$. By the assumption $T_{(i'_j,v_j)}\in\Lambda(\mathscr{C}_j)$ is transverse to $K(\mathscr{C}_j)$. Thus $T_{(i_j,v_j)}\Lambda(\mathscr{C}_j)$ is transverse to $K(\mathscr{C}_j)$. The rest of the proof is the same as that of Theorem 1.

3. Well-posedness.

We will discuss the problem when the connected circuit is well-posed. Henceforth, we assume all the circuits have proper trees and are transverse. First we show a variant of Theorem in [2], which gives a necessary and sufficient condition for a circuit to be well-posed in terms of transversality. Let us recall some notations from [2]. For $(\boldsymbol{i}_L, \boldsymbol{v}_C) \in C_1(G_L) \times C^1(G_C)$, put

$$K(\boldsymbol{i}_L, \boldsymbol{v}_C) = \pi_{LC}^{-1}(\boldsymbol{i}_L, \boldsymbol{v}_C) \cap K ,$$

where π_{LC}: $C_1(G) \times C^1(G) \to C_1(G_L) \times C^1(G_C)$ is the natural projection and K is the Kirchhoff space. By (4), we can see the space $K(\boldsymbol{i}_L, \boldsymbol{v}_C)$ is parallel translation of the space $K(0, 0)$ to a point $b(\boldsymbol{i}_L, \boldsymbol{v}_C)$ in $C_1(G) \times C^1(G)$. Here, $b(\boldsymbol{i}_L, \boldsymbol{v}_C) = \begin{bmatrix} B^t, & 0 \\ 0, & Q^t \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{i}_L \\ 0 \\ \boldsymbol{v}_C \end{bmatrix}$.

Since π_R: $C_1(G) \times C^1(G) \to C_1(G_R) \times C^1(G_R)$ is a linear map, the space $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ is also the parallel translation of $K_0 = \pi_R(K(0, 0))$ to the point $(\boldsymbol{i}_R, \boldsymbol{v}_R) = \pi_R(b(\boldsymbol{i}_L, \boldsymbol{v}_C))$.

Theorem A. *A circuit* $\mathscr{C}$ *is well-posed if and only if the affine subspace* $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ *is transverse to the characteristic submanifold* Λ_R *for all* $(\boldsymbol{i}_L, \boldsymbol{v}_C) \in C_1(G_L) \times C^1(G_C)$.

Proof of Theorem A. Assume $\mathscr{C}$ is not well-posed. Then there exist a singular point $(\boldsymbol{i}, \boldsymbol{v}) = (\boldsymbol{i}_L, \boldsymbol{i}_C, \boldsymbol{i}_R, \boldsymbol{v}_L, \boldsymbol{v}_C, \boldsymbol{v}_R) \in \Sigma$. By Theorem in [2], the space $K_0 + (\boldsymbol{i}_R, \boldsymbol{v}_R) = \pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ is not transverse to Λ_R at $(\boldsymbol{i}_R, \boldsymbol{v}_R)$. This proves the 'if' part of Theorem A.

Let $\mathscr{T}$ be a proper tree and $\mathscr{L}$ a corresponding link. Assume $(\boldsymbol{i}_L, \boldsymbol{v}_C) \in C_1(G_L) \times C^1(G_C)$ be a point such that $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ and Λ_R have non-transversal intersection. Let $(\boldsymbol{i}_R, \boldsymbol{v}_R) = (\boldsymbol{i}_{R(\mathscr{T})}, \boldsymbol{i}_{R(\mathscr{L})}, \boldsymbol{v}_{R(\mathscr{T})}, \boldsymbol{v}_{R(\mathscr{L})}) \in \pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C)) \cap \Lambda_R$ be a non-transversal point, where $\boldsymbol{i}_{R(\mathscr{T})}$ and $\boldsymbol{i}_{R(\mathscr{L})}$ ($\boldsymbol{v}_{R(\mathscr{T})}$ and $\boldsymbol{v}_{R(\mathscr{L})}$) denote the currents (voltages) of tree resistors and link reistors, respectively. Since $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ belongs to $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$, the following holds

$$\begin{aligned} \boldsymbol{i}_{R(\mathscr{T})} &= A_{R\mathscr{T}}^t \boldsymbol{i}_{R(\mathscr{L})} + A_{L\mathscr{T}}^t \boldsymbol{i}_L , \\ \boldsymbol{v}_{R(\mathscr{L})} &= -A_{R\mathscr{T}} \boldsymbol{v}_{R(\mathscr{T})} - A_{RC} \boldsymbol{v}_C , \end{aligned}$$

where $A_{R\mathscr{T}}$, $A_{L\mathscr{T}}$ and A_{RC} are the submatrices of B given by the following form;

$$B = \begin{array}{c} \begin{array}{cccc} R(\mathscr{L}) & L & R(\mathscr{T}) & C \end{array} \\ \left[\begin{array}{cccc} 1 & 0 & A_{R\mathscr{T}} & A_{RC} \\ 0 & 1 & A_{L\mathscr{T}} & A_{LC} \end{array} \right] \end{array} \begin{array}{l} \\ R(\mathscr{L}) \\ L \end{array} . \tag{6}$$

Put

$$\boldsymbol{i}_C=A_{RC}^t\boldsymbol{i}_{R(\mathscr{L})}+A_{LC}^t\boldsymbol{i}_L\ ,$$
$$\boldsymbol{v}_L=-A_{L\mathscr{T}}\boldsymbol{v}_{R(\mathscr{T})}-A_{LC}\boldsymbol{v}_C\ .$$

Then

$$\begin{bmatrix}\boldsymbol{i}_{R(\mathscr{T})}\\ \boldsymbol{i}_C\end{bmatrix}=A^t\begin{bmatrix}\boldsymbol{i}_{R(\mathscr{L})}\\ \boldsymbol{i}_L\end{bmatrix},$$
$$\begin{bmatrix}\boldsymbol{v}_{R(\mathscr{L})}\\ \boldsymbol{v}_L\end{bmatrix}=-A\begin{bmatrix}\boldsymbol{v}_{R(\mathscr{T})}\\ \boldsymbol{v}_C\end{bmatrix},$$

this means

$$(\boldsymbol{i},\boldsymbol{v})=(\boldsymbol{i}_L,\boldsymbol{i}_C,\boldsymbol{i}_R,\boldsymbol{v}_L,\boldsymbol{v}_C,\boldsymbol{v}_R)\in K(\boldsymbol{i}_L,\boldsymbol{v}_C)\ .$$

Since $(\boldsymbol{i}_R,\boldsymbol{v}_R)$ belongs to Λ_R, $(\boldsymbol{i},\boldsymbol{v})$ belongs to $\Lambda=\pi_R^{-1}(\Lambda_R)$. Applying Theorem in [2] for a point $(\boldsymbol{i},\boldsymbol{v})\in K(\boldsymbol{i}_L,\boldsymbol{v}_C)\cap\Lambda\subset K\cap\Lambda=\Sigma$, we see that $(\boldsymbol{i},\boldsymbol{v})$ is a singular point. This proves the 'only if' part of Theorem A.

According to Theorem A, we can reduce the well-posedness problem to the transversality problem. Now we define a stronger condition than well-posedness, corresponding to the 'everywhere transversality' in section 2. A circuit is called *strongly well-posed* if for all $(\boldsymbol{i}_R,\boldsymbol{v}_R)\in\Lambda_R$, $T_{(\boldsymbol{i}_R,\boldsymbol{v}_R)}\Lambda_R$ is transverse to K_0.

Theorem 3. *Suppose the circuit $\mathscr{C}_i$ is strongly well-posed, $i=1,2$. Then the new cicuit $\mathscr{C}$ is also strongly well-posed.*

Theorem 4. *Suppose the circuit $\mathscr{C}_i$ is well-posed, $i=1,2$. If $p(r)$ is L-connected with $q(s)$ in $\mathscr{C}_1(\mathscr{C}_2)$, then the new circuit $\mathscr{C}$ is also well-posed.*

Proof of Theorem 3. This is essentially the same as that of Theorem 1.

Case 1. The element e_1 is a capacitor and e_2 is an inductor.

We can see the following holds by direct verification or by noting that the space K_0 agrees with the Kirchhoff space of the resistive circuit obtained by open-circuitting the inductor branches and short-circuitting the capacitor branches,

$$(7)\qquad K_0(\mathscr{C})=K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2)\ .$$

Clearly,

$$(8)\qquad \Lambda_R(\mathscr{C})=\Lambda_R(\mathscr{C}_1)\times\Lambda_R(\mathscr{C}_2)\subset C_1(G_R(\mathscr{C}))\times C^1(G_R(\mathscr{C}))\ .$$

Take a point $(\boldsymbol{i}_R,\boldsymbol{v}_R)\in\Lambda_R$. Put

$$(\boldsymbol{i}_R,\boldsymbol{v}_R)=(\boldsymbol{i}_{R_1},\boldsymbol{v}_{R_1},\boldsymbol{i}_{R_2},\boldsymbol{v}_{R_2})\in C_1(G_R(\mathscr{C}_1))\times C^1(G_R(\mathscr{C}_1))\times C_1(G_R(\mathscr{C}_2))\times C^1(G_R(\mathscr{C}_2))$$

and by (8) $(\boldsymbol{i}_{R_j},\boldsymbol{v}_{R_j})\in\Lambda_R(\mathscr{C}_j)$. Since $\mathscr{C}_j$ is strongly well-posed, $T_{(\boldsymbol{i}_{R_j},\boldsymbol{v}_{R_j})}(\Lambda_R(\mathscr{C}_j))$ and $K_0(\mathscr{C}_j)$ are transverse by (7) and (8).

Case 2. The element e_1 is a linear resistor and the element e_2 is an inductor. In this case, we have

$$\Lambda_R(\mathscr{C})=\Lambda_R(\mathscr{C}_1)\times\Lambda_R(\mathscr{C}_2)\times\Lambda_R(e_1) \tag{9}$$

where

$$\Lambda_R(e_1)=\{(i_{e_1}, v_{e_1}):\ v_{e_1}=i_{e_1}\cdot r_{e_1}\}\subset C_1(e_1)\times C^1(e_1)\ .$$

And

$$K_0(\mathscr{C})=K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2)\times K_0(e_1)\ , \tag{10}$$

where

$$K_0(e_1)=\{(i_{e_1}, v_{e_1}):\ i_{e_1}=0\}\subset C_1(e_1)\times C^1(e_1)\ .$$

Unless $r_{e_1}=0$, $\Lambda_R(e_1)$ is everywhere transverse to $K_0(e_1)$. The rest of the proof is quite similar to that of case 1.

Case 3. The element e_2 is a linear resistor and e_1 is a capacitor. In this case, instead of (10) the following holds;

$$\begin{aligned}K_0(\mathscr{C})=\{(\boldsymbol{i}_{R_1}, \boldsymbol{i}_{R_2}, i_{e_2}, \boldsymbol{v}_{R_1}, \boldsymbol{v}_{R_2}, v_{e_2}):\ &\boldsymbol{i}_{R(\mathscr{T}_j)}=A^t_{R_j\mathscr{T}_j}\boldsymbol{i}_{R(\mathscr{L}_j)},\\ &\boldsymbol{v}_{R(\mathscr{L}_j)}=-A_{R_j\mathscr{T}_j}\boldsymbol{v}_{(\mathscr{T}_j)}, v_{e_2}=-\tilde{A}_{21}\boldsymbol{v}_{R(\mathscr{T}_1)}-\tilde{A}_{22}\boldsymbol{v}_{R(\mathscr{T}_2)}\}\end{aligned} \tag{11}$$

where $A_{R_j\mathscr{T}_j}$ is determined by the following form, $j=1, 2$;

$$B_j=\begin{array}{c} \\ \left[\begin{array}{cccc} R(\mathscr{L}_j) & L_j & R(\mathscr{T}_j) & C_j \\ 1 & & A_{R_j\mathscr{T}_j} & A_{R_jC_j} \\ & 1 & A_{L_j\mathscr{T}_j} & A_{L_jC_j} \end{array}\right]\begin{array}{l} \\ R(\mathscr{L}_j) \\ L_j \end{array}\end{array} \tag{12}$$

and $\tilde{A}_{2j}$ is the submatrix of A_{2j} consisting of the column corresponding to resistive tree elements, $j=1, 2$. In other words, $K_0(\mathscr{C})$ is the graph of the map,

$$F:\ K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2)\times C_1(e_2)\rightarrow C^1(e_2)$$

defined by

$$F(\boldsymbol{i}_{R_1}, \boldsymbol{v}_{R_1}, \boldsymbol{i}_{R_2}, \boldsymbol{v}_{R_2}, i_{e_2})=-\tilde{A}_{21}\boldsymbol{v}_{R(\mathscr{T}_1)}-\tilde{A}_{22}\boldsymbol{v}_{R(\mathscr{T}_2)}\ .$$

While we have;

$$\Lambda_R(\mathscr{C})=\Lambda_R(\mathscr{C}_1)\times\Lambda_R(\mathscr{C}_2)\times\Lambda_R(e_2) \tag{13}$$

where

$$\Lambda_R(e_2)=\{(i_{e_2}, v_{e_2}):\ v_{e_2}=i_{e_2}\cdot r_{e_2}\}\ .$$

We assert that $\Lambda_R(\mathscr{C})$ is everywhere transverse to $K_0(\mathscr{C})$ unless $r_{e_2}=0$ and $\tilde{A}_{21}=\tilde{A}_{22}=0$. This can be verified by the straightforward calculation similar to the proof of Theorem 1, or we can convince ourselves by observing Fig. 1.

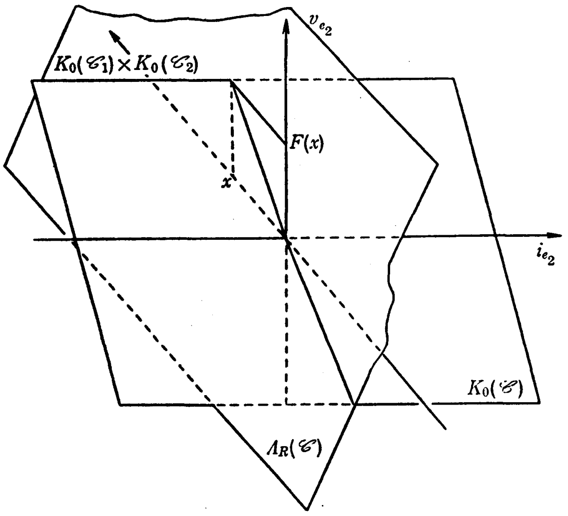

Fig. 1.

(The space $K_0(\mathscr{C})$ is the rotated $(i_{e_2}, K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2))$-space with i_{e_2}-axis fixed. On the other hand, the space $\Lambda_R(\mathscr{C})$ is the rotated $(i_{e_2}, K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2))$-space with $K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2)$-space fixed.) The rest of the proof is quite the same as before.

Proof of Theorem 4. According to Theorem A, we will show that for each $(\boldsymbol{i}_L, \boldsymbol{v}_C)\in C_1(G_L(\mathscr{C}))\times C^1(G_C(\mathscr{C}))$ the space $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ is transverse to Λ_R.

Case 1. The element e_1 is a capacitor and e_2 is an inductor. As before, the following holds;

$$\Lambda_R(\mathscr{C})=\Lambda_R(\mathscr{C}_1)\times\Lambda_R(\mathscr{C}_2) \tag{14}$$

and

$$K_0(\mathscr{C})=K_0(\mathscr{C}_1)\times K_0(\mathscr{C}_2)\ . \tag{15}$$

For a point $(\boldsymbol{i}_R, \boldsymbol{v}_R)\in\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))\cap\Lambda_R(\mathscr{C})$, write

$$(\boldsymbol{i}_R, \boldsymbol{v}_R)=(\boldsymbol{i}_{R_1}, \boldsymbol{i}_{R_2}, \boldsymbol{v}_{R_1}, \boldsymbol{v}_{R_2})$$

and

$$\boldsymbol{i}_{R_j}=(\boldsymbol{i}_{R(\mathscr{T}_j)}, \boldsymbol{i}_{R(\mathscr{L}_j)})\ , \qquad \boldsymbol{v}_{R_j}=(\boldsymbol{v}_{R(\mathscr{T}_j)}, \boldsymbol{v}_{R(\mathscr{L}_j)})\ , \quad j=1,2\ .$$

Then the fact that $(\boldsymbol{i}_R, \boldsymbol{v}_R)$ belongs to $\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ is equivalent to the following:

$$\begin{aligned}\boldsymbol{i}_{R(\mathscr{T}_j)}&=A^t_{R_j\mathscr{T}_j}\boldsymbol{i}_{R(\mathscr{L}_j)}+A^t_{L_j\mathscr{T}_j}\boldsymbol{i}_{L_j}+\widetilde{A}^t_{2j}\boldsymbol{i}_{e_2}\ ,\\ \boldsymbol{v}_{R(\mathscr{L}_j)}&=-A_{R_j\mathscr{T}_j}\boldsymbol{v}_{R(\mathscr{T}_j)}-A_{R_jC_j}\boldsymbol{v}_{C_j}\ .\end{aligned}$$

The assumption of L-connectedness implies the existence of inductor currents $\boldsymbol{i}'_{L_j}\in C_1(G_L(\mathscr{C}_j))$ such that

$$\boldsymbol{i}_{R(\mathscr{T}_j)}=A^t_{R_j\mathscr{T}_j}\boldsymbol{i}_{R(\mathscr{L}_j)}+A^t_{L_j\mathscr{T}_j}\boldsymbol{i}'_{L_j}\ ,\quad j=1,2\ .$$

This in turn means that the point $(\boldsymbol{i}_{R_j}, \boldsymbol{v}_{R_j})=(\boldsymbol{i}_{R(\mathscr{T}_j)}, \boldsymbol{i}_{R(\mathscr{L}_j)}, \boldsymbol{v}_{R(\mathscr{T}_j)}, \boldsymbol{v}_{R(\mathscr{L}_j)})$ belongs to $\pi_R(K(\boldsymbol{i}'_{L_j}, \boldsymbol{v}_{C_j}))$, $j=1,2$. Thus we have,

$$(\boldsymbol{i}_{R_j}, \boldsymbol{v}_{R_j})\in\pi_{R_j}(K(\boldsymbol{i}'_{L_j}, \boldsymbol{v}_{C_j}))\cap\Lambda_R(\mathscr{C}_j)\ ,\quad j=1,2\ .$$

By the assumption, $T_{(\boldsymbol{i}_{R_j},\boldsymbol{v}_{R_j})}\Lambda_R(\mathscr{C}_j)$ is transverse to $\pi_{R_j}(K(\boldsymbol{i}'_{L_j}, \boldsymbol{v}_{C_j}))$ at $(\boldsymbol{i}_{R_j}, \boldsymbol{v}_{R_j})$ in $C_1(G_R(\mathscr{C}_j))\times C^1(G_R(\mathscr{C}_j))$, $j=1,2$. Since $T_{(\boldsymbol{i}_{R_j},\boldsymbol{v}_{R_j})}(\pi_R(K(\boldsymbol{i}_{L_j}, \boldsymbol{v}_{C_j})))=K_0(\mathscr{C}_j)$, $j=1,2$, noting (14) and (15) we see $T_{(\boldsymbol{i}_R,\boldsymbol{v}_R)}\Lambda_R$ is transverse to $T_{(\boldsymbol{i}_R,\boldsymbol{v}_R)}(\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C)))$ $(=K_0(\mathscr{C}))$ at $(\boldsymbol{i}_R, \boldsymbol{v}_R)$.

Case 2. The element e_1 is a linear resistor and e_2 is an inductor. In this case, $\pi_R(\Lambda)$ and $K_0(\mathscr{C})$ have the forms (9) and (10), respectively. For a point $(\boldsymbol{i}_R, \boldsymbol{v}_R)\in\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))\cap\Lambda_R(\mathscr{C})$, write

$$(\boldsymbol{i}_R, \boldsymbol{v}_R)=(\boldsymbol{i}_{R_1}, \boldsymbol{i}_{R_2}, i_{e_1}, \boldsymbol{v}_{R_1}, \boldsymbol{v}_{R_2}, v_{e_1})$$

and

$$\boldsymbol{i}_{R_j}=(\boldsymbol{i}_{R(\mathscr{T}_j)}, \boldsymbol{i}_{R(\mathscr{L}_j)})\ ,\qquad \boldsymbol{v}_{R_j}=(\boldsymbol{v}_{R(\mathscr{T}_j)}, \boldsymbol{v}_{R(\mathscr{L}_j)})\ ,\quad j=1,2\ .$$

since $(\boldsymbol{i}_R, \boldsymbol{v}_R)\in\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$, the following holds;

$$\begin{aligned}&\boldsymbol{i}_{R(\mathscr{T}_j)}=A^t_{R_j\mathscr{T}_j}\boldsymbol{i}_{R(\mathscr{L}_j)}+A^t_{L_j\mathscr{T}_j}\boldsymbol{i}_{L_j}+\widetilde{A}^t_{2j}i_{e_2}\ ,\quad j=1,2\ ,\\ &i_{e_1}=-i_{e_2}\ ,\\ &\boldsymbol{v}_{R(\mathscr{L}_j)}=-A_{R_j\mathscr{T}_j}\boldsymbol{v}_{R(\mathscr{T}_j)}-A_{R_jC_j}\boldsymbol{v}_{C_j}\ ,\quad j=1,2\ .\end{aligned}$$

Then the proof proceeds in the same way as in case 1.

Case 3. The element e_1 is a capacitor and e_2 is a linear resistor. In this case, $\Lambda_R(\mathscr{C})$ and $K_0(\mathscr{C})$ have the forms (13) and (11), respectively. For a point $(\boldsymbol{i}_R, \boldsymbol{v}_R)\in\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))\cap\Lambda_R(\mathscr{C})$, write

$$(\boldsymbol{i}_R, \boldsymbol{v}_R)=(\boldsymbol{i}_{R_1}, \boldsymbol{i}_{R_2}, i_{e_2}, \boldsymbol{v}_{R_1}, \boldsymbol{v}_{R_2}, v_{e_2})$$

and

$$\boldsymbol{i}_{R_j}=(\boldsymbol{i}_{R(\mathscr{T}_j)}, \boldsymbol{i}_{R(\mathscr{L}_j)})\ ,\qquad \boldsymbol{v}_{R_j}=(\boldsymbol{v}_{R(\mathscr{T}_j)}, \boldsymbol{v}_{R(\mathscr{L}_j)})\ ,\quad j=1,2\ .$$

The condition that $(\boldsymbol{i}_R, \boldsymbol{v}_R)\in\pi_R(K(\boldsymbol{i}_L, \boldsymbol{v}_C))$ is equivalent to the following;

$$\begin{aligned}&\boldsymbol{i}_{R(\mathscr{T}_j)}=A^t_{R_j\mathscr{T}_j}\boldsymbol{i}_{R(\mathscr{L}_j)}+A^t_{L_j\mathscr{T}_j}\boldsymbol{i}_{L_j}+\widetilde{A}^t_{2j}i_{e_2}\ ,\quad j=1,2\ ,\\ &\boldsymbol{v}_{R(\mathscr{L}_j)}=-A_{R_j\mathscr{T}_j}\boldsymbol{v}_{R(\mathscr{T}_j)}-A_{R_jC_j}\boldsymbol{v}_{C_j}\ ,\\ &v_{e_2}=-A_{21}\boldsymbol{v}_{\mathscr{T}_1}-A_{22}\boldsymbol{v}_{\mathscr{T}_2}+v_{e_1}\ ,\end{aligned}$$

where

$$\boldsymbol{v}_{\mathscr{T}_j}=(\boldsymbol{v}_{R(\mathscr{T}_j)}, \boldsymbol{v}_{C_j})\ ,\quad j=1,2\ .$$

The rest of proof proceeds as before.

4. Examples.

In our Theorems, we have treated the case of connection by different kinds of elements. If e_1 and e_2 are both capacitors or both inductors, then the new circuit may have no proper tree. If e_1 and e_2 are both resistors, Theorems 3 and 4 do not hold as shows the following example.

Example 1. Consider the circuits $\mathscr{C}_1$ and $\mathscr{C}_2$ shown in Fig. 2. Here the element R_1 is non-linear current-controlled resistor having the characteristics shown in Fig. 4. The essential point is that the characteristic curve has a portion of negative inclination.

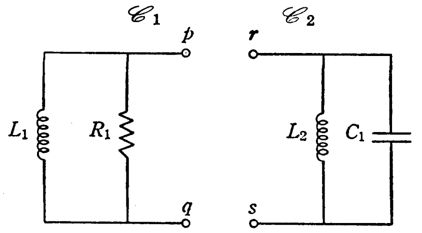

Fig. 2.

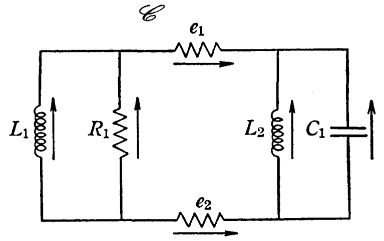

Fig. 3.

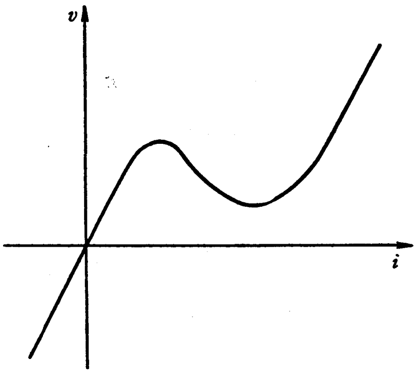

Fig. 4.

Connecting $\mathscr{C}_1$ and $\mathscr{C}_2$ by the resistors e_1 and e_2, we obtain the new circuit $\mathscr{C}$. Although the circuit $\mathscr{C}_1$ and $\mathscr{C}_2$ are strongly well-posed, the new circuit $\mathscr{C}$ is not necessary well-posed. Take the branches $\mathscr{T}=\{R_1, e_1, C_1\}$ as a tree for $\mathscr{C}$ and denote the fundamenta loop matrix associated with $\mathscr{T}$ by B. Then we have,

$$B=\begin{array}{cccccc|c} L_1 & e_2 & L_2 & R_1 & e_1 & C_1 & \\ \hline 1 & 0 & 0 & -1 & 0 & 0 & L_1 \\ 0 & 1 & 0 & -1 & -1 & 1 & e_2 \\ 0 & 0 & 1 & 0 & 0 & -1 & L_2 \end{array},$$

$$B_R=\begin{array}{cccl} e_2 & R_1 & e_1 & \\ [1 & -1 & -1] & e_2 \end{array},$$

$$Q_R=\begin{array}{c|ccc|c} & e_2 & R_1 & e_1 & \\ & 1 & 1 & 0 & R_1 \\ & 1 & 0 & 1 & e_1 \end{array}.$$

Thus, $(\dot{i}_R, \dot{v}_R)\in T_{(i_R,v_R)}K_0(\mathscr{C})=K_0(\mathscr{C})$ if and only if

$$B_R\dot{v}_R=0\,,\qquad Q_R\dot{i}_R=0\,.$$

While, $(i_R, v_R)\in T_{(i_R,v_R)}\Lambda_R$ if and only if

$$R\begin{bmatrix}\dot{i}_R\\ \dot{v}_R\end{bmatrix}=0\,,$$

where

$$R=\begin{array}{c|cccccc|c} & e_2 & R_1 & e_1 & e_2 & R_1 & e_1 & \\ & -r_2 & 0 & 0 & 1 & 0 & 0 & \Lambda_{e_2} \\ & 0 & -f'(i_{R_1}) & 0 & 0 & 1 & 0 & \Lambda_{R_1} \\ & 0 & 0 & -r_1 & 0 & 0 & 1 & \Lambda_{e_1} \end{array}.$$

Therefore $T_{(i_R,v_R)}\Lambda_R$ is transverse to $K_0(\mathscr{C})$ if and only if the enlarged matrix

$$J=\begin{bmatrix} Q & 0 \\ 0 & B \\ \multicolumn{2}{c}{R} \end{bmatrix}$$

has the full rank 2ρ. By elementary operations, we can see rank $J<2\rho$ if and only if

$$\det\begin{bmatrix} 1 & 1 \\ r_2+r_1 & -f'(i_{R_1}) \end{bmatrix}=0$$

i.e.,

$$f'(i_{R_1})=-(r_1+r_2)\,.$$

Thus, this is a counterexample for Theorems 3 and 4. Next, we will give some examples of strongly well-posed circuits.

Example 2. (Linear circuits)

Any circuit whose resistors are all linear, positive and uncoupled is strongly well-posed. Each resistor R_j has the characteristics; $v_{R_j}=i_{R_j}\cdot r_j$, $r_j>0$, $j=1, 2, 3, \cdots, 2\rho$. Let $\mathscr{T}$ be a tree, $\mathscr{L}$ a link, $R(\mathscr{T})$ tree resistors and $R(\mathscr{L})$ link resistors. Then the fundamental loop matrix B is decomposed as follows;

$$B=\begin{array}{c|cccc|c} & R(\mathscr{L}) & \mathscr{L}_{LC} & R(\mathscr{T}) & \mathscr{T}_{LC} & \\ & 1 & 0 & F_1 & F_2 & R(\mathscr{L}) \\ & 0 & 1 & F_3 & F_4 & \mathscr{L}_{LC} \end{array}.$$

And

$$B_R = \begin{array}{cc} R(\mathscr{L}) & R(\mathscr{T}) & \\ [\ 1 & F_1\ ] & R(\mathscr{L}) \end{array}.$$

Thus the enlarged matrix J is given by

$$J = \begin{array}{c} \\ \left[\begin{array}{cccc} R(\mathscr{L}) & R(\mathscr{T}) & R(\mathscr{L}) & R(\mathscr{T}) \\ -F_1^t & 1 & 0 & 0 \\ 0 & 0 & 1 & F_1 \\ 0 & 0 & 0 & -\boldsymbol{r}_{\mathscr{T}}^{-1} \\ -\boldsymbol{r}_{\mathscr{L}} & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ R(\mathscr{T}) \\ R(\mathscr{L}) \\ R(\mathscr{T}) \\ R(\mathscr{L}) \end{array},$$

where $\boldsymbol{r}_{\mathscr{T}}$ and $\boldsymbol{r}_{\mathscr{L}}$ are diagonal matrices with positive entries. The matrix J has the full rank if and only if

$$\det\begin{bmatrix} -F_1^t & \boldsymbol{r}_{\mathscr{T}}^{-1} \\ \boldsymbol{r}_{\mathscr{L}} & F_1 \end{bmatrix} \neq 0 .$$

This is equivalent to $\det[F_1F_1^t + \boldsymbol{r}_{\mathscr{L}}\boldsymbol{r}_{\mathscr{T}}^{-1}] \neq 0$. And this is always assumed because $F_1^tF_1$ is positive semi-definite and $\boldsymbol{r}_{\mathscr{L}}\boldsymbol{r}_{\mathscr{T}}^{-1}$ is positive definite.

Of course, if the resistors are not linear but monotone increasing, then the result remains valid. More generally, the following holds.

Example 3. If the characteristics of the resistors have the following form with respect to some tree,

$$\Lambda_R = \Lambda_R(\mathscr{T}) \times \Lambda_R(\mathscr{L})$$

and

$$\Lambda_R(\mathscr{T}) = \{(\boldsymbol{i}_{\mathscr{T}}, \boldsymbol{v}_{\mathscr{T}}) : \boldsymbol{i}_{\mathscr{T}} = f(\boldsymbol{v}_{\mathscr{T}})\}, \qquad \Lambda_R(\mathscr{L}) = \{(\boldsymbol{i}_{\mathscr{L}}, \boldsymbol{v}_{\mathscr{L}}) : \boldsymbol{v}_{\mathscr{L}} = g(\boldsymbol{i}_{\mathscr{L}})\}$$

and if at each point $(\boldsymbol{i}_R, \boldsymbol{v}_R) \in \Lambda_R$

$$\det\begin{bmatrix} -F_1^t & f'(\boldsymbol{v}_{\mathscr{T}}) \\ g'(\boldsymbol{i}_{\mathscr{L}}) & F_1 \end{bmatrix} \neq 0 .$$

then the circuit is strongly well-posed.

Next, we consider a rather old-fashioned electron tube amplifier.

Example 4. (A unit amplifier using a triode.)

Consider the circuit A_1 shown in Fig. 5. The coupled resistors R_b and R_c consists a triode. The resistors R_g and R_k are uncoupled positive linear resistors. The resistor R_p is a linear resistor with a voltage source, its characteristics is represented as shown in Fig. 6. The electron tube $\{R_b, R_c\}$ has the following characteristics;

$$\Lambda_{R_{b,c}} = \{(i_b, i_c, v_b, v_c) : i_b = f_b(v_b, v_c),\ i_c = f_c(v_b, v_c)\}$$

where the function f_b and f_c are supposed to be smooth and at each point to satisfy the condition;

$$\frac{\partial f_b}{\partial v_b} > 0\,, \qquad \frac{\partial f_c}{\partial v_c} > 0\,.$$

This is satisfied at least within the region usually used. A typical form of f_b and f_c are illustrated as follows.

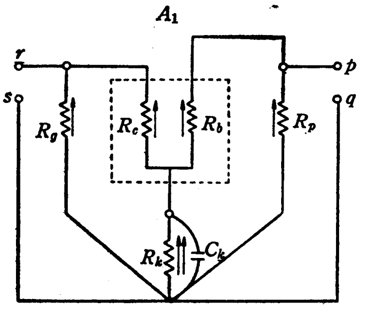

Fig. 5.

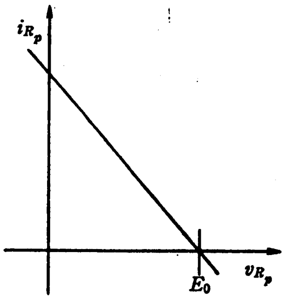

Fig. 6.

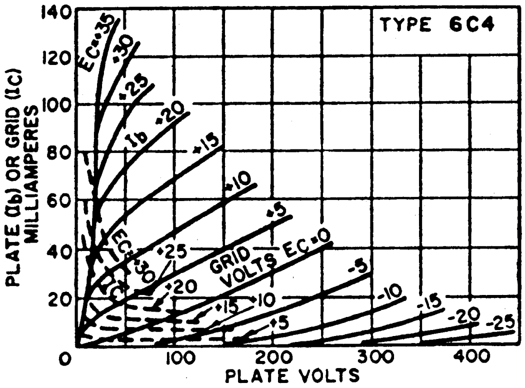

Fig. 7. (From RCA tube manual.)

If we take $\{C_k, R_b, R_c\}$ as a tree, then the condition of example 3 is satisfied. For the condition

$$\det \begin{bmatrix} 0 & 0 & 1 & \dfrac{\partial f_b}{\partial v_b} & \dfrac{\partial f_b}{\partial v_c} \\ 1 & 0 & 1 & \dfrac{\partial f_c}{\partial v_b} & \dfrac{\partial f_c}{\partial v_c} \\ r_g & 0 & 0 & 0 & -1 \\ 0 & r_k & 0 & 0 & 0 \\ 0 & 0 & r_p & -1 & 0 \end{bmatrix} = 0$$

is equivalent to

$$1+\frac{\partial f_b}{\partial v_b}\cdot r_p+\frac{\partial f_c}{\partial v_c}\cdot r_g=0 .$$

Hence for any $r_p>0$ and $r_g>0$, the circuit is strongly well-posed.

Applying our results inductively, we obtain,

Example 5. (Triode amplifier)

Let A_i be a copy of the circuit A_1, $i=1, 2, 3, \cdots, n$. Connect A_i's as shown in the following. Then this circuit is strongly well-posed.

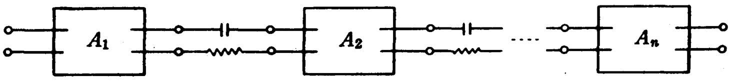

Fig. 8.

Now, consider a circuit for which we can take a tree consisting only capacitors, and we have,

$$B_r=\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & & 1 \end{bmatrix}, \quad J=\begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ D_{i_R}g & & D_{v_R}g & & \end{bmatrix}$$

where

$$\Lambda_R=\{(\boldsymbol{i}_R, \boldsymbol{v}_R) : g(\boldsymbol{i}_R, \boldsymbol{v}_R)=0\} .$$

Therefore the circuit is strongly well-posed provided $\det D_{i_R}g\neq 0$. This argument shows the usual transistor flip-flop circuit with stray capacitors as shown in Fig. 9 is strongly well-posed.

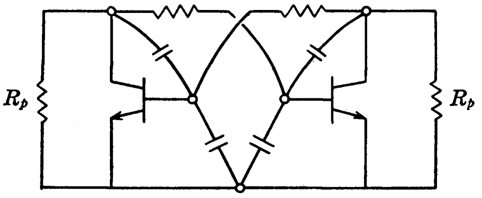

Fig. 9.

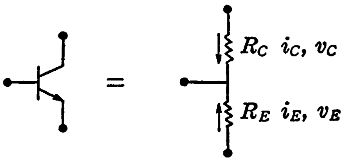

Fig. 10.

The characteristics of the transistor is given by

$$i_C=f_C(v_C)-\alpha_{21}i_E ,$$
$$i_E=f_E(v_E)-\alpha_{12}i_C ,$$

where

$$0<\alpha_{12}\,,\quad \alpha_{21}<1\,.$$

Since R_1 and R_p are linear uncoupled (rigourously saying, R_p is affine, for R contains a voltage source), it is sufficient to verify the determinant of the following 2×2 matrix is non-zero,

$$\begin{array}{cc} i_C & i_E & \\ \left[\begin{array}{c} 1 \\ \alpha_{12} \end{array}\right. & \left.\begin{array}{c} \alpha_{21} \\ 1 \end{array}\right] & \begin{array}{c} R_C \\ R_E \end{array} \end{array}.$$

Since $0<\alpha_{12}$, $\alpha_{21}<1$, the circuit is strongly well-posed.

More generally, we have.

Example 6. Any transistor circuit consisting of transistors, uncoupled strictly monotone resistors, capacitors and inductors having a tree consisting of only capacitors is strongly well-posed.

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