YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

NOTE ON AN ALMOST SURE INVARIANCE PRINCIPLE FOR SOME EMPIRICAL PROCESSES

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(Received June 20, 1979)

1. Summary. Let $\{\xi_i\}$ be a strictly stationary sequence of random variables which are distributed uniformly over the interval [0, 1] and satisfy the strong mixing (s.m.) condition

(1.1) $$\alpha(n)=\sup_{A\in\mathscr{M}_{-\infty}^{k},B\in\mathscr{M}_{k+n}^{\infty}}|P(A\cap B)-P(A)P(B)|\downarrow 0$$

as $n\rightarrow\infty$, where $\mathscr{M}_a^b$ is the σ-algebra generated by $\xi_a, \cdots, \xi_b$ $(a\leqq b)$.

Recently, Berkes and Philipp (1977) proved an almost sure invariance principle for some empirical processes by which a functional law of the iterated logarithm for the functions of s.m. sequences, a two-dimensional functional law of the iterated logarithm, etc., are easily obtained. In this note, we shall prove that Theorem 1 in Berkes and Philipp [1] remains true under the less restrictive s.m. condition.

2. The main result. Let $F_N(s)$ $(0\leqq s\leqq 1)$ be the empirical distribution function defined by $\xi_1, \cdots, \xi_N$.

Let

(2.1) $$R(s, t)=[t](F_{[t]}(s)-s)\ ,\qquad 0\leqq s\leqq 1\ ,\qquad t\geqq 0$$

where $[t]$ denotes the largest integer not exceeding t. Write

(2.2) $$g_t(\alpha)=I_{[0,t)}(\alpha)-t$$

where $I_{[s,t)}(.)$ denotes the indicator function of the interval $[s, t)$ and for fixed s and t with $0\leqq s<t\leqq 1$, put

(2.3) $$x_n(s, t)=g_t(\xi_n)-g_s(\xi_n)\ .$$

Then, we can rewrite $R(s, t)$ as

(2.4) $$R(s, t)=\sum_{j=1}^{[t]}x_j(0, s)\ .$$

Consider the covariance function

$$\Gamma(s, t)=Eg_s(\xi_1)g_t(\xi_1)+\sum_{n=2}^{\infty} Eg_s(\xi_1)g_t(\xi_n)+\sum_{n=2}^{\infty} Eg_t(\xi_1)g_s(\xi_n) \qquad (0\leqq s,\ t\leqq 1)\ . \tag{2.5}$$

(It is known that under the conditions of Theorem (below) the two series in (2.5) converge absolutely for $0\leqq s,\ t\leqq 1$).

Let

$$\sigma^2(s, t)=\Gamma(s, s)+\Gamma(t, t)-2\Gamma(s, t)\ . \tag{2.6}$$

It is clear that if $\Gamma(s, t)$ is positive definite, then $\sigma^2(s, t)>0$ for $0\leqq s<t\leqq 1$. Further, let $\{K(s, t),\ 0\leqq s\leqq 1,\ t\geqq 0\}$ be a Kiefer process, i.e., a separable Gaussian process $K(s, t)$ on $[0, 1]\times[0, \infty)$ such that $K(0, t)=K(1, t)=K(s, 0)$ for all $0\leqq s\leqq 1$, $t\geqq 0$,

$$EK(s, t)=0 \tag{2.7}$$

and

$$EK(s, t)K(s', t')=\min(t, t')\Gamma(s, s')\ . \tag{2.8}$$

We prove the following

Theorem. *Let $\{\xi_i\}$ be random variables defined above. Suppose that $\alpha(n)=O(n^{-3/a})$ for some a $(0<a<1)$. Suppose that $\Gamma(s, s')$ is positive definite. Then, without changing the distribution of the empirical process $R(s, t)$ of $\{\xi_n\}$ we can redefine R on a richer probability space on which there exists a Kiefer process with covariance* $\min(t, t')\Gamma(s, s')$ *such that*

$$\sup_{0\leqq s\leqq 1}\ \sup_{0\leqq t\leqq T} |R(s, t)-K(s, t)|=O(T^{1/2}(\log T)^{-\lambda}) \quad a.s. \tag{2.9}$$

for some $\lambda>0$.

3. Proof. To prove Theorem, we need some lemmas. In what follows, we denote by the letter C, with or without subscript, various absolute constants.

Lemma 1. *Let X be a random variable with finite first moment. Let $\varphi(t)$ be the characteristic function of X. Further, let Z be the standardized normal random variable. If there exist two numbers L and $T(>1)$ such that for all $t(|t|\leqq T)$*

$$|\varphi(t)-e^{-t^2/2}|\leqq L\ , \tag{3.1}$$

then for all $M(>1)$

$$\sup_u |P(X<u)-\Phi(u)|\leqq C\left[M^{-1}\{E|X|+E|Z|\}+L\log MT+\frac{1}{T}\right]. \tag{3.2}$$

where

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\ .$$

Proof. Let $M(>1)$ be an arbitrary number. Since $E|X|<\infty$, so for all t $(|t|\leqq M^{-1})$

$$|\varphi(t)-e^{-t^2/2}|\leqq E|e^{i(X-Z)t}-1|\leqq C|t|\{E|X|+E|Z|\}\,.$$

Hence, we have

$$\begin{aligned}\int_{|t|\leqq T}\left|\frac{\varphi(t)-e^{-t^2/2}}{t}\right|dt&=\left\{\int_{|t|\leqq M^{-1}}+\int_{M^{-1}<|t|\leqq T}\right\}\left|\frac{\varphi(t)-e^{-t^2/2}}{t}\right|dt\\&\leqq C\left[\{E|X|+E|Z|\}\int_{|t|\leqq M^{-1}}dt+L\int_{M^{-1}<t\leqq T}\left|\frac{1}{t}\right|dt\right]\\&\leqq C[M^{-1}\{E|X|+E|Z|\}+L\log MT]\,.\end{aligned}$$

Now, (3.2) follows from Theorem 2 in [2, Chap. 5, §1], which completes the proof.

We put $l=t-s$ for any pair (s, t) $(0\leqq s<t\leqq 1)$.

Lemma 2. *Suppose that the conditions of Theorem are satisfied. Then there exist positive numbers γ, ρ, μ and C_0 such that $1/2<\rho<\gamma$ and*

$$\begin{aligned}P(|\sum_{j=H+1}^{H+N}x_j(s, t)|\geqq 3Al^{(\gamma-\rho)/2}(2N\log\log N)^{1/2})\\ \leqq C[\exp(-A^2C_0^{-2}l^{\rho}\log\log N)+A^{-2}l^{\rho}N^{-1/4}]\end{aligned}\tag{3.4}$$

uniformly for all pairs (s, t) $(l\geqq N^{-1/2-\mu})$ and for all $H\geqq 0$, $A>0$ as $N\to\infty$.

Proof. Firstly, we note that if $\alpha(n)=O(n^{-3/a})$ then we can easily find positive numbers C_0 and $\gamma'(\geqq 5/9)$ such that for s, t $(0\leqq s, t\leqq 1)$ and for all n sufficiently large

$$\sigma_n^2(s, t)=\frac{1}{n}E|\sum_{j=1}^{n}x_j(s, t)|^2\leqq C_0^2l^{\gamma'}\tag{3.5}$$

since $|x_0(s, t)|\leqq 1$ and $E|x_0(s, t)|\leqq Cl$.

Secondly, as $\{\xi_i\}$ is strictly stationary, we shall prove Lemma 2 in the case $H=0$.

Let

$$\gamma=\min(5/9, (7-3a)/8)\tag{3.6}$$

and choose ρ so that $1/2<\rho<\gamma$. Let N be a sufficiently large number. Let $p=[N^{1/2}(\log N)^{-3}]$ and $k=[N/2p]$. Choose a number μ so that

$$0<\mu<(1-a)/2(3+a)\,.\tag{3.7}$$

For brevity, we put

$$\chi_N=(2\log\log N)^{1/2}\tag{3.8}$$

and $\sigma=\sigma_p(s, t)$ (>0).

For any pair (s, t) such that $l \geqq N^{-1/2-\mu}$, put

$$y_j = p^{-1/2}\sigma^{-1} \sum_{j=1}^{p} x_{2(j-1)p+i}(s, t) \qquad (j=1, \cdots, k)$$

and

$$y^*_{k+1} = p^{-1/2} \sum_{i=2kp+1}^{N} x_i(s, t) .$$

As $\{x_j(s, t)\}$ is strictly stationary, so

$$\text{(3.9)} \qquad \text{LHS of (3.4)} \leqq 2P(|\sum_{j=1}^{k} y_j| \geqq A l^{(\gamma-\rho)/2}\sigma^{-1}k^{1/2}\chi_N) + P(|y^*_{k+1}| \geqq A l^{(\gamma-\rho)/2}k^{1/2}\chi_N) = 2I_1 + I_2 , \qquad \text{(say)} .$$

It follows from (3.5) that

$$\text{(3.10)} \qquad I_2 \leqq A^{-2}l^{-(\gamma-\rho)}k^{-1}\chi_N^{-2}E|y^*_{k+1}|^2 \leqq CA^{-2}l^{-(\gamma-\rho)}k^{-1}\chi_N^{-2}p^{-1}(N-2kp)l^{\gamma'} \leqq CA^{-2}N^{-1/4}l^{\rho} .$$

Now, we proceed to estimate I_1. From (3.7), (3.9) and Lemma 1 in Yoshihara [3] we have

$$E|y_1|^4 \leqq C\sigma^{-4}\{l^{4/3} + l^{1-\alpha}p^{-1}(\log p)\} \leqq C\sigma^{-4}l^{2\gamma}$$

and so from Schwartz's inequality and the fact $E|y_1|^2 = 1$ we have

$$\text{(3.11)} \qquad E|y_1|^3 \leqq \{E|y_1|^4\}^{1/2}\{E|y_1|^2\}^{1/2} \leqq C\sigma^{-2}l^{\gamma} .$$

Hence, by Lemma 1, [2, Chap. 5, §2] and (3.11) we have that for all t $(|t| \leqq (1/4)T_N)$

$$|E\{\exp(ik^{-1/2}t\sum_{j=1}^{k} y_j)\} - e^{-t^2/2}|$$
$$\leqq |E\{\exp(ik^{-1/2}t\sum_{j=1}^{k} y_j)\} - \prod_{j=1}^{k} E\{\exp(ik^{-1/2}ty_j)\}| + |\prod_{j=1}^{k} E\{\exp(ik^{-1/2}ty_j)\} - e^{-t^2/2}|$$
$$\leqq C\{k\alpha(p) + T_N^{-1}\}$$

where

$$T_N = k^{1/2}\{E|y_1|^2\}^{3/2}\{E|y_1|^3\}^{-1} \geqq Ck^{1/2}\sigma^2 l^{-\gamma} .$$

Since for all N sufficiently large

$$E|k^{-1/2}\sum_{j=1}^{k} y_j| \leqq k^{1/2}E|y_1| \leqq k^{1/2}\{E|y_1|^2\}^{1/2} \leqq k^{1/2}$$

so using Lemma 1 (with $M=N^2$), we have

$$\sup_z |P(k^{-1/2}\sum_{j=1}^{k} y_j < z) - \Phi(z)| \leqq CN^{-1/4}\sigma^{-2}l^{\gamma} .$$

Hence, from the non-uniform estimate of the central limit theorem and (3.5)

(3.12)
$$I_1 \leqq C\left[\{1-\Phi(Al^{(\gamma-\rho)/2}\sigma^{-1}\chi_N)\}+\frac{N^{-1/4}\sigma^{-2}l^{\gamma}}{1+A^2 l^{(\gamma-\rho)}\sigma^{-2}}\right]$$
$$\leqq C[\exp(-A^2C_0^{-2}l^{\rho}\log\log N)+A^{-2}N^{-1/4}l^{\rho}] .$$

Combining (3.9), (3.10) and (3.12), we have (3.4) and the proof is completed.

The following two lemmas correspond Lemmas 5.1 and 5.2 in Berkes and Philipp [1].

Lemma 3. *If* (3.4) *holds, then as* $k\to\infty$

(3.13)
$$P(\max_{1\leqq j\leqq 2^r}\sup_{s_j\leqq s\leqq s_{j+1}}|R(s, t_k)-R(s_j, t_k)|\geqq t_k^{1/2}(\log t_k)^{-4\varepsilon})\leqq C\exp(-k^{4\varepsilon})$$

where $r=r_k=[\log k/\log 4]$, $t_k=[\exp(k^{1-\varepsilon})]$ *and* $\varepsilon=(\gamma-\rho)/16$.

Proof. We write for $0\leqq s<s'\leqq 1$ and integers $P(\geqq 0)$, $Q(\geqq 1)$

$$F(P, Q, s, s')=|\sum_{j=P+1}^{P+Q} x_j(s, s')| .$$

Put $m=[(1/2+\mu)\log t_k/\log 2]$ and write for $s_j\leqq s<s_{j+1}$

$$s=s_j+\sum_{\nu=r+1}^{m}\beta_\nu 2^{-\nu}+\theta 2^{-m}$$

where $\beta_\nu=0, 1$ and $0\leqq\theta\leqq 1$. We define the following events:

$$E_k(\nu, a)=\{F(0, t_k, a2^{-\nu}, (a+1)2^{-\nu})\geqq 2C_0 2^{-(\gamma-\rho)\nu/2}t_k^{1/2}\chi_{t_k}\}$$
$$E_k=\bigcup_{r<\nu\leqq m+1}\ \bigcup_{0\leqq a<2^\nu} E_k(\nu, a) .$$

Then, applying the same method in the proof of Lemma 5.1 in [1] and using Lemma 2 we have

$$F(0, t_k, s_j, s)\leqq Ct_k^{1/2}(\log t_k)^{-(\gamma-\rho)/4} \qquad \text{a.s.}$$

and the proof is completed.

Lemma 4. *If* (3.4) *holds, then as* $k\to\infty$

$$P(\operatorname{map}_{t_k\leqq t\leqq t_{k+1}}\sup_{0\leqq s\leqq 1}|R(s, t)-R(s, t_k)|\geqq t_k^{1/2}(\log t_k)^{-\varepsilon})\leqq Ck^{-2} .$$

Proof. Put $p=[(1-\mu)\log t_k/\log 4]$ and $q=[\log(t_{k+1}-t_k)/\log 2]$. We write each integer t $(t_k\leqq t\leqq t_{k+1})$ in the form

$$t=t_k+\sum_{0\leqq j\leqq q}\tau_j 2^j=t_k+\sum_{p<j\leqq q}\tau_j 2^j+\theta t_k^{(1-\mu)/2}$$

where $\tau_j=0, 1$ and $0\leqq\theta\leqq 1$. Also, we write $s(0\leqq s\leqq 1)$ in the form

$$s=\sum_{\nu=1}^{\infty}\sigma_\nu 2^{-\nu}=\sum_{\nu\leqq m}\sigma_\nu 2^{-\nu}+\theta 2^{-m} .$$

Further, let

$$H_k(\nu, a, j, h) = \{F(t_k + h2^{j+1}, 2^j, a^j, a2^{-\nu}, (a+1)2^{-\nu}) \geqq 2C_0 2^{-(\gamma-\rho)\nu/2}(q-j)^{-2}2^{q/2}\chi_{2^q}\}$$

$$H_k = \bigcup_{p<j\leq q} \bigcup_{0\leq\nu\leq(1+\mu)j/2} \bigcup_{0\leq a<2^\nu} \bigcup_{0\leq h<2^{q-j}} H_k(\nu, a, j, h) .$$

Then, applying the same method in the proof of Lemma 5.2 in [1] and using Lemma 2, we have the lemma.

From Lemmas 3 and 4, we have the following lemma.

Lemma 5. *If the conditions of Theorem are satisfied, then*

$$\max_{t_k\leq t\leq t_{k+1}} \max_{1\leq j\leq 2^{r_k}} \max_{s_j\leq s\leq s_{j+1}} |R(s, t) - R(s_j, t_k)| \leqq C t_k^{1/2}(\log t_k)^{-\lambda} \quad a.s. \tag{3.14}$$

where t_k and r_k are the ones defined in Lemma 3.

Proof of Theorem. Since under the conditions of Theorem the corresponding result to Proposition 3.1 in [1] is proved by the method used there, so using the Berkes and Philipp method in [1] and Lemma 5, we have the theorem.

References

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