YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

A CONSTRUCTION METHOD OF HOMOLOGY 3-SPHERES IN S^4

By

TATSUO HOMMA and TERUO NAGASE[1)]

(Received June 19, 1979)

Introduction.

Mazur gives a construction method of a contractible 4-manifold which collapses to the Dance hat (hence the double of the 4-manifold is the 4-sphere), and whose boundary is not the 3-sphere [Mz]. In this paper, we give a construction method of a homology 3-sphere in S^4 corresponding to each embedding of a polyhedron into the 3-sphere. More precisely, let X be a polyhedron which possesses a cell subdivision with only one 0-cell P. Let $K=\overline{X-N(P;X)}$. Let $f:K\to S^3$ be an embedding. Then we can associate a 4-manifold in S^4 which has a spine X. Hence we can get a homology 3-sphere in S^4 if X is aspherical, and a contractible 4-manifold in S^4 if X is contractible. Since the Dance hat is such a polyhedron, we can construct a Mazur's example (see Example 1). Our construction method stands on the h-cobordism theory [Ml]. We use a special Morse function to respect that the object which we deal with lies in S^4.

§ 1. Preliminaries.

We work in the piecewise linear category.

Maps are all piecewise linear maps. The interior, closure, and boundary of $(\cdots)$ are denoted by $\mathrm{Int}(\cdots)$, $\overline{(\cdots)}$, and $\partial(\cdots)$ respectively. The n-dimensional sphere is denoted by S^n. For intervals, we use the common notations $[a, b]$, $(a, b]$, $[a, b)$, and (a, b). The unit interval will be denoted by I.

For two simplical complexes K and L, we denote by $K*L$ the abstract join of K and L (for the definition of "join", see [Zm] or [Hd]). If they lie in S^n and they are joinable in the space, then we assume that the join $K*L$ also lies in S^n.

For a regular neighborhood N of a subpolyhedron X in a polyhedron P, we always assume that N is small, compared to things previously defined. That is to say, we construct N as follows: Choose a triangulation L in which all sub-

1) The partial results in this article are contained in the second author's Master thesis written at University of Tokyo under the direction of Professor I. Tamura.

spaces, previously mentioned in the argument, are subcomplexes. Let L'' be a second derived subdivision of L, and let N be the simplical neighborhood of X with respect to L''. We often denote N by $N(X; P)$.

We consider S^4 as $S^3\times[-1,1]/\{S^3\times(-1), S^3\times 1\}$. We denote $S^3\times(-1)$ and $S^3\times 1$ by $-\infty$ and $+\infty$ respectively when we regard $S^3\times(-1)$ and $S^3\times 1$ as the points in S^4. Let $p: S^4\to[-1,1]$ be the natural projection onto the "second" factor $[-1,1]$, and π: $S^4-\{+\infty, -\infty\}\to S^3$ be the natural projection onto the first factor S^3.

In this paper, a "cell" means the closure of an open cell.

We use the sign $\square$ to indicate the end of proofs.

§ 2. Main theorem.

To state main theorem, we introduce several notations and definitions.

We define a *layer* $(L; V, W)_a^b$ as follows. Let V^* be a 3-manifold in S^3 and let $A_1\cong I^3, A_2\cong I^3, \cdots, A_m\cong I^3, B_1\cong I^3, \cdots, B_n\cong I^3$ be mutually disjoint 3-balls in S^3 such that for each $i=1,\cdots,m$, $j=1,\cdots,n$

(1) $A_i\subset V^*$ and $B_j\subset\overline{(S^3-V^*)}$

(2) $A_i\cap\partial V^*\cong(\partial I^{\alpha(i)})\times I^{3-\alpha(i)}$, $\alpha(i)=0,1,2,3$, and

(3) $B_j\cap\partial V^*\cong(\partial I^{\beta(j)})\times I^{3-\beta(j)}$, $\beta(j)=0,1,2,3$.

Let $W^*=(V^*-\bigcup_{i=1}^m A_i)\cup(\bigcup_{j=1}^n B_j)$. For integers $-1<a<c<b<1$, let $L^*=V^*\times[a,c]\cup W^*\times[c,b]\subset S^3\times[a,b]\subset S^4$. Let $(L; V, W)$ be a triad with $L\subset S^4$, $V=L\cap p^{-1}(a)$, and $W=L\cap p^{-1}(b)$. If there is a homeomorphism φ: $S^4\to S^4$ such that (1) $p=p\circ\varphi$, and (2) $\varphi(L^*)=L$, then we call $(L; V, W)_a^b$ a *layer*. Simply $(L; V, W)_a^b$ will be denoted by L. For each $i=1,\cdots,m$, $\varphi(A_i)$ is called an *inside critical handle of index* $\alpha(i)$, and $\varphi(B_j)$ is called an *outside* critical handle of index $\beta(j)$ for each $j=1,\cdots,n$. The number c is called the *critical level* (of the layer).

Suppose that $(L_1; V_0, V_1)_{a(0)}^{a(1)}$, $(L_2; v_1, v_2)_{a(1)}^{a(2)}$, $\cdots$, $(L_k; V_{k-1}, V_k)_{a(k-1)}^{a(k)}$ are layers such that V_0 and V_k are disjoint unions of 3-balls. Then the union $\bigcup_{i=1}^k L_i$ is called a *stratum*. We call L_i the *ith layer* of the stratum.

We can show that every 4-manifold in S^4 is sent to a stratum by an ambient isotopy on S^4.

Let $\mathscr{S}$ be the set of strata. Let $\mathscr{N}$ be the subset of $\mathscr{S}$ such that the boundary of each stratum is connected, each stratum consists of four layers, and its critical handles of index i lie on the $i+1st$ layer for each $i=0,1,2,3$.

Two strata are said to be *equivalent*, if there exists an ambient isotopy on S^4 which sends one to the other. For each stratum N, we denote by $[N]$ the equivalent class of N. Then there is a natural projection of $\mathscr{S}$ onto the set of all equivalence classes. For each subset $\mathscr{X}$ of $\mathscr{S}$, we denote by $\mathscr{X}^*$ the image of $\mathscr{X}$ under the projection.

Let $\mathscr{C}$ be the subset of 2-dimensional polyhedra such that each element X possesses a cell subdivision $C(X)$ with only one 0-cell. For each $X \in \mathscr{C}$, the 0-cell of $C(X)$ is denoted by $P(X)$. Let $\mathscr{K}(X)=\overline{X-N(P(X); X)}$, and let $\mathscr{K}=\{\mathscr{K}(X)|X \in \mathscr{C}\}$. Let

$$\mathscr{F}=\{(f: K)|K \in \mathscr{K}, f: K \to S^3 \text{ is an embedding.}\}$$

Suppose that $K=\mathscr{K}(X)$. Then we denote X by $\mathscr{C}(K)$. And we denote by $\dot{K}$ the intersection $K \cap N(P(X); X)$.

Our main theorem is the following:

Theorem. *There exist two maps $\Phi: \mathscr{N}^* \to \mathscr{F}$ and $\Psi: \mathscr{F} \to \mathscr{N}^*$ with $\Psi \circ \Phi = 1$.*

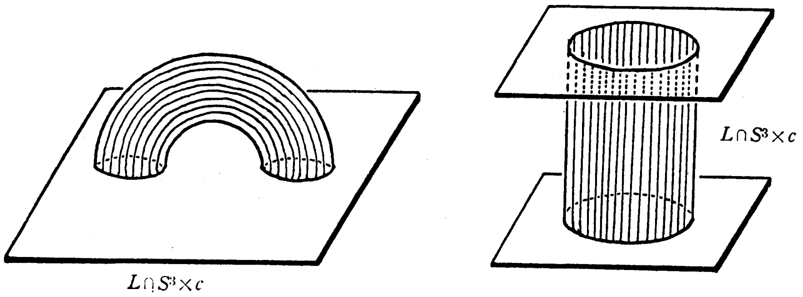
§ 3. Definition of Φ.

In this section, we define the map $\Phi: \mathscr{N}^* \to \mathscr{F}$.

Proposition 3.1. *Suppose $N \in \mathscr{N}$. Then N is equivalent to an element $N' \in \mathscr{N}$ such that N' possesses no critical handles of index* 0 *and* 3, *and that the first layer and the 4th layer are connected.*

Proof. Let $L_1=N \cap p^{-1}([a_0, a_1])$ and $L_2=N \cap p^{-1}([a_1, a_2])$ be the first and second layers, and let c_0 and c_1 be the critical levels of index 0 and 1 respectively. Let $W=\overline{\partial N \cap p^{-1}([-1, c_1))}$. If W is connected, then N possesses no critical handles of index 0 and the first layer is conected. Suppose that W is disconnected. Since ∂N is connected, there is a handle H of index 1 which connects two connected components of W. Suppose that the handle H is an outside handle. Then we may assume that $\pi(H) \times [a_0, c_1] \cap N = H \cup \pi(H \cap \partial W) \times [a_0, c_1]$, which is a 3-ball. Thus there is an ambient isotopy on S^4 which sends N to $N \cup \pi(H) \times [a_0, c_1]$, an element of $\mathscr{N}$. Suppose that the handle H is an inside handle. Then we may assume that $\pi(H) \times [c_0, c_1] \subset N$ and that $\pi(H) \times [c_0, c_1] \cap \partial N = H \cup \pi(H \cap \partial W) \times [c_0, c_1]$, which is a 3-ball. Thus there is an ambient isotopy on S^4 which sends N to $\overline{N-\pi(H) \times [c_0, c_1]}$, an element of $\mathscr{N}$. This process lessens the number of connected components of W. Thus there exists an element N'' in the class $[N]$ which possesses no critical handles of index 0. The above argument holds for the top layer. Hence the result follows. $\square$

Definition. An element N in $\mathscr{S}$ is said to be *admissible* if N possesses no



The fine line part is Annulus(H) for an outside critical handle.

The fine line part is Annulus(H) for an inside critical handle.

Fig. 3.1.

critical handles of index 0 and 3. If an element N in $\mathscr{N}$ is admissible, then N possesses a stratum consisting of two layers such that the first layer contains critical handles of index 1 and that the second layer contains critical handles of index 2.

Let L be a layer with a critical handle H of index i on the critical level c. Let $h: I^i\times I^{3-i}\to L$ be an embedding such that $\mathrm{Im}\, h=H$ and $\mathrm{Im}\, h\cap\overline{(\partial L\cap p^{-1}([-1, c)))}=h(\partial I^i\times I^{3-i})$. Then for any point $q\in \mathrm{Int}\, I^{3-i}$, we call $h(\partial I^i\times q)$ and $h(I^i\times q)$ a *left hand sphere* and a *left hand disk* respectively. Similarly, we call $h(r\times\partial I^{3-i})$ and $h(r\times I^{3-i})$ a *right hand sphere* and a *right hand disk* for each point $r\in\mathrm{Int}\, I^i$. We denote $h(\partial I^i\times I^{3-i})$ by Attach (H). For the case $i=1$, we denote $h(I^1\times\partial I^2)$ by Annulus (H) (see Fig. 3.1).

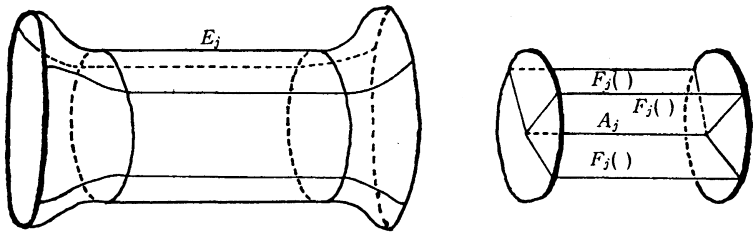
Now we define the map Φ as follows: For each class of $\mathscr{N}^*$, we specify an admissible element $N\in\mathscr{N}$ in the class. Then N is a stratum consisting of two layers such that the first layer L_1 contains critical handles of index 1 on the level c_1, and that the second layer L_2 contains critical handles of index 2 on the level c_2. Let $B_1, \cdots, B_m$ be the outside critical handles of index 2, and $E_1, \cdots, E_n$ the outside critical handles of index 1. We may assume that for each $i=1, \cdots, m$

$$\pi(B_i)\times[c_1, c_2]\cap N=B_i\cup(\pi(\mathrm{Attach}\ (B_i)))\times[c_1, c_2]\ ,$$

which is a 3-ball. We may further assume that for each $i=1, \cdots, m$, there exists a left hand sphere S_i of B_i such that for each $j=1, \cdots, n$

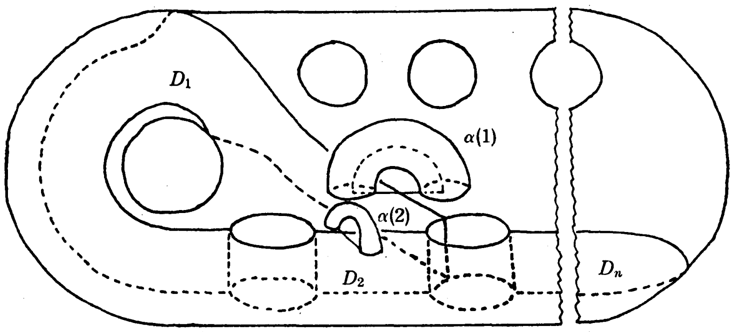
$$\pi(S_i)\times c_1\cap E_j=\pi(S_i)\times c_1\cap \mathrm{Annulus}\ (E_j)\ ,$$

which consists of proper arcs on Annulus (E_j). Let $N'=N\cup(\bigcup_{i=1}^{m}\pi(B_i)\times[c_1, c_2])$. Then there is an ambient isotopy on S^4 which sends N to N' on S^4. For each $i=1, \cdots, m$, let D_i be a left hand disk of B_i whose boundary is S_i, and let $\hat{D}_i=\pi(D_i)\times c_1$. Now



The fine lines are a part of the S_i's $\qquad$ $Z_j \cup \mathrm{Attach}(E_j)$

Fig. 3.2.



$\overline{W-\bigcup_{j=1}^{m} E_j}$

Fig. 3.3.

N' collapses to $N'\cap p^{-1}([-1, c_1])$. Note that

$$N'\cap p^{-1}([-1, c_1])=(N\cap p^{-1}([-1, c_1]))\cup(\bigcup_{i=1}^{m} \pi(B_i)\times c_1) .$$

Since B_i collapses to the union $D_i \cup \mathrm{Attach}\ (B_i)$, $N'\cap p^{-1}([-1, c_1])$ collapses to the union $(N\cap p^{-1}([-1, c_1]))\cup(\bigcup_{i=1}^{m} \hat{D}_i)$. We collapse as follows each outside handle E_j of index 1 to a set Z_j (see Fig. 3.2). Suppose that $(\bigcup_{i=1}^{m} \hat{D}_i)\cap \mathrm{Annulus}\ (E_j)$ consists of proper arcs $\alpha_j(1), \cdots, \alpha_j(t_j)$ on Annulus (E_j). Let A_j be a left hand disk of E_j which is an unknotted proper arc in E_j. Then there are 2-disks $F_j(1), \cdots, F_j(t_j)$ in E_j such that for each $k, h=1, \cdots, t_j$,

(1) the 2-disk $F_j(k)$ contains the arcs A_j and $\alpha_j(k)$,

(2) $F_j(k)\cap \mathrm{Int}\, E_j = A_j \cup \mathrm{Int}\, F_j(k)$, and

(3) $F_j(k)\cap F_j(h)=A_j$.

Let $Z_j=\bigcup_{k=1}^{t_j} F_j(k)$. Then E_j collapses to the union $Z_j\cup \mathrm{Attach}\ (E_j)$. Let $K=(\bigcup_{i=1}^{m} \hat{D}_i)\cup(\bigcup_{j=1}^{m} Z_j)$, $W=\overline{N\cap p^{-1}([-1, c_1))}$, and $\dot{K}=K\cap W$. Let $\mathscr{V}$ be a point in S^4 with $-1< p(\mathscr{V})<c_1$. Then $\mathscr{V}*\dot{K}\subset S^4$. Let $X=K\cup(\mathscr{V}*\dot{K})$. Since W is a 4-ball and $W\cap$

$p^{-1}(c_1)$ is a 3-ball, there is an ambient isotopy on S^4 which ships N' to a regular neighborhood of X in S^4. Now $K \subset S^3 \times c_1$ implies $\pi(K) \subset S^3$. This defines an element in $\mathscr{F}$ to be $\Phi([N])$.

§ 4. Definition of Ψ.

In this section, we show a construction method of a closed 3-manifold in S^4 starting from an element $(f\colon K)$ of $\mathscr{F}$. Specially if $\mathscr{C}(K)$ is aspherical, then the constructed closed 3-manifold is a homology 3-sphere. We use the idea of neutralizer in [Cr].

A handle body in S^3 is said to be *unknotted* if the closure of its complementary domain is also a handle body.

Let $(f\colon K)$ be an element of $\mathscr{F}$. Let C be a cell subdivision of $\mathscr{C}(K)$ with only one 0-cell. Let $\tilde{S}_1, \cdots, \tilde{S}_m$ be the 1-cells in C, and $\tilde{D}_1, \cdots, \tilde{D}_n$ the 2-cells in C. For each $i=1, \cdots, m$, $j=1, \cdots, n$, let $S_i=f(\tilde{S}_i \cap K)$ and $D_j=f(\tilde{D}_j \cap K)$. Then $f(K)-\bigcup_{i=1}^{m} S_i$ consists of n connected components, and each component is contained in one of the D_j's. We will define the element $\Psi(f\colon K)$ in $\mathscr{N}$ in three steps.

Step 1. In this step, we will construct an unknotted handle body W in S^3 such that (1) $f(K) \subset W$, (2) $f(K) \cap \partial W = f(\dot{K})$, and (3) there is a complete system of meridinal disks $A_1, \cdots, A_g$ of W which miss the S_i's, where "complete" means that $W-\bigcup_{k=1}^{g} N(A_k; W)$ is a 3-ball.

For each $i=1, \cdots, m$, $j=1, \cdots, n$, let s_i be an interior point of S_i, and d_j be an interior point of D_j. Let $Q=\{(i, j) | S_i \subset D_j\}$. Let $\{J(i, j) | (i, j) \in Q\}$ be the set of simple arcs in $f(K)$ such that (1) $J(i, j) \subset D_j$, (2) $J(i, j) \cap J(i', j) = d_j$ $(i \neq i')$, and (3) $J(i, j) \cap \partial D_j = s_i$ (see Fig. 4.1). Let $J = \cup \{J(i, j) | (i, j) \in Q\} \cup (\bigcup_{k=1}^{m} s_k) \cup (\bigcup_{h=1}^{n} d_h)$. Then K

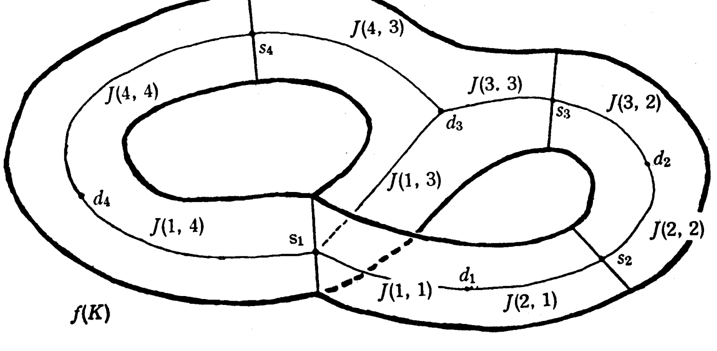


Fig. 4.1.

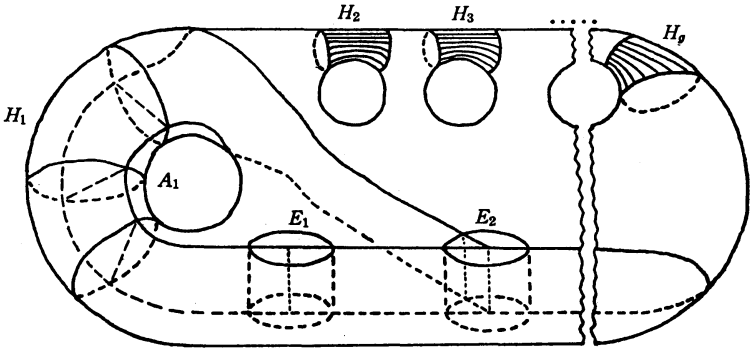


Fig. 4.2.

collapses to J. Since J is a 1-dimensional polyhedron in S^3, there exists a 1-dimensional polyhedron P in S^3 such that (1) $J \subset P$ and (2) a regular neighborhood W^* of J in S^3 is an unknotted handle body. We may assume that W^* does not contain $f(\dot{K})$. Then there are proper disks $A_1^*, \cdots, A_g^*$ in W^* such that (1) $W^* - \bigcup_{k=1}^{g} N(A_k^*; W^*)$ is a 3-ball, (2) for each $h=1, \cdots, g$, $A_h^* \cap J$ consists of at most one point at which J pierces A_h^*, and (3) A_h^* misses $\bigcup_{k=1}^{m} S_k$. Now expand W^* along the D_j's to get a desired handle body W. The proper disks $A_1^*, \cdots, A_g^*$ assure the existence of a complete system of meridinal disks $A_1, \cdots, A_g$ of W.

Step 2. In this step, we construct the first layer of the stratum $\Psi(f\colon K)$ (see Fig. 4.2, 4.3).

For each $h=1, \cdots, g$, let H_h be a regular neighborhood of A_h in W which misses $\bigcup_{k=1}^{m} S(K)$. For each $i=1, \cdots, m$, let E_j be regular neighborhood of $S(i)$ in W which misses $\bigcup_{h=1}^{g} H_h$. For each $j=1, \cdots, n$, let $F_j = \overline{D - \bigcup_{i=1}^{n} E_i}$. We may assume that each F_j is a proper disk in $\overline{W - \bigcup_{i=1}^{m} E_i}$ (see Fig. 4.2). Let $\Delta(1), \cdots, \Delta(n)$ be mutually disjoint disk in S^3 such that (1) each disk misses $(\bigcup_{i=1}^{m} E_i) \cup (\bigcup_{h=1}^{g} H_h) \cup \operatorname{Int} W$, and (2) for each $t=1, \cdots, n$, $\Delta(t) \cap \partial W$ is an arc in $\partial\Delta(t)$ and $\Delta(t) \cap (\bigcup_{j=1}^{n} \partial F_j) = \Delta(t) \cap \partial F_t =$ a crossing point (see Fig. 4.3). For each $t=1, \cdots, n$, let $\alpha(t)$ be a regular neighborhood of $\overline{(\partial\Delta(t) - W)}$ in $\overline{S^3 - W}$ which misses $(\bigcup_{i=1}^{m} E_i) \cup (\bigcup_{h=1}^{g} H_h)$. These $\alpha(t)$'s correspond to neutralizers in [Cr]. Let

$$V_0^* = \overline{S^3 - W} \cup \left(\bigcup_{h=1}^{g} H_h\right), \quad \text{and}$$

$$V_1^* = \overline{S^3 - \left(W \cup \bigcup_{t=1}^{n} \alpha(t)\right)} \cup \left(\bigcup_{i=1}^{m} E_i\right).$$

Let $L_1 = V_0^* \times [-(1/2), -(1/4)] \cup V_1^* \times [-(1/4), 0]$. Then L_1 is a layer which possesses

$(g+n)$ inside critical handles and m outside critical handles of index 1 on the level $-(1/4)$.

Step 3. In this step, we construct the second layer.

Let $W'=W-\bigcup_{i=1}^{m} E_i$ and $W''=S^3-\operatorname{Int} V_1^*$. Then W' and W'' are unknotted handle bodies in S^3. Note that $W''=W'\cup(\bigcup_{t=1}^{n}\alpha(t))$. For each $j=1,\cdots,n$, let U_j be a regular neighborhood of F_j in W''. Thanks to the neutralizers, there is an ambient isotopy on S^3 which sends $W''-\bigcup_{j=1}^{n} U_j$ to W'. Hence there exist mutually disjoint 3-balls $G_1,\cdots,G_{g+m}$ in W'' such that

(1) each 3-ball meets $\partial W''$ by an annulus,

(2) each 3-ball misses $\bigcup_{j=1}^{n} U_j$, and

(3) $\overline{V_1^*-\bigcup_{r=1}^{g+m} G_r}\cup(\bigcup_{j=1}^{n} U_j)$ is a 3-ball.

Let $V_2^*=\overline{V_1^*-\bigcup_{r=1}^{g+m} G_r}\cup(\bigcup_{j=1}^{n} U_j)$. Let $L_2=V_1^*\times[0,1/4]\cup V_2^*\times[1/4,1/2]$. Then L_2 is a layer which possesses $(g+m)$ inside critical handles and n outside critical handles of index 2 on the level 1/4. We define $\Psi(f\colon K)$ to be the equivalence class of the stratum $L_1\cup L_2$. The definition of $\Psi(f\colon K)$ is well-defined in the following sense: Let Y be the cone on $f(\dot{K})\times(-(1/4))$ with a vertex on $S^3\times(-(1/2))$. Then the stratum $L_1\cup L_2$ has a spine which is isotopic to $Y\cup(f(\dot{K})\times(-(1/4)))$ by an ambient isotopy on S^4.

5. Corollaries.

Corollary 1. *Suppose that $(f\colon K)$ is an element of $\mathscr{F}$ with $\mathscr{C}(K)$ being aspherical. Let N be a stratum in the class $\Psi(f\colon K)$. Then ∂N is a homology 3-sphere.*

Proof. Since N is of the same homotopy type to $\mathscr{C}(K)$, N is aspherical. Hence $S^4-\operatorname{Int} N$ is also aspherical by Alexander duality theorem. Therefore, ∂N is a homology 3-sphere by the Mayer-Vietoris sequence of N and $S^4-\operatorname{Int} N$. $\square$

Corollary 2. *Suppose that N is an admissible stratum whose boundary is a homology 3-sphere. Then the number of outside critical handles of index 1 is equal to the number of outside critical handles of index 2. The same statement holds for inside handles.*

Proof. Let $\Phi([N])=(f\colon K)$. By the definition of Φ, there exists a cell subdivision of $\mathscr{C}(K)$ whose 1-cells correspond to outside critical handles of index 1 and whose 2-cells correspond to outside critical handles of index 2. Since N is homotopy equivalent to $\mathscr{C}(K)$, then $\mathscr{C}(K)$ is aspherical. Since $\mathscr{C}(K)$ possesses only one 0-cell, the first statement holds. For the second statement, let $N^*=\overline{p^{-1}p(N)-N}$. Then there is an ambient isotopy on S^4 which sends $S^4-\mathrm{Int}\,N$ to N^*. Each inside critical handle of N is now an outside critical handle of the stratum N^*. Since N^* is admissible, the first statement implies the second statement. □

Corollary 3. *If N is a connected stratum without outside critical handles, then N is a 4-ball. The same statement holds for inside.*

Proof. Since each layer does not have any outside critical handles, each layer collapses from the top to the bottom. Hence N collapses from the top to the bottom. Since N is connected, the bottom must be a 3-ball. Hence N is collapsible. Hence the first statement holds. Considering $\overline{p^{-1}p(N)-N}$, the first statement implies the second statement. □

For each element $X\in\mathscr{C}$, a cell subdivision C of X is said to be *regular*, provided that (1) C possesses only one 0-cell $P(X)$, (2) each 1-cell is contained in at most three 2-cells, (3) each 2-cell contains at most three 1-cells and (4) each 2-cell D possesses a characteristic map ϕ: $(I^2, \partial I^2)\to(D, \dot{D})$ with $\{x\in\partial I^2|\phi^{-1}\phi(x)\neq\{x\}\}=\phi^{-1}(P(X))$.

Let $(f: K)$ and $(f': K')$ be elements in $\mathscr{F}$. We say that $(f: K)$ and $(f': K')$ are *equivalent* if $\Psi(f: K)=\Psi(f': K')$. Then by an argument similar to the one in Lemma 2.2 of [Wd], we can show that each element $(f: K)$ in $\mathscr{F}$ is equivalent to an element $(f': K')$ such that $\mathscr{C}(K')$ possesses a regular cell subdivision. For each regular cell subdivision C, we define as follows two kinds of complexity of C:

Complexity $(1:C)$=the number of 1-cells contained in three 2-cells

Complexity $(2:C)$=the number of 2-cells containing three 1-cells.

Corollary 4. *Let $(f: K)$ be an element in $\mathscr{F}$ with $\mathscr{C}(K)$ being aspherical. Suppose that $\mathscr{C}(K)$ possesses a regular cell subdivision C with Complexity $(1; C)=0$ or Complexity $(2; C)=0$. Then $\Psi(f: K)$ is the class containing a 4-ball.*

Proof. Suppose that $K_1, \cdots, K_n$ are the connected components of K. For each $j=1, \cdots, n$, let $f_j=f|K_j$ and $N_j\in\Psi(f_j: K_j)$. Then $\Psi(f: K)$ is the class of a stratum homeomorphic to the boundary connected sum $N_1\# N_2\#\cdots\# N_n$. Hence we may assume that K is connected.

Note that $H_i(K, \dot{K})=0$ for all $k=0, 1, 2, \cdots$.

Suppose that Complexity $(1; C)=0$. Then K is a surface. Since $\mathscr{C}(K)$ is aspherical, $\dot{K}$ is connected by the Mayer-Vietoris sequence. Since each component of ∂K meets $\dot{K}$, ∂K is connected. Furthermore, $H_1(K, \dot{K})=0$ implies that K is of genus 0. Hence K is a disk or a Möbius band. Now asphericity of $\mathscr{C}(K)$ implies that $\partial K \neq \dot{K}$. Hence K must be a disk. Thus $\partial K \neq \dot{K}$ and the connectness of $\dot{K}$ imply that $\mathscr{C}(K)$ is a disk. Since a stratum in $\Psi(f: K)$ has a spine homeomorphic to $\mathscr{C}(K)$, $\Psi(f: K)$ is the class containing a 4-ball.

Suppose that Complexity $(2; C)=0$. Let C' be a subcomplex of C such that C cellwisely collapses to C' and that C' possesses no free face. We claim that C' possesses no 2-cell. Suppose that C' possesses a 2-cell. Then C' satisfies the following property $(*)$:

$(*)$ there is only one 0-cell, there is at least one 2-cell, there is no free face, and each 2-cell possesses a characteristic map whose preimage of the 0-cell consists of two points.

Let D be a 2-cell in C' containing two 1-cells S_1 and S_2. Let C'' be the cell complex obtained from $C'-D$ by identifying the two 1-cells S_1 and S_2. Then C' and C'' are of the same homotopy type. Hence C'' is aspherical. Thus C'' satisfies the property $(*)$. If C'' contains a 2-cell containing two 1-cells, we can repeat the above process to get an aspherical cell complex which satisfies the property $(*)$ and possesses fewer 2-cells. Thus we get an aspherical cell complex C^* which satisfies the property $(*)$ and contains no 2-cell containing two 1-cells. Since C^* is aspherical, C^* contains a 1-cell S. By the property $(*)$, each 2-cell contains at most one 1-cell. Hence each 2-cell containing S is a 2-sphere or a projective plane. This is impossible since C^* is aspherical. Therefore, C' possesses no 2-cell. Since C' is aspherical, C' possesses no 1-cell, either. Hence C' consists of only the 0-cell. This means that C is collapsible. Therefore, $\Psi(f: K)$ is the class containing a 4-ball. $\square$

§ 6. Two examples.

In [Mz], Mazur constructed a 4-manifold M such that M collapses to the Dance hat $\mathscr{D}$, ∂M is not S^3, and the double of M is S^4. In our word, he constructed a stratum M such that M collapses to $\mathscr{D}$ and that M is not a 4-ball. Since $\mathscr{D} \in \mathscr{C}$, we can construct Mazur stratum from embeddings of $\mathscr{K}(\mathscr{D})$ to S^3. In example 1, we give a Mazur stratum in $\mathscr{N}$ from the embedding of $\mathscr{K}(\mathscr{D})$ to S^3 as in Fig. 6.1. By Corollary 2 and 3, any contractible stratum with at most one critical handle of index 1 is a 4-ball. In example 2, we give a contractible stratum N in $\mathscr{N}$ such that N possesses only two critical handles of index 1 and that N is not a 4-ball.

Example 1. The Dance hat possesses a cell subdivision consisting of one 0-cell, one 1-cell $\tilde{S}$, and one 2-cell $\tilde{D}$. Let f be the embedding of $\mathscr{K}(\mathscr{D})$ to S^3 as in Fig. 6.1. Let $S=f(\tilde{S}\cap\mathscr{K}(\mathscr{D}))$ and $D=f(\hat{D}\cap\mathscr{K}(\mathscr{D}))$. Let W be an unknotted handle body of genus 2 in S^3 as in Fig. 6.2. Let H_1 and H_2 be disjoint 3-balls in W such that $\overline{W-(H_1\cup H_2)}$ is a 3-ball and that each 3-ball misses the proper arc S. Let H_3 be a regular neighborhood of S in W such that $D\cap\overline{(W-H_3)}$ is a proper 2-disk and that H_3 misses H_1 and H_2. Let $V_0=(S^3-\mathrm{Int}\,W)\cup H_1\cup H_2$ and $V_1=(S^3-\mathrm{Int}\,W)\cup H_3$. Let $L_1=V_0\times[-(1/2),-(1/4)]\cup V_1\times[-(1/4),0]$. Then L_1 is a layer which possesses an outside critical handle and two inside critical handles of index 1 on the level $-(1/4)$. Let a, b, c, x, y, z be oriented simple closed curves on ∂V_1 as in Fig. 6.3 which represent a generated system of the fundamental group of ∂V_1. Let e_1 and e_2 be disjoint simple closed curves on ∂V_1, missing $\partial(D\cap V_1)$, represented by

$$r_1=byb^{-1}x^{-1}a^{-1}xby^{-1}b^{-1}z^{-1}c^{-1}c^{-1}y^{-1}czbyb^{-1}x^{-1}axby^{-1}b^{-1}z^{-1}c^{-1}ycz\,,$$
$$r_2=byb^{-1}x^{-1}abyb^{-1}a^{-1}z^{-1}c^{-1}y^{-1}cz \qquad \text{(see Fig. 6.4).}$$

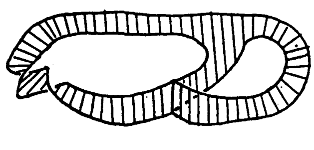


Fig. 6.1.

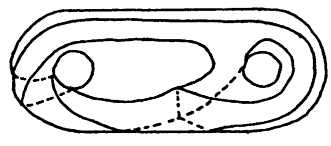


Fig. 6.2.

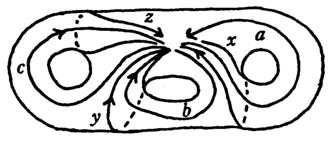


Fig. 6.3.

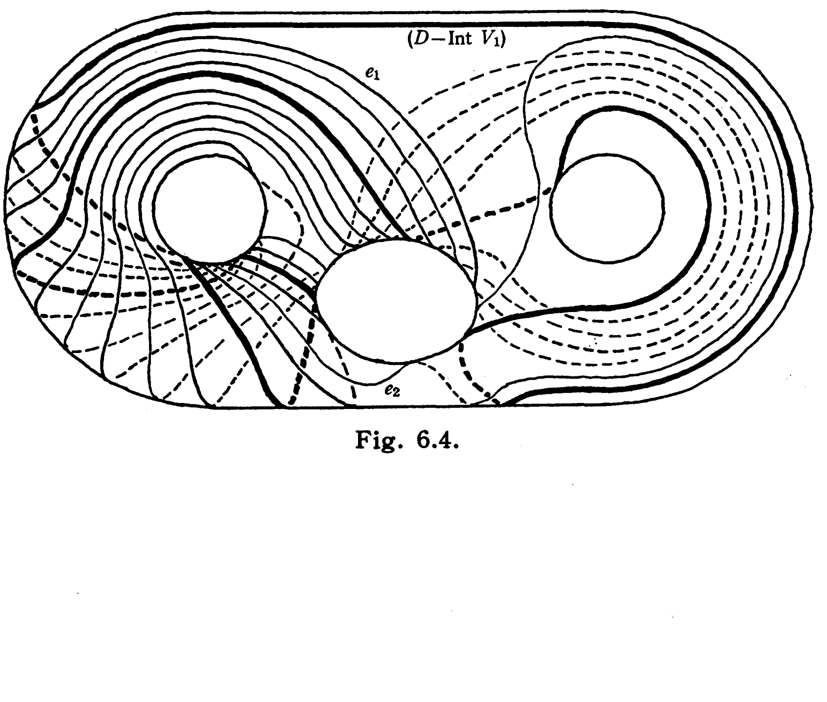


Fig. 6.4.

Then e_1 and e_2 are homotopic to zero in V_1. It follows from Dehn's lemma and the irreducibility of the handle body V_1 that the curves e_1 and e_2 bound disjoint disks B_1 and B_2 in V_1. Let J_1 be a regular neighborhood of the union $B_1 \cup B_2$ in V_1, missing the disk $D-\mathrm{Int}\, V_1$. Let J_2 be a regular neighborhood of the disk $D-\mathrm{Int}\, V_1$ in $(S^3-\mathrm{Int}\, V_1)$, missing J_1. Let $V_2=\overline{(V_1-J_1)}\cup J_2$. Then V_2 is a 3-ball. Let $L_2=V_1\times[0, 1/4]\cup V_2\times[1/4, 1/2]$. Then $L_1\cup L_2$ is a stratum in $\mathscr{N}$ which collapses to the Dance hat. Since $\partial(D-\mathrm{Int}\, V_1)$ is represented by

$$r_3=byb^{-1}x^{-1}a^{-1}xby^{-1}b^{-1}c^{-1}y^{-1}czbyb^{-1}x^{-1}axby^{-1}b^{-1}z^{-1}c^{-1}yb^{-1}a^{-1}$$

$\pi_1(\partial(L_1\cup L_2))$ has the following group presentation G.

$$\begin{aligned}
G&=\{a, b, c, x, y, z: b, x, z, r_1, r_2, r_3\}\\
&=\{a, c, y: r_1', r_2', r_3'\}, \quad \text{where } r_1'=ya^{-1}y^{-1}c^{-1}y^{-1}cyay^{-1}c^{-1}ya^{-1},\\
&\qquad r_2'=ya^{-1}y^{-1}c^{-1}y^{-1}cyay^{-1}c^{-1}yc, \quad r_3'=yaya^{-1}c^{-1}y^{-1}c\\
&=\{a, c, y: r_1'', r_2'', r_3'\}, \quad \text{where } r_1''=ya^{-1}y^{-1}ay^{-1}y^{-1}c^{-1}ya^{-1},\\
&\qquad r_2''=ya^{-1}y^{-1}c^{-1}ay^{-1}y^{-1}c^{-1}y^{-1}c\\
&=\{a, c, y: ya^{-1}y^{-1}ay^{-1}y^{-1}c^{-1}ya^{-1}, ya^{-1}y^{-1}c^{-1}ay^{-1}aya^{-1}, r_3'\}\\
&=\{a, y: r_4, r_5\}, \quad \text{where } r_4=ya^{-1}ya^{-1}yay^{-1}ay^{-1}ay^{-1}aya^{-1},\\
&\qquad r_5=aya^{-1}yya^{-1}yay^{-1}ay^{-1}a^{-1}ya^{-1}y^{-1}ay^{-1}.
\end{aligned}$$

Replacing ay^{-1} by v, the above group presentation becomes:

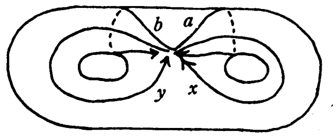


Fig. 6.5.

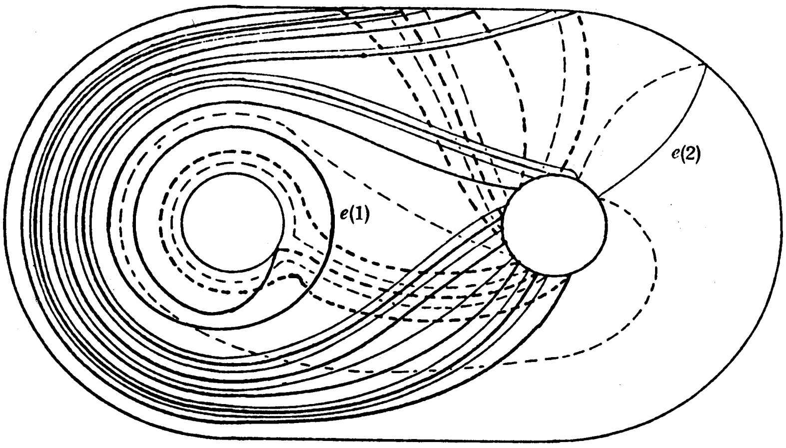


Fig. 6.6.

$$(v, y: v^{-1}v^{-1}yvvvvyv^{-1}, vyv^{-1}yv^{-1}yvvy^{-1}v^{-1}v^{-1}y^{-1}v)$$
$$=(v, y: v^{-1}v^{-1}yvvvvyv^{-1}, vyv^{-1}yv^{-1}yvvy^{-1}v)$$

Replacing yv^{-1} by w, the above group presention becomes:

$$(v, w: v^{-1}v^{-1}wvvvvvvw, vwwwwwvvw^{-1}v)=(v, w: v^{-2}wv^5w, w^5v^7) .$$

Let S_7 be the symmetric group of order 7. Let φ be the homomorphism of G to S_7 defined by $\varphi(v)=(1526374)$ and $\varphi(w)=(34672)$. Then $\varphi(G)$ is a non-trivial subgroup of S_7. Hence G is not a trivial group. Therefore, the stratum $L_1 \cup L_2$ is not a 4-ball.

Example 2. Let V_1 be an unknotted handle body in S^3 with genus 2. Let a, b, x, y be oriented simple closed curves on ∂V_1 as in Fig. 6.5. Let D_1 be a proper disk in V_1 whose boundary is homotopic to b on ∂V_1. Let D_2 be a proper disk in $\overline{S^3-V_1}$ whose boundary is homotopic to x on ∂V_1. Let B_1 be a regular neighborhood of D_1 in V_1, and B_2 a regular neighborhood of D_2 in $\overline{S^3-V_1}$ with $B_1 \cap B_2=\emptyset$. Let $V_0=\overline{V_1-B_1} \cup B_2$ and $L(1)=V_0\times[-(1/2), -(1/4)] \cup V_1\times[-(1/4), 0]$. Then $L(1)$ is a layer. Let $e(1)$ and $e(2)$ be simple closed curves on ∂V_1, represented by

$$r(1)=ya^{-1}ya^{-1}by^{-1}b^{-1}ay^{-1}ayyb^{-1}ay^{-1}ay^{-1}a^{-1}$$
$$r(2)=ya^{-1}ya^{-1}by^{-1}b^{-1}ay^{-1}ay^{-1}a^{-1}ya^{-1}yx^{-1}aa \qquad \text{(see Fig. 6.6).}$$

Since $e(1)$ is homotopic to zero in V_1, then $e(1)$ bounds a proper disk E_1 in V_1. Since $e(2)$ is homotopic to zero in $\overline{S^3-V_1}$, then $e(2)$ bounds a proper disk E_2 in $\overline{S^3-V_1}$. Let J_1 be a regular neighborhood of E_1 in V_1, and J_2 a regular neighborhood of E_2 in $\overline{S^3-V_1}$ with $J_1 \cap J_2=\emptyset$. Let $V_2=\overline{V_1-E_1} \cup E_2$. Then V_2 is a 3-ball. Let $L(2)=V_1\times[0, 1/4] \cup V_2\times[1/4, 1/2]$. Then $L(1) \cup L(2)$ is a stratum in $\mathscr{N}$ which possesses one outside critical handle of index 1, and one inside critical handle of index 1, and whose boundary is a homology 3-sphere. Let M be the boundary of the stratum. Then $\pi_1(M)$ has the following group presentation G.

$$G=\{a, b, x, y: b, x, r(1), r(2)\}$$
$$=\{a, y: ya^{-1}ya^{-1}y^{-1}ay^{-1}ayyay^{-1}ay^{-1}a^{-1}, ya^{-1}ya^{-1}y^{-1}ay^{-1}ay^{-1}a^{-1}ya^{-1}yaa\}$$

Replacing ya^{-1} by v, the above group presentation becomes:

$$(a, v: vva^{-1}v^{-1}v^{-1}avavav^{-1}v^{-1}a^{-1}, vva^{-1}v^{-1}v^{-1}v^{-1}a^{-1}vvaaa) .$$

Let S_5 be the symmetric group of order 5. Let ψ be a homomorphism of G to S_5 defined by $\psi(a)=(35241)$ and $\psi(v)=(241)$. Then $\psi(G)$ is a non-trivial subgroup of S_5. Hence G is not a trivial group. This means that M is not the 3-sphere. Since the stratum is simply connected and aspherical, the stratum is contractible.

References

[Cr] R. Craggs: *Stable representations for 3- and 4-manifolds*, (mimeographed manuscript).

[Hd] J. F. P. Hudson: *Piecewise Linear Topology* (Mathematics Lecture Note Series) New York: W. A. Benjamin, Inc. 1969.

[Ml] J. Milnor: *Lectures on h-cobordism theory*, Preliminary Informal Notes of University Courses and Seminars in Mathematics, Princeton Mathematical Notes.

[Mz] B. Mazur: *A note on some contractible 4-manifolds*, Ann. Math. **73**, 221–228 (1969).

[Zm] E. C. Zeeman: *Seminor on combinatorial topology*, Publ. Hautes Sci. Paris, 1963 (mimeographed note).

Department of Information Science
Tokyo Institute of Technology
Ookayama, Meguro-ku
Tokyo Japan

1–8–6 Hongo, Seya-ku
Yokohama Japan