YOKOHAMA MATHEMATICAL
JOURNAL VOL. 27, 1979

ON THE ASYMPTOTIC TAIL BEHAVIORS OF INFINITELY DIVISIBLE DISTRIBUTIONS

By

HIKARU OHKUBO

(Received April 21, 1979)

SUMMARY. We investigate the relationship between the asymptotic tail behavior of an infinitely divisible distribution function and the Lévy spectral function in the Lévy canonical representation of the distribution.

1. Introduction and theorems.

Let $F(x)$ be an infinitely divisible ditribution function and $f(t)$ be its characteristic function with the Lévy canonical representation

$$\log f(t)=i\gamma t-\sigma^2 t^2/2+\int_{|x|>0}(e^{itx}-1-itx(1+x^2)^{-1})dM(x)\ , \tag{1.1}$$

where γ is real, $\sigma^2\geqq 0$ and $M(x)$ is a nondecreasing function over each of $(-\infty, 0)$ and $(0, \infty)$ vanishing at $\mp\infty$ with $\int_{0<|x|<1} x^2 dM(x)<\infty$.

Chatterjee and Pakshirajan (1956) obtained the following result on the tail $T(x)=1-F(x)+F(-x)$, $x>0$:

Theorem A. *$T(x)=0$ for all large $x>0$ if and only if $M(x)=0$ for all $x\neq 0$ and $\sigma^2=0$.*

Dealing with the one-sided tail $F(-x)$, $x>0$, Baxter and Shapiro (1960) proved

Theorem B. *$F(-x)=0$ for all large $x>0$ if and only if $M(-x)=0$ for all $x>0$, $\sigma^2=0$ and $\int_{0+}^{1} xdM(x)<\infty$.*

In addition to this, Tucker (1961), Esseen (1965) and Ramachandran (1966) showed that $\mathrm{l.ext.}\,F=\inf\{x\colon F(x)>0\}$ is given by $\mathrm{l.ext.}\,F=\gamma-\int_{0+}^{\infty} x(1+x^2)^{-1}dM(x)$.

On the other hand, the asymptotic conditions on the tail $T(x)$ for $F(x)$ to be normal or degenerate were investigated by Ruegg (1970), Horn (1972) and Steutel (1974). Actually, Steutel gave the most general result:

Theorem C.

(i) $\lim_{x\to\infty} -x^{-2}\log T(x)=\infty$ *if and only if $M(x)=0$ for all $x\neq 0$ and $\sigma^2=0$.*

(ii) $\lim_{x\to\infty} -(x\log x)^{-1}\log T(x)=\infty$ *if and only if* $M(x)=0$ *for all* $x\neq 0$.

Ruegg (1971) and Horn (1972) also gave some results containing the one-sided tail.

From these results, it naturally arises the problem: What explicit relationships exist between the Lévy spectral function $M(x)$, σ^2 and the asymptotic behavior of the one-sided tail $F(-x)$ when $x\to\infty$? The purpose of this paper is to prove the following theorems, answering this question.

Throughout this paper, we make the convention that $-\log 0=\infty$, $0^{-1}=\infty$ and $\infty^{-1}=0$.

Theorem 1. *If* $M(-x)=0$ *for all* $x>0$ *and* $\sigma^2=0$, *then*

$$\liminf_{x\to\infty} -x^{-1-a}\log F(-x)\geqq k_a \liminf_{x\to 0+} x^{-1-a}|M(x)|^{-a}$$

and

$$\liminf_{x\to\infty} -x^{-1-a}\log F(-x)\leqq h_a \liminf_{x\to 0+} x^{-1-a}|M(x)|^{-a},$$

where $1<a<\infty$, *and* $k_a>0$ *and* $h_a>0$ *are constants which are independent of* $F(x)$ *and* $M(x)$. *In addition to the assumptions if* $\lim_{x\to 0+} x^{1+(1/a)}|M(x)|$ *exists, then*

$$\liminf_{x\to\infty} -x^{-1-a}\log F(-x)=k_a \lim_{x\to 0+} x^{-1-a}|M(x)|^{-a}.$$

Theorem 2.

(i) $\lim_{x\to\infty} -x^{-2}\log F(-x)=(2\sigma^2)^{-1}$, *provided that* $M(-x)=0$ *for all* $x>0$.

(ii) $\lim_{x\to\infty} -(x\log x)^{-1}\log F(-x)=(-\text{l.ext.}\,M)^{-1}$, *where* l.ext. M *is defined by* l.ext. $M=\inf\,[\{x<0\colon M(x)>0\}\cup\{0\}]$.

Theorem 3. *If* $M(-x)>0$ *for all* $x>0$, *then*

(i) $\liminf_{x\to\infty} -x^{-1}(\log x)^{-a}\log F(-x)=c_a \liminf_{x\to\infty} x^{-1}[-\log M(-x)]^{1-a}$, *where* $0<a<1$, *and* $c_a>0$ *is a constant which is independent of* $F(x)$ *and* $M(x)$.

(ii) $\liminf_{x\to\infty} -(xs(\log x))^{-1}\log F(-x)=\liminf_{x\to\infty} -(xs(x))^{-1}\log M(-x)$, *where* $s(x)$ *is any strictly increasing continuous function defined over* (A,∞) *for some* $A>0$ *with* $s(\infty)=\infty$ *such that* $(\log\log x)^{-1}\log s(x)$ *is nonincreasing.*

(iii) $\liminf_{x\to\infty} -(u(x))^{-1}\log F(-x)=\liminf_{x\to\infty} -(u(x))^{-1}\log M(-x)$ *where* $u(x)$ *is any strictly increasing continuous function defined over* (A,∞) *for some* $A>0$ *with* $u(\infty)=\infty$ *such that* $x^{-1}u(x)$ *is nonincreasing.*

As a consequence of Theorem 2, we have

Corollary. *The following three conditions are equivalent:*

(i) $\lim_{x\to\infty} -x^{-2}\log F(-x)>0$

(ii) $\lim_{x\to\infty} -(x \log x)^{-1} \log F(-x)=\infty$

(iii) $M(-x)=0$ *for all* $x>0$.

Moreover the following two conditions are equivalent:

(iv) $\lim_{x\to\infty} -x^{-2} \log F(-x)=\infty$

(v) $M(-x)=0$ *for all* $x>0$ *and* $\sigma^2=0$.

Since the duals of our results are true, Theorems 2, 3 and Corollary remain true if $F(-x)$, $M(-x)$ and $-\mathrm{l.ext.}\,M$ are replaced by $T(x)=1-F(x)+F(-x)$, $L(x)=|M(x)|+M(-x)$ and $\max\{-\mathrm{l.ext.}\,M, \mathrm{r.ext.}\,M\}$, respectively. (The notation r.ext. M will be clear.) For, the equations

$$\liminf_{x\to\infty} -v(x) \log T(x)=\min\{\liminf_{x\to\infty} -v(x) \log F(-x), \liminf_{x\to\infty} -v(x) \log(1-F(x))\}$$

and the similar equations for $L(x)$, $M(-x)$ and $|M(x)|$ hold, where $v(x)=x^{-2}$, $(x \log x)^{-1}, \cdots$, etc. Hence Corollary is thought of as a generalization of Theorem C. Theorem 2 (ii) and Theorem 3 (i) are generalizations and simplifications of Ruegg (1971, Remark 3), but Theorem 2 (ii) is reduced to the one-dimensional case of the result due to Sato (1973, Theorem 3). However our proof is different from his. Theorem 3 (iii) is close to Wolfe's (1971, Theorem 4). Corollary and the dual of it imply that the complicated conditions in all theorems of Horn (1972) can be dropped.

Theorems 1, 2 and 3 will be proved in Section 4. Their proofs except of Theorem 3 (iii) are based on the theory of Fourier-Stieltjes transform analytic in the upper half-plane which will be discussed in Section 2.

2. Fourier-Stieltjes transform analytic in the upper half-plane.

The contents of this section relates deeply to the work of Ramachandran (1962). His results play important roles in the proofs of our theorems.

Throughout this section, we suppose that $G(x)$ is a bounded nondecreasing function defined on $(-\infty, \infty)$ with $G(-\infty)=0$ and write $g(z)=\int_{-\infty}^{\infty} e^{izx} dG(x)$ for some complex number z, when it exists.

We begin with the following Lemma 1. The special case with $r(x)=\log x$ and $r(x)=x$ are owing to Ramachandran (1962, Theorem 3.1 and Theorem 4.1).

Lemma 1. *Let* $0\leqq p\leqq\infty$ *and* $r(x)$ *be a strictly increasing continuous function defined over* $[0, \infty)$ *with* $r(\infty)=\infty$. *Then*

$$\int_{-\infty}^{0} \exp(yr(-x))dG(x)<\infty \quad (0\leqq y<p), \tag{2.1}$$

$$\int_{-\infty}^{0} \exp(yr(-x))dG(x)=\infty \quad (y>p)$$

if and only if

$$\liminf_{x\to\infty} -(r(x))^{-1}\log G(-x)=p\,.$$

When $p=0$, the first relation of (2.1) *is dropped out and when $p=\infty$, the second relation is neglected.*

Proof. It suffices to prove the "if" part. Suppose $p>0$. For any $0<y<p$, there exists a finite number q such that $y<q<p$. By the assumption, we obtain $G(-x)\leqq\exp(-qr(x))$ for all $x>A>0$. Using integration by parts, we have that

$$\int_{-x}^{0} \exp(yr(-u))dG(u)=[G(-u)\exp(yr(u))]_x^A+\int_A^x G(-u)d(\exp(yr(u))$$
$$+\int_{-A}^{0} \exp(yr(-u))dG(u)$$

for all $x>A$ in which

$$G(-x)\exp(yr(x))\leqq\exp((y-q)r(x))\to 0 \quad \text{as} \quad x\to\infty$$

and

$$\int_A^x G(-u)d(\exp(yr(u))\leqq\int_A^x \exp(-qr(u))d(\exp(yr(u))$$
$$\leqq\int_{\exp(yr(A))}^{\infty} v^{-q/y}dv<\infty\,.$$

The above shows the first relation of (2.1) to be true. In order to prove the second relation of (2.1) we may suppose $p<\infty$. If it were true that $\int_{-\infty}^{0}\exp(qr(-u))dG(u)<\infty$ for some $q>p$, there should exist a constant $K>0$ such that $K>\int_{-\infty}^{-x}\exp(qr(-u))dG(u)\geqq \exp(qr(x))G(-x)$ for all large $x>0$. We then have

$$\liminf_{x\to\infty} -(r(x))^{-1}\log G(-x)\geqq q>p\,.$$

This contradicts the assumption.

From Lemma 1, we immediately have well known result:

(2.2) $\quad g(iy)$ exists for $y>0$ if and only if $\lim_{x\to\infty} -x^{-1}\log G(-x)=\infty$.

When $g(iy)$ exists for $0<y<p$ $(0<p\leqq\infty)$, we easily see that

$$e^{yx}G(-x)\leqq g(iy)\leqq y\int_0^\infty e^{yu}G(-u)du+G(\infty) \tag{2.3}$$

for all $x>0$ and for all $0<y<p$.

In the following, we shall consider the relationship between the asymptotic behaviors of $\log g(iy)$ when $y\to\infty$ and of $-\log G(-x)$ when $x\to\infty$.

We start with the result of Ramachandran (1966, Theorem 1). Esseen (1965) also has noticed it.

Lemma 2.

$$\limsup_{y\to\infty} y^{-1}\log g(iy)=b$$

if and only if $\lim_{x\to\infty}-x^{-1}\log G(-x)=\infty$ *and*

$$-\mathrm{l.ext.}\, G=b\ ,$$

where $\mathrm{l.ext.}\, G=\inf\{x\colon G(x)>0\}$ *and* $-\infty<b\leqq\infty$.

The following lemma is essentially due to Ramachandran (1962, Theorem 7.1).

Lemma 3.

$$\limsup_{y\to\infty} y^{-1-(1/a)}\log g(iy)=b^{1/a}$$

if and only if $\lim_{x\to\infty}-x^{-1}\log G(-x)=\infty$ *and*

$$\liminf_{x\to\infty}-x^{-1-a}\log G(-x)=r_a b^{-1}\ ,$$

where $0<a<\infty$, $0\leqq b\leqq\infty$ *and* $r_a=a^a(1+a)^{-1-a}$.

Proof. It suffices to prove that for every $0<b<\infty$,

$$\limsup_{y\to\infty} y^{-1-(1/a)}\log g(iy)\leqq b^{1/a} \tag{2.4}$$

if and only if $\lim_{x\to\infty}-x^{-1}\log G(-x)=\infty$ and

$$\liminf_{x\to\infty}-x^{-1-a}\log G(-x)\geqq r_a b^{-1}\ . \tag{2.5}$$

The proof of this is carried out without any change of the way of proof of Ramachandran.

If (2.4) holds, then necessarily $\lim_{x\to\infty}-x^{-1}\log G(-x)=\infty$ by virtue of (2.2), and for an arbitrary $\varepsilon>0$

$$g(iy)\leqq\exp\left((b^{1/a}+\varepsilon)y^{1+(1/a)}\right)$$

for all large $y>0$. From the first inequality of (2.3), it follows that

$$-\log G(-x) \geqq yx-(b^{1/a}+\varepsilon)y^{1+(1/a)}$$

for all $x>0$ and for all large $y>0$. Take $y=a^a(1+a)^{-a}(b^{1/a}+\varepsilon)^{-a}x^a$ for large $x>0$, we then have

$$-x^{-1-a}\log G(-x) \geqq r_a(b^{1/a}+\varepsilon)^{-a}$$

which yields (2.5).

We prove the converse. The assumption $\lim_{x\to\infty} -x^{-1}\log G(-x)=\infty$ and the statement (2.2) ensure that $g(iy)$ exists for any $y>0$. If (2.5) holds, then for an arbitrary $\varepsilon>0$

$$G(-x) \leqq \exp(-(b+\varepsilon)^{-1}r_a x^{1+a})$$

for all $x>B=B(\varepsilon)$. Hence from the second inequality of (2.3) we have that

$$\begin{aligned} g(iy) &\leqq yG(\infty)\int_0^B \exp(yx)dx+y\int_B^\infty \exp(yx-(b+\varepsilon)^{-1}r_a x^{1+a})dx+G(\infty) \\ &\leqq G(\infty)\exp(By)+Ky^{1+(1/a)}\exp((b+\varepsilon)^{1/a}y^{1+(1/a)}) \\ &\qquad\qquad \text{(see Kawata (1972), Lemma 11.11.1)} \\ &\leqq Cy^{1+(1/a)}\exp((b+\varepsilon)^{1/a}y^{1+(1/a)}) \end{aligned}$$

for all large $y>0$, where $K>0$ and $C>0$ are constants. Therefore

$$\log g(iy) \leqq (1+o(1))(b+\varepsilon)^{1/a}y^{1+(1/a)} \quad \text{as} \quad y\to\infty\,.$$

This yields (2.4).

Lemma 4.

$$\limsup_{y\to\infty} y^{-1}s(\log g(iy))=b$$

if and only if $\lim_{x\to\infty} -x^{-1}\log G(-x)=\infty$ *and*

$$\liminf_{x\to\infty} -(xs(x))^{-1}\log G(-x)=b^{-1}$$

where $0\leqq b\leqq\infty$ *and* $s(x)$ *is any strictly increasing continuous function defined on* $(-\infty,\infty)$ *with* $-\infty<s(-\infty)$ *and* $s(\infty)=\infty$ *such that* $(\log\log x)^{-1}\log s(x)$ *is nonincreasing over* (A,∞) *for some* $A>0$.

Proof. It suffices to prove that for every $0<b<\infty$,

(2.6) $$\limsup_{y\to\infty} y^{-1}s(\log g(iy)) \leqq b$$

if and only if $\lim_{x\to\infty} -x^{-1}\log G(-x)=\infty$ and

(2.7) $$\liminf_{x\to\infty} -(xs(x))^{-1}\log G(-x) \geqq b^{-1}\,.$$

To prove this, we first observe that $s(u)$ has the property that

$$s(cus(u)+d)=(1+o(1))s(u) \quad \text{as} \quad u\to\infty \tag{2.8}$$

for any costant $c>0$ and d.

If (2.6) holds, then again from (2.2) necessarily $\lim_{x\to\infty} -x^{-1}\log G(-x)=\infty$, and for an arbitrary $\varepsilon>0$

$$g(iy)\leqq \exp s^{-1}((b+\varepsilon)y)$$

for all large $y>0$. Therefore, from the first inequality of (2.3), it follows that

$$-\log G(-x)\geqq yx-s^{-1}((b+\varepsilon)y)$$

for all $x>0$ and for all large $y>0$. If we take $y=(b+\varepsilon)^{-1}s(x)$ for large $x>0$, then we have

$$-(xs(x))^{-1}\log G(-x)\geqq(b+\varepsilon)^{-1}-(s(x))^{-1}$$

which yields (2.7).

Conversely, if (2.7) holds, then for an arbitrary $\varepsilon>0$

$$G(-x)\leqq \exp(-(b+\varepsilon)^{-1}xs(x))$$

for all $x\geqq B=B(\varepsilon)$. From (2.2), (2.3) and the assumption $\lim_{x\to\infty}-x^{-1}\log G(-x)=\infty$, we have that for any $C>0$ and for any $y>0$

$$g(iy)\leqq yG(\infty)\int_0^C \exp(yx)dx+y\int_C^\infty G(-x)\exp(yx)dx+G(\infty)\ .$$

Fix $\delta>0$ and set $C=C(y)=s^{-1}((b+\varepsilon)(y+\delta))$ for $y>(b+\varepsilon)^{-1}s(B)-\delta$. Then $C>B$. Hence we obtain

$$g(iy)\leqq G(\infty)\exp(Cy)+y\int_C^\infty \exp(yx-(b+\varepsilon)^{-1}xs(x))dx\ .$$

Since $(b+\varepsilon)^{-1}s(x)\geqq y+\delta$ for $x\geqq C$, we have

$$g(iy)\leqq G(\infty)\exp(Cy)+\delta^{-1}y=(G(\infty)+o(1))\exp(Cy) \quad \text{as} \quad y\to\infty\ .$$

Consequently

$$\log g(iy)\leqq ys^{-1}((b+\varepsilon)(y+\delta))+O(1) \quad \text{as} \quad y\to\infty\ .$$

From (2.8) with $u=s^{-1}((b+\varepsilon)(y+\delta))$, it follows that

$$s(\log g(iy))\leqq(1+o(1))(b+\varepsilon)(y+\delta) \quad \text{as} \quad y\to\infty\ .$$

This yields (2.6).

3. Further lemmas.

In order to prove our theorems, we need more lemmas.

Let $F(x)$ be an infinitely divisible distribution function whose characteristic function $f(t)$ has the representation (1.1). The following lemma is an immediate consequence of Lemma 1, the theorem due to Marcinkiewicz (for instance see Lukacs (1970), Theorem 11.11.1 or Kawata (1972), Theorem 11.10.1) and the result of Esseen (1965, Theorem 2).

Lemma 5.

$$\liminf_{x\to\infty} -x^{-1}\log F(-x)=\liminf_{x\to\infty} -x^{-1}\log M(-x)\,.$$

Cosider the case when $M(-x)>0$ for some $x>0$ and $\lim_{x\to\infty} -x^{-1}\log M(-x)=\infty$. For any constant $0<c<-\mathrm{l.ext.}\,M$, by choosing suitable constants c and d, we may write $\log f(iy)$ as

$$\log f(iy)=\int_{-\infty}^{-c} e^{-yx}dM(x)+\int_{-c}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)+\sigma^2y^2/2+cy+d\,.$$

As is easily seen

$$\log f(iy)=(1+o(1))\int_{-\infty}^{-c} e^{-yx}dM(x)\quad\text{as}\quad y\to\infty$$

and hence

$$\log\log f(iy)=(1+o(1))\log\int_{-\infty}^{-c} e^{-yx}dM(x)\quad\text{as}\quad y\to\infty\,.$$

Thus we obtain

Lemma 6. *If $M(-x)>0$ for some $x>0$ and $\lim_{x\to\infty} -x^{-1}\log M(-x)=\infty$, then for any constant $0<c<-\mathrm{l.ext.}\,M$,*

$$\log\log f(iy)=(1+o(1))\log m_c(iy)\quad\text{as}\quad y\to\infty\,,$$

where $m_c(iy)=\int_{-\infty}^{-c} e^{-yx}dM(x)$.

Lemma 7. *Let $1<a<\infty$. Then*

$$\text{(3.1)}\qquad e^{-1}\limsup_{x\to0+} x^{1+(1/a)}|M(x)|\le\limsup_{y\to\infty} y^{-1-(1/a)}\int_{0+}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)$$
$$\le a\Gamma(1-(1/a))\limsup_{x\to0+} x^{1+(1/a)}|M(x)|.$$

Moreover if $\lim_{x\to0+} x^{1+(1/a)}|M(x)|$ exists, then

(3.2) $$\lim_{y\to\infty} y^{-1-(1/a)}\int_{0+}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)=a\Gamma(1-(1/a))\lim_{x\to 0+} x^{1+(1/a)}|M(x)|\ .$$

Proof. Since the condition $\int_{0+}^{1} x^2 dM(x)<\infty$ implies that $x^2M(x)\to 0$ as $x\to 0+$, we have that for every $a>0$,

(3.3) $$\begin{aligned} &y^{-1-(1/a)}\int_{0+}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)\\ &=y^{-1-(1/a)}\int_{0+}^{1}(e^{-yx}-1+yx)dM(x)+o(1)\\ &=y^{-1-(1/a)}[(e^{-yx}-1+yx)M(x)]_{0+}^{1}+y^{-1/a}\int_{0+}^{1}M(x)(e^{-yx}-1)dx+o(1)\\ &=y^{-1/a}\int_{0+}^{1}|M(x)|(1-e^{-yx})dx+o(1)\\ &=\int_{0+}^{y}(y^{-1}v)^{1+(1/a)}|M(y^{-1}v)|v^{-1-(1/a)}(1-e^{-v})dv+o(1)\quad\text{as}\quad y\to\infty\ . \end{aligned}$$

As is easily seen

$$\int_{0}^{\infty} v^{-1-(1/a)}(1-e^{-v})dv=a\Gamma(1-(1/a))$$

when $a>1$. Hence we obtain the second inequality of (3.1) and the equation (3.2) by applying the Fatou's Lemma and the dominated convergence theorem, respectively. On the other hand, the inequalities

$$\begin{aligned} (3.3)&\geqq y^{-1/a}\int_{0}^{1/y}|M(x)|(1-e^{-yx})dx+o(1)\\ &\geqq e^{-1}y^{-1-(1/a)}|M(y^{-1})|+o(1)\quad\text{as}\quad y\to\infty \end{aligned}$$

ensure that the first inequality of (3.1) holds.

In the following, $F(x)$ is supposed to be the covolution of two distribution functions $F_1(x)$ and $F_2(x)$. The infinite divisibility for $F(x)$ is not necessarily assumed. For any small $\varepsilon>0$ and for any small $\delta>0$, choosing $b>0$ such that $F_2(b)\geqq 1-\varepsilon$, we have

(3.4) $$(1-\varepsilon)F_1(-(1+\delta)x)\leqq F_1(-(1+\delta)x)F_2(b)\leqq F_1(-(1+\delta)x)F_2(\delta x)\leqq F(-x)$$

for all $x\geqq\delta^{-1}b$.

When $F_1(x)$ is a normal distribution function with variance $\sigma^2\geqq 0$, it obviously follows that

(3.5) $$\lim_{x\to\infty} -x^{-2}\log F_1(-x)=(2\sigma^2)^{-1}\ .$$

When $F_1(x)$ is a compound Poisson distribution function defined by

$$F_1(x)=e^{-\lambda}\sum_{k=0}^{\infty}\lambda^k(k!)^{-1}G^{*k}(x)\,,\qquad \lambda>0\,, \tag{3.6}$$

where $G(x)$ is a distribution function with $-\infty\leqq \text{l. ext.}\,G<0$ and $G^{*k}(x)$ indicates the k-fold convolution of $G(x)$ with itself, we necessarily have

$$\limsup_{x\to\infty}-(x\log x)^{-1}\log F_1(-x)\leqq(-\text{l. ext.}\,G)^{-1}\,. \tag{3.7}$$

This follows from the inequalities

$$F_1(-nc)\geqq e^{-\lambda}\lambda^n(n!)^{-1}G^{*n}(-nc)\geqq e^{-\lambda}[\lambda G(-c)n^{-1}]^n$$

for any constant $0<c<-\text{l. ext.}\,G$ and for every positive integer n.

From (3.4), (3.5) and (3.7), we have

Lemma 8. *Suppose that $F=F_1*F_2$, where $F_2(x)$ is any distribution function.*

(i) *If $F_1(x)$ is a normal distribution function with variance $\sigma^2\geqq0$, then*

$$\limsup_{x\to\infty}-x^{-2}\log F(-x)\leqq(2\sigma^2)^{-1}\,.$$

(ii) *If $F_1(x)$ is a compound Poisson distribution function defined by* (3.6), *then*

$$\limsup_{x\to\infty}-(x\log x)^{-1}\log F(-x)\leqq(-\text{l. ext.}\,G)^{-1}\,,$$

provided that $-\infty\leqq\text{l. ext.}\,G<0$.

4. Proofs of theorems.

Now we go back to the proofs of theorems.

Proof of Theorem 1. Write $\limsup_{x\to0+}x^{1+\alpha}|M(x)|^\alpha=p$. Since

$$\log f(iy)=\int_{0+}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)-\gamma y$$

when $M(-x)=0$ for all $x>0$ and $\sigma^2=0$, it follows from Lemma 7 that

$$e^{-1}p^{1/\alpha}\leqq\limsup_{y\to\infty}y^{-1-(1/\alpha)}\log f(iy)\leqq\alpha\Gamma(1-(1/\alpha))p^{1/\alpha}\,.$$

By Lemma 3, this is equivalent to that

$$\alpha^\alpha(1+\alpha)^{-1-\alpha}e^\alpha p^{-1}\geqq\liminf_{x\to\infty}-x^{-1-\alpha}\log F(-x)\geqq(1+\alpha)^{-1-\alpha}[\Gamma(1-(1/\alpha))]^{-\alpha}p^{-1}\,.$$

Thus the first assertion is shown to hold with $k_\alpha=(1+\alpha)^{-1-\alpha}[\Gamma(1-(1/\alpha))]^{-\alpha}$ and $h_\alpha=\alpha^\alpha(1+\alpha)^{-1-\alpha}e^\alpha$. The second assertion is verified in the same way.

Proof of Theorem 2.

(i) We first observe that the equation (3.3) with $a=1$ implies that

$$\int_{0+}^{\infty}(e^{-yx}-1+yx(1+x^2)^{-1})dM(x)=o(y^2) \quad \text{as} \quad y\to\infty .$$

Therefore we see that $\lim_{y\to\infty} y^{-2}\log f(iy)=\sigma^2/2$ if $M(-x)=0$ for all $x>0$. Hence we have

$$\liminf_{x\to\infty} -x^{-2}\log F(-x)=(2\sigma^2)^{-1}$$

by Lemma 3. On the other hand, we have

$$\limsup_{x\to\infty} -x^{-2}\log F(-x)\leqq(2\sigma^2)^{-1}$$

by Lemma 8 (i).

(ii) Write $b=-\text{l. ext.}\, M$. When $b=0$, our assertion is trivial because of (i). When $0<b<\infty$ or $b=\infty$ and $\lim_{x\to\infty} -x^{-1}\log M(-x)=\infty$, choosing a constant $0<c<b$ and then using Lemma 2, we have

$$\limsup_{y\to\infty} y^{-1}\log m_c(iy)=b , \qquad m_c(iy)=\int_{-\infty}^{-c} e^{-yx}dM(x) .$$

From Lemma 6 and Lemma 4 with $s(x)=\log x$ for large $x>0$, it follows that

$$\liminf_{x\to\infty} -(x\log x)^{-1}\log F(-x)=b^{-1}$$

When $b=\infty$ and $\liminf_{x\to\infty} -x^{-1}\log M(-x)<\infty$, noticing Lemma 5, we obtain

$$\liminf_{x\to\infty} -(x\log x)^{-1}\log F(-x)=0 .$$

Therefore the equation

$$\liminf_{x\to\infty} -(x\log x)^{-1}\log F(-x)=b^{-1}$$

holds when $0<b\leqq\infty$. On the other hand, by virtue of Lemma 8 (ii), the inequality

$$\limsup_{x\to\infty} -(x\log x)^{-1}\log F(-x)\leqq b^{-1}$$

also holds when $0<b\leqq\infty$.

Proof of Theorem 3. (iii) is an immediate consequence of Lemma 1 together with the theorem due to Kruglov (1970). Since (i) and (ii) are evident by Lemma 5 when $\lim_{x\to\infty} -x^{-1}\log M(-x)<\infty$, we assume that $\lim_{x\to\infty} -x^{-1}\log M(-x)=\infty$.

(i) Write $\liminf_{x\to\infty} -x^{-1-a}\log M(-x)=r_a b^{-1}$, $r_a=a^a(1+a)^{-1-a}$, $0<a<\infty$. It follows from Lemma 3 and Lemma 6 that

$$\limsup_{y\to\infty} y^{-1-(1/a)} \log\log f(iy) = b^{1/a} ,$$

i.e.,

$$\limsup_{y\to\infty} y^{-1}[\log\log f(iy)]^{a/(1+a)} = b^{1/(1+a)} .$$

Applying Lemma 4 with $s(x)=(\log x)^{a/(1+a)}$ for large $x>0$, we obtain

$$\liminf_{x\to\infty} -x^{-1}(\log x)^{-a/(1+a)} \log F(-x) = b^{-1/(1+a)} .$$

Thus the assertion is shown to hold with $c_a = a^{-a}(1-a)^{a-1}$ by replacing $a/(1+a)$ with a.

(ii) Writing $\liminf_{x\to\infty} -(xs(x))^{-1} \log M(-x) = b^{-1}$, we obtain

$$\limsup_{y\to\infty} y^{-1}s(\log\log f(iy)) = b$$

from Lemma 4 and Lemma 6. Again from Lemma 4 we have

$$\liminf_{x\to\infty} -(xs(\log x))^{-1} \log F(-x) = b^{-1} .$$

Thus our theorems are all proved.

Acknowledgment

I would like to express my thanks to Professor T. Kawata of Keio University for his continuing guidance.

References

[1] Baxter, G. and Shapiro, J. M.: *On bounded infinitely divisible random variables*, Sankhya, **22**, 253–260 (1960).
[2] Chatterjee, S. D. and Pakshirajan, R. P.: *On the unboundedness of infinitely divisible laws*, Sankhya, **17**, 349–350 (1956).
[3] Esseen, C. G.: *On infinitely divisible one-sided distributions*, Math. Scand., **17**, 65–76 (1965).
[4] Horn, R. A.: *On necessary and sufficient conditions for an infinitely divisible distribution to be normal or degenerate*, Z. Wahrscheinlichkeitstheorie verw. Geb., **21**, 179–187 (1972).
[5] Kawata, T.: *Fourier Analysis in Probability Theory*, Academic Press, New York, (1972).
[6] Kruglov, V. M.: *A note on infinitely divisible distributions*, Theory Prob. Appl., **15**, 319–324 (1970).
[7] Lukacs, E.: *Characteristic Functions*, 2nd ed., Griffin, London.
[8] Ramachandran, B.: *On the order and the type of entire characteristic functions*, Ann. Math. Statist., **33**, 1238–1255 (1962).
[9] Ramachandran, B.: *On one-sided distribution functions*, Sankhya Ser. A, **28**, 315–318 (1966).
[10] Ruegg, A. F.: *A characterization of certain infinitely divisible laws*, Ann. Math.

Statist., **41**, 1354–1356 (1970).

[11] Ruegg, A. F.: *A necessary condition on the infinite divisibility of probability distributions*, Ann. Math. Statist., **42**, 1681–1685 (1971).

[12] Sato, K.: *A note on infinitely divisible distributions and their Lévy measures*, Sc. Rep. T. K. D. Sect. A, **12**, 101–109 (1973).

[13] Steutel, F. W.: *On the tails of infinitely divisible distributions*, Z. Wahrscheinlichkeitstheorie verw. Geb., **28**, 273–276 (1974).

[14] Tucker, H. G.: *Best one-sided bounds for infinitely divisible random variables*, Sankhya Ser. A, **23**, 387–396 (1961).

[15] Wolfe, S. J.: *On moments of infinitely divisible distribution functions*, Ann. Math. Statist., **42**, 2036–2043 (1971).

Department of Mathematics
Keio University
Hiyoshi, Kohoku-ku
Yokohama 223, Japan