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TOPOLOGICAL ENTROPY OF CIRCLE ENDOMORPHISMS

By

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1. The purpose of this paper is to investigate the topological entropy $h(f)$ of a piecewise monotone and continuous transformation f of the unit circle S^1 and to relate it to the mapping degree $\deg(f)$ of f. For such a transformation f, $L(f)$ denotes the number of the maximal intervals where f is monotone. Then the limit $G(f)=\lim_{n\to\infty}(1/n)\log L(f^n)$ exists and is called the *growth number* of f. Our main results are the following Theorems 1 and 2.

Theorem 1. *Let f be a piecewise monotone and continuous transformation of S^1. Suppose that f is not a local homeomorphism. Then $h(f)=G(f)$.*

Theorem 2. (1) *For a piecewise monotone and continuous transformation f, $h(f)\geqq\log|\deg(f)|$.*

(2) *For a local homeomorphism f of S^1, $h(f)=\log|\deg(f)|$.*

Remark. Theorem 2 (1) is the simplest case of a general theorem of Manning ([4]).

In the case of transformations of the interval, the concept of growth number is introduced and extensively studied by Milnor and Thurston ([3]). Now let us recall the definition of the topological entropy of a continuous transformation of a compact space X ([1]):

$$h(f)=\sup\{H(f,\mathfrak{A})\mid\mathfrak{A}\text{ is a open covering of }X\}$$

$$H(f,\mathfrak{A})=\lim_{n\to n}\frac{1}{n}\log N(\mathfrak{A}\vee f^{-1}\mathfrak{A}\vee\cdots\vee f^{-n+1}\mathfrak{A})\ .$$

Here for open coverings $\mathfrak{A}$, $\mathfrak{B}$ of X, $\mathfrak{A}\vee\mathfrak{B}$ denotes the open covering $\{A\cap B\mid A\in\mathfrak{A}, B\in\mathfrak{B}\}$, and $N(\mathfrak{A})$ is the smallest cardinality of subcoverings of $\mathfrak{A}$.

2. A nonempty finite subset $\varDelta$ of the unit circle S^1 is called a *partition*; its cardinality is denoted by $m(\varDelta)$; the closure of a connected component of $S^1-\varDelta$ is called a *small interval* of $\varDelta$; for a piecewise monotone and continuous transformation f, $\varDelta_n$ denotes the partition $\varDelta\cup f^{-1}\varDelta\cup f^{-2}\varDelta\cup\cdots\cup f^{-n+1}\varDelta$. A continuous transformation f is called *piecewise monotone* if there exists a partition $\varDelta$ such that

f is strictly monotone on each small interval of Δ. Such a partition Δ is called a *monotone partition* for f. Henceforth in this paper, any transformation of S^1 is to be piecewise monotone and continuous.

Lemma 3. *For a monotone partition Δ for f,*

$$h(f)\leqq\liminf_{n\to\infty}\frac{1}{n}\log m(\Delta_n)$$

Proof. Given an arbitrary covering $\mathfrak{A}$ of S^1, one can choose a partition Δ' such that (1) Δ' contains as a subset the prescribed monotone partition Δ for f, (2) each small interval of Δ' is contained in some member of $\mathfrak{A}$. Then $m(\Delta'_n)\geqq N(\mathfrak{A}\vee f^{-1}\mathfrak{A}\vee f^{-2}\mathfrak{A}\vee\cdots\vee f^{-n+1}\mathfrak{A})$. (Each small interval of Δ'_n is contained in at least one member of $\mathfrak{A}\vee f^{-1}\mathfrak{A}\vee\cdots\vee f^{-n+1}\mathfrak{A}$.) Let J be a small interval of Δ_n. Then because Δ is a monotone partition for f, f^i maps J homeomorphically into some small interval of Δ for each $i\in\{0, 1, \dots, n-1\}$. Thus $m(\Delta'_n\cap\mathring{J})\leqq nb$, where b is so chosen that $b\geqq m(\Delta'\cap\mathring{I})$ for each small interval I of Δ. Hence $m(\Delta'_n)\leqq(nb+1)m(\Delta_n)$. Thus

$$N(\mathfrak{A}\vee f^{-1}\mathfrak{A}\vee\cdots\vee f^{-n+1}\mathfrak{A})\leqq(nb+1)m(\Delta_n)\,.$$

Letting $n\to\infty$, we have

$$H(f, \mathfrak{A})\leqq\liminf_{n\to\infty}\frac{1}{n}\log m(\Delta_n)\,.$$

Now as $\mathfrak{A}$ was an arbitrary covering of S^1 we have done with the proof.

3. A partition Δ is called a fine partition for a transformation f of S^1, in case (1) f embeds each small interval of Δ into S^1, (2) the length of each small interval of Δ, as well as that of its image by f, is smaller than 1/3 of the whole length of S^1. Thus a fine partition for f is automatically a monotone partion for f. The next lemma is a converse of lemma 3.

Lemma 4. *For a fine partition Δ for f,*

$$h(f)\geqq\limsup_{n\to\infty}\frac{1}{n}\log m(\Delta_n)\,.$$

Proof. Our proof is analogous to the argument employed by Bowen in his paper [2].

Given a positive integer N, we shall construct an open covering, say $\mathfrak{A}_N$, out of the partition $\Delta=\{x_1, x_2, \dots, x_r\}$. For each point x_j of Δ, consider the open interval U_j whose endpoints are those points of Δ_N which are next to x_j. These U_j's, together with interiors of small intervals $\mathring{I}_j$'s of Δ $(1\leqq j\leqq r)$, constitute an open covering $\mathfrak{A}_N$ of S^1. For each member $B=A_0\cap f^{-1}A_1\cap\cdots\cap f^{-n+1}A_{n-1}$ of $\mathfrak{A}_N\vee f^{-1}\mathfrak{A}_N\vee\cdots\vee f^{-n+1}\mathfrak{A}_N$ (A_i is either some U_j or some $\mathring{I}_j$), we shall count the number of the small intervals of Δ_n which intersects B. Notice that a small interval of Δ_n is of the form $I_{i_0}\cap f^{-1}I_{i_1}\cap f^{-2}I_{i_2}\cap\cdots\cap f^{-n+1}I_{i_{n-1}}$. (Here we use the assumption that Δ is fine.) Now look at the sequence $A_0, A_1, \dots, A_{n-1}$ which defines B. We shall construct a subsequence in the following way. Let i_1 be the smallest number, if any, such that A_{i_1} is equal to some U_j. Then delete the $(N-1)$ terms $A_{i_1+1}, \dots, A_{i_1+N-1}$ from the sequence. Consider the new sequence, and let i_2 be the next number, if any, such that $A_{i_2}=U_j$ for some U_j. Then delete the $N-1$ terms $A_{i_2+1}, \dots, A_{i_2+N-1}$ from the sequence. Proceeding in this fashion, one obtains finally a subsequence, say, $A_{j_1}, A_{j_2}, \dots, A_{j_s}$. Notice that if $A_{j_k}=U_j$, then A_{j_k} is contained in a union of two small intervals of Δ_N. Thus $f^{-j_1}(A_{j_1})\cap\cdots\cap f^{-j_s}(A_{j_s})$, hence *a fortiori* B, is contained in a union of at most $2^{n/N+1}$ small intervals of Δ_n. Thus

$$N(\mathfrak{A}_N\vee\cdots\vee f^{-n+1}\mathfrak{A}_N)\cdot 2^{n/N+1}\geqq m(\Delta_n)\ .$$

Letting $n\to\infty$, one gets

$$H(f,\mathfrak{A}_N)+\frac{1}{N}\log 2\geqq\limsup_{n\to\infty}\frac{1}{n}\log m(\Delta_n)\ .$$

As N is arbitrary, one obtains $h(f)\geqq\limsup_{n\to n}(1/n)\log m(\Delta_n)$, as is desired.

4. In this paragraph we shall complete the proof of Theorems 1 and 2. To begin with one has the following easy consequence of Lemmas 3 and 4.

Proposition 5. *Let Δ be an arbitrary monotone partition for a transformation f of S^1. Then the following limit exists and is equal to $h(f)$:*

$$\lim_{n\to\infty}\frac{1}{n}\log m(\Delta_n)\ .$$

Proof. Given a monotone partition Δ for f, there exists a fine partition Δ' for f such that $\Delta'\supset\Delta$. Then

$$h(f)\geqq\limsup_{n\to\infty}\frac{1}{n}\log m(\Delta'_n)\geqq\liminf_{n\to\infty}\frac{1}{n}\log m(\Delta'_n)$$
$$\geqq\liminf_{n\to\infty}\frac{1}{n}\log m(\Delta_n)\geqq h(f)\ .$$

Hence $h(f)=\lim_{n\to\infty}(1/n)\log m(\Delta_n)$, as is desired.

Proof of Theorem 1. Let f be a transformation of S^1, which is not a local homeomorphism. Let Δ_f be a monotone partition for f with the smallest cardinality. Of course such a partition is unique and each small interval of Δ_f is a maximal interval where f is monotone. Then clearly $L(f^n)=m((\Delta_f)_n)$. Thus Theorem 1 is implied by Proposition 5.

Proof of Theorem 2. (1) Notice that $m(f^{-1}\{x\})\geqq|\deg(f)|$ for each point x of S^1. Thus for a monotone partition Δ for f, one has $m(\Delta_n)\geqq m(f^{-n+1}\Delta)\geqq m(f^{-n+1}\{x_0\})\geqq |\deg(f)|^{n-1}$, where $x_0\in\Delta$. Hence $h(f)=\lim_{n\to\infty}(1/n)\log m(\Delta_n)\geqq\log|\deg(f)|$.

Proof of Theorem 2. (2) Let x be an arbitrary point of S^1. Then $\{x\}$ is a monotone partition for a local homeomorphism f. Thus $h(f)=\lim_{n\to\infty}(1/n)\log m(\{x\}_n)$. Now $m(f^{-i}\{x\})=|\deg(f)|^i$. Thus $|\deg(f)|^{n-1}\leqq m(\{x\}_n)\leqq 1+|\deg(f)|+|\deg(f)|^2+\cdots+|\deg(f)|^{n-1}$. Hence we get $h(f)=\log|\deg(f)|$.

References

[1] R. Adler, A. Konheim and M. Mc'Andrew: *Topological Entropy*, Trans. A.M.S. **114**, 309–319 (1965).
[2] R. Bowen: *Topological Entropy and Axiom A*, Proc. Symp. Pure Math. **14**, A.M.S. 23–41 (1970).
[3] J. Milnor and W. Thurston: *On Iterated Maps of the Interval*, Preprint.
[4] A. Manning: *Topological Entropy and the First Homology Group*, Dynamical Systems-Warwick, Springer Lecture Note 468, 185–190.

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